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Research Article

Enhancing Lifetime Data Modeling With the Modified Version of Weibull Distribution: A Comparative Approach

[1]Arjun Kumar Gaire, [2]Shahid Mohammad and [3]Yogendra Bahadur Gurung

[1]*Department of Science and Humanities, Khwopa Engineering College, Purbanchal University, Bhaktapur, Nepal*
[2]*Department of Mathematics, University of Wisconsin, Oshkosh, United States*
[3]*Central Department of Population Studies, Tribhuvan University, Kirtipur, Nepal*

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Corresponding Author:
Arjun Kumar Gaire
Department of Science and Humanities, Khwopa Engineering College, Purbanchal University, Bhaktapur, Nepal
Email: arjun.gaire@gmail.com

Abstract: This study introduces a new two-parameter univariate distribution, a modified version of the Weibull distribution, for a non-negative realm. Originating from a framework introduced by Azzalini and Capitanio, the distribution exhibits a monotonically increasing hazard rate. It offers closed-form expressions for key statistical properties, including reliability functions, moments, moment-generating functions, and order statistics. An expression for random number generation is formulated. The maximum likelihood is employed for parameter inference. A simulation study shows the reliability of these estimators under various conditions. To assess the model's practical applicability, it is fitted to two real-world datasets: waiting times and tensile strength of carbon fiber. Results from quantile-quantile plots and validity tools tests show that the proposed model outperforms compared to selected alternatives, which shows its superior flexibility and accuracy to fit real-world problems.

Keywords: Carbon Fiber, Hazard Function, Maximum Likelihood, Skew, Waiting Time

Introduction

Over the past few years, researchers have observed a notable increase in the development of univariate probability distributions through generalization, modification, or extension of existing models. These efforts primarily aim to introduce additional parameters to improve the flexibility of probability distributions, particularly in accurately modeling real-world data characterized by skewness, varying hazard rates, and other complex behaviors. Among such distributions, the Weibull model has enjoyed widespread application in fields like reliability engineering, survival analysis, and environmental studies because of its adaptability to increasing or decreasing hazard rates.

However, traditional Weibull distributions and their many modifications—such as the generalized Weibull and extended Weibull families—often face limitations. These include issues like over-parameterization, mathematical intractability, or limited ability to model diverse data structures with precision. Existing Weibull extensions frequently cannot capture the full spectrum of real-world hazard behaviors or lack the flexibility needed for datasets exhibiting strong skewness or tail behavior. This creates a clear gap for a simpler, more robust alternative.

The present study introduces a novel two-parameter distribution termed the Modified Version of Weibull (MVW) distribution, derived from a general transformation approach proposed by Azzalini and Capitanio (2013). The MVW distribution is constructed to preserve the mathematical simplicity of the original Weibull model while enhancing its modeling capacity. This paper systematically explores the statistical and mathematical properties of the MVW distribution, including moments and order statistics. Additionally, an expression for random number generation and employing maximum likelihood for parameter inference. To validate the model's performance and practical utility, a simulation study is performed and applied to fit the two real-world datasets: Customer waiting times in a bank, and tensile strength measurements of carbon fiber. The results, supported by quantile-quantile (Q-Q) plots and fitted density curves, highlighted that the MVW model provides a better capture than a selective alternative. This underscores the model's potential for broader application to model the non-negative real-life data.

Motivating Examples

The distributional patterns of data in real-world problems often deviate from the normal distribution. Researchers continually seek modified probability models capable of fitting such data accurately. Our proposed model is one such modification, designed to capture real-world data in the positive realm and to accommodate deviations from normality. To illustrate

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this, we selected two real-world datasets. One dataset comprises 100 observations of waiting time (in minutes) for customers at a bank before receiving service (Ghitany *et al.*, 2008). Another data set comprises ″69 data points on the tensile strength of carbon fibers tested under tension at gauge lengths of 20 mm, measured in GPa units" (Bader and Priest, 1982). Both sets exhibit significant deviations from the normal distribution, as evidenced by their descriptive summaries and plots. Further, the histograms and box plots of both data sets are presented in Figures 1 and 2. The descriptive statistics presented in Table 1 and the graphical representations demonstrate deviations from normality, thereby supporting the selection of the MVW distribution for analysis.

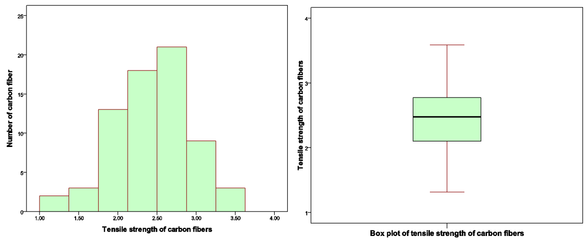

Fig. 1: Histogram and box plot of tensile strength of carbon fiber data

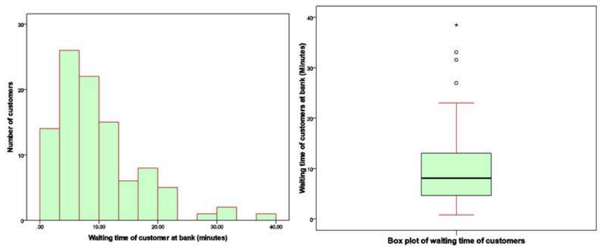

Fig. 2: Histogram and box plot of waiting time for customer data

Table 1: Fundamental statistical measures of both data sets

Data	N	Max.	Min.	Mean	SD	Sk	Ku
Waiting Time	100	.080	38.50	9.877	7.237	1.495	2.735
Tensile Strength	69	1.31	3.59	2.4513	0.4951	-0.029	0.028

Materials and Methods

In this part, we have incorporated the information about materials and methods used in this study. The general skewing method is used to formulate the new probability distribution. The R programming software is used to prepare the graphical illustration of the unique characteristics. For assessing the goodness of fit, we employed likelihood-based criteria, including ″Negative Log-Likelihood” (NLL), ″Akaike Information Criterion" (AIC), and ″Bayesian Information Criterion” (BIC). In addition, goodness of fit was evaluated using the tests based on the empirical distribution functions, such as the ″Kolmogorov-Smirnov” (K-S) test, ″Anderson-Darling" (A-D) test, and ″Cramer-Von Mises” (C-M) criterion. For Parameter estimation is conducted using the ″standard likelihood method” available tool *nlmixed* procedure in SAS softwar.

Preliminaries

In this section, background information on the two-parameter Weibull, skew-normal, and the general formula of skew distributions has been presented.

Weibull Distribution

Two-parameter Weibull distribution for a random variable X is defined through its probability density function (PDF) and cumulative distribution function (CDF) in Equations (1) and (2), respectively.

$$g(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right)}, \text{ for } x > 0 \quad (1)$$

$$G(x) = 1 - e^{\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right)} \quad (2)$$

Here, $\alpha > 0$ represents the shape of the distribution, while $\beta > 0$ denotes the scale. This distribution was used by researchers across disciplines (Johnson *et al.*, 1994; Murthy *et al.*, 2004; Lai *et al.*, 2006). Recently, the Weibull distribution was also used to analyze the data of age at menopause of Nepalese women (Gaire *et al.*, 2023b).

Numerous generalizations of distributions have emerged within the realm of univariate probability distributions. Researchers have extended, modified, and generalized the Weibull distribution by incorporating scale, location, or threshold parameters. Some notable examples of modified distributions including, but not restricted to, the complementary Weibull (Drapella,1993); generalized Weibull (Mudholkar and Kollia, 1994); extended Weibull (Xie *et al.*, 2002); Marshall–Olkin Weibull (Ghitany *et al.*, 2005); Beta exponential Weibull (Nadarajah and Kotz, 2006); extended flexible Weibull (Bebbington *et al.*, 2007); Beta Weibull (Lee *et al.*, 2007); generalized modified Weibull (Carrasco *et al.*, 2008); modified Weibull (Sarhan and Zaindin, 2009); Beta modified Weibull (Silva *et al.*, 2010; Nadarajah *et al.*, 2011); Kumaraswamy Weibull (Cordeiro *et al.*, 2010); Transmuted Weibull (Aryal and Tsokos, 2011); Gamma-exponentiated Weibull (Pinho *et al.*, 2012); transmuted modified Weibull (Khan *et al.*, 2018); Weibull-G family (Bourguignon *et al.*, 2021), and three-parameter modification of Weibull (Tashkandy and Emam, 2023). In alignment with these advancements, the MVW distribution has been introduced.

The General Skew Distribution

Azzalini (1985, 2005) initially introduced the method of skewing the normal distribution, where an additional asymmetry parameter $\lambda > 0$ was incorporated to extend the ″standard normal distribution". The PDF of the resulting ″skew-normal" distribution was defined as:

$$f(z) = 2\phi(z)\,\phi(\lambda z), \text{ for } z, \lambda \in R \quad (3)$$

where *φ(z)* and *Φ(λz)* in Equation (3) represent the "Standard normal" PDF and CDF, respectively.

After identifying the utility of this approach, Azzalini and Capitanio (2013) generalized the formulation by replacing the standard normal components with an arbitrary base distribution. This led to a more flexible family of distributions, where the PDF is expressed as:

$$f(x) = 2g(x)\,\mathrm{G}(x),\ x \in R, \tag{4}$$

In Equation (4), $g(x)$ and $G(x)$ are the functions to be chosen as the baseline distribution. Such formulation retains the skewing mechanism of Equation (3) while broadening its applicability to non-normal settings. Gupta *et al.* (2002) introduced skew-uniform, skew-t, skew-Cauchy, skew-Laplace, and skew-logistic models utilizing this concept. Generalized skew-Cauchy model induced by Huang and Chen (2007). Subsequently, Nadarajah (2009) conducted a detailed study on the skew-logistic distribution. In all aforementioned cases, symmetrical base distributions were selected. The skew log-logistic (SLLog) distribution, introduced by Gaire *et al.* (2019) and further investigated by Gaire and Gurung (2024b), considered the log-logistic distribution as a base distribution. This choice diverges from symmetrical distributions, as advocated by (Shaw and Buckley, 2009). Recently, the SLLog distribution was applied to model data of age at first marriage of women (Gaire *et al.*, 2024a), age-specific fertility rate, and age at menarche (Gaire *et al.*, 2024b). These applications underscored the significance of such modifications. In our context, we opt for a Weibull distribution as the base to formulate the MVW distribution.

A Modified Version of the Weibull Distribution

In this section, we introduced the MVW distribution and presented different probability functions along with basic properties of this distribution.

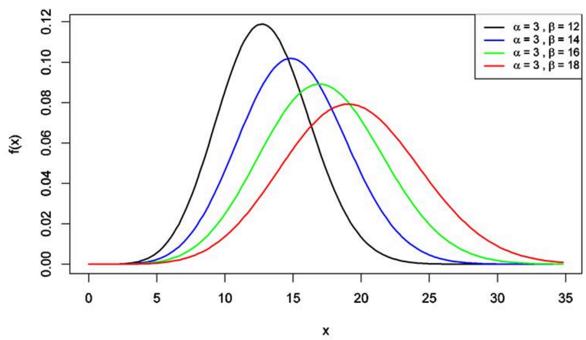

Fig. 3: Graphical illustration of the PDF of the MVW distribution

Probability Functions of the MVW Distribution

Substituting the value of *g(x)* and *G(x)* of the Weibull distribution, in Equation (4) yields the PDF of the MVW distribution expressed in Equation (5) as:

$$f(x) = \frac{2\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right)}\left(1 - e^{\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right)}\right);\ x > 0 \tag{5}$$

Figure 3 illustrates different curves of the PDF of the MVW distribution for selected values of parameters. The cumulative distribution function of the MVW distribution is defined and given in Equation (6) as:

$$F(x) = \left(1 - 2e^{\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right)} + e^{\left(-2\left(\frac{x}{\beta}\right)^{\alpha}\right)}\right) \tag{6}$$

Figure 4 illustrates the CDF of the MVW distribution for various parameter values, demonstrating a monotonically increasing with the random variable X.

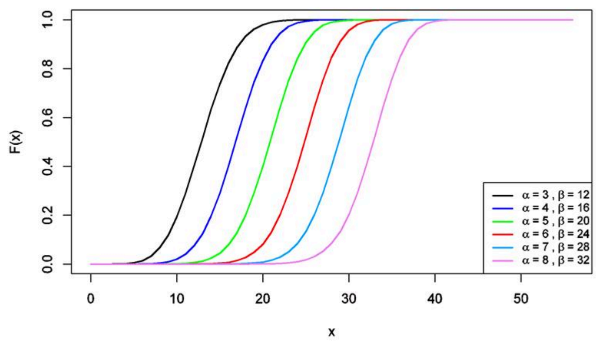

Fig. 4: Graph of the CDF of the MVW distribution

Moments

For the MVW distribution, the k^{th} moment is defined in Equation (7) as:

$$\mu_k^{'} = \frac{k\beta^k}{\alpha}\Gamma\left(\frac{k}{\alpha}\right)\left(2 - 2^{-\frac{k}{\alpha}}\right),\ \text{for } k > \alpha \tag{7}$$

In particular, the first two moments of the MVW distributions are expressed as:

$$\mu_1^{'} = \frac{\beta}{\alpha}\Gamma\left(\frac{1}{\alpha}\right)\left(2 - 2^{-\frac{1}{\alpha}}\right) \text{ and } \mu_2^{'} = \frac{2\beta^2}{\alpha}\Gamma\left(\frac{2}{\alpha}\right)\left(2 - 2^{-\frac{2}{\alpha}}\right)$$

The values of the moment about the origin for different values of parameters can easily be obtained and used to compute the value of skewness of the distribution. Similarly, the k^{th} incomplete moment of the MVW distribution is given in Equation (8) as follows:

$$m_k(t) = \left(\frac{\beta^{\alpha}}{2}\right)^{\frac{k}{\alpha}}\left[\Gamma\left(\frac{k+\alpha}{\alpha}, \frac{2t^{\alpha}}{\beta^{\alpha}}\right) - 2^{\frac{k}{\alpha}+1}\Gamma\left(\frac{k+\alpha}{\alpha}, \frac{t^{\alpha}}{\beta^{\alpha}}\right) + \left(2^{\frac{k}{\alpha}+1} - 1\right)\Gamma\left(\frac{k+\alpha}{\alpha}, \frac{2t^{\alpha}}{\beta^{\alpha}}\right)\right] \tag{8}$$

Where $\Gamma(\mathrm{s}, \mathrm{x}) = \int_x^{\infty} t^{s-1}e^{-t}dt$ is an "upper incomplete gamma function".

Moment Generating and Characteristics Function

Using the power series of the exponential function e^{tx}, the "moment generating function" $M_x(t)$ is obtained in the form:

$$M_x(t) = \sum_{n=0}^{\infty}\frac{t^n}{n!}\mu_n^{'}$$

where $\mu_n^{'}$ is the n^{th} moment expressed in Equation (7).

The function $\varphi_x(t)$, known as the "characteristic function of a random variable X", is defined as the expected value of e^{itx}. It can be expressed as:

$$\varphi_x(t) = \mathbb{E}\left(e^{itx}\right) = \mathbb{E}\left(Cos\left(tx\right)\right) + i\mathbb{E}\left(Sin\left(tx\right)\right)$$

Where i is the imaginary unit. Using the power series expansion of the cosine and sine functions as:

$\cos(tX) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} X^{2n}$ and

$\sin(tX) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} X^{2n+1}$

The characteristic function $\varphi_x(t)$, corresponding to the MVW distribution, can be expressed through a series expansion. Utilizing the previous simplification, it takes the following form:

$$\varphi_x(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \mu'_{2n} + i \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \mu'_{2n+1}$$

Where μ'_{2n} and μ'_{2n+1} are the moments obtained from Equation (7).

Quantile Function and Random Number Generation

Assume that the random variable X follows the MVW distribution whose CDF is in Equation (6) and $p \in (0,1)$, where p is a uniformly distributed variable. Inversion of the CDF yields the quantile function for the MVW distribution, which is expressed as follows:

$$F(x) = \left(1 - e^{\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right)}\right)^2 = p$$

$$X = \beta\left[-ln\left(1 - \sqrt{p}\right)\right]^{\frac{1}{\alpha}} \quad (9)$$

This derivation assumes that ($0 < p < 1$) to ensure that all terms are well-defined and that the inverse exists. It also assumes that the MVW distribution parameters α and β are strictly positive, which is necessary for the monotonicity of the CDF and valid interpretation of the logarithm and square root operations involved. Hence, Equation (9) provides a valid method for generating pseudo-random variables from the MVW distribution, provided these regularity conditions are met. The generated sets of random numbers can describe the future scenario of a continuous random variable of any social event with given α and β. Such data sets generated by the method of inversion can help to anticipate future situations. From Equation (9), quartiles corresponding to one-quarter, one-half, and three-quarters of the MVW distribution are also obtained by putting $p = \frac{1}{4}, p = 1/2$, and $p = 3/4$, respectively.

Reliability Analysis of the MVW Distribution

The probability that an item continues to function beyond time x is captured by the reliability function R(x), which is formulated in Equation (10), and visual illustrations are presented in Figure 5.

$$R(x) = 1 - F(x) = \left(2e^{\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right)} - e^{\left(-2\left(\frac{x}{\beta}\right)^{\alpha}\right)}\right) \quad (10)$$

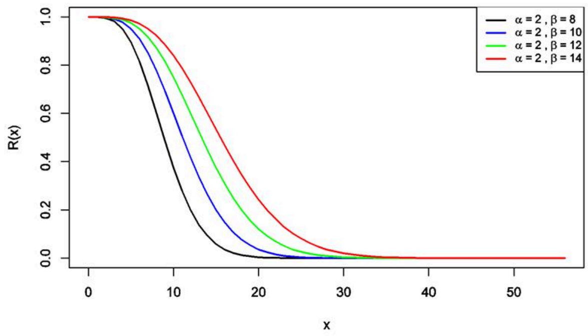

Fig. 5: Graphical illustration of the reliability function of the MVW distribution

Similarly, the hazard rate function represents the conditional likelihood of failure, assuming the system has functioned without failure until time x.

This function for the MVW distribution is formulated in Equation (11).

$$h(x) = \frac{2\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}\left(1 - e^{\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right)}\right)^{-1} \quad (11)$$

Now, the MVW distribution's cumulative hazard rate is formulated and presented in Equation (12) as:

$$H(x) = \left(\frac{x}{\beta}\right)^{\alpha} - ln\left(2 - e^{\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right)}\right) \quad (12)$$

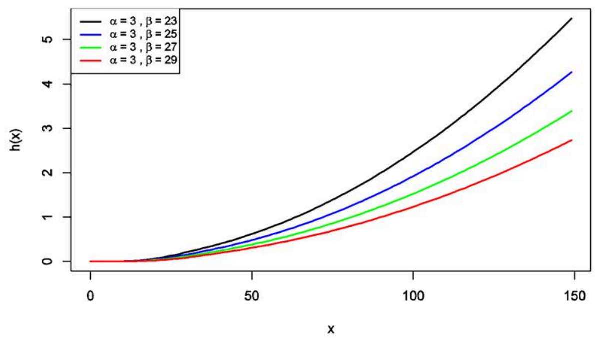

Fig. 6: Graphical illustration of the hazard rate function of the MVW distribution

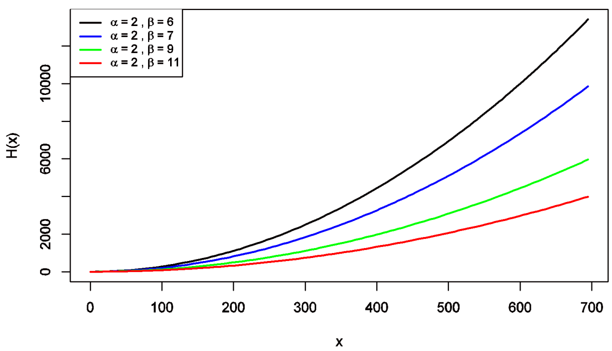

Fig. 7: Graph of the cumulative hazard rate function

Figures 6 and 7 illustrate a visual illustration of both the hazard and cumulative function of the proposed MVW distribution, choosing suitable parameter values.

It is found that the hazard rate function increases with time.

Order Statistics

A sample $X_1, X_2, \ldots, X_n$, consisting of n random variables, is taken from a two-parameter MVW distribution defined by CDF $F(x)$ and the PDF $f(x)$ in Equations (6) and (5). The corresponding order statistics of these samples are $X_{1:n} \leq X_{2:n} \leq \ldots X_{n:n}$. Then, the PDF of r^{th} order statistics for MVW distribution is formulated in Equation (13) as:

$$g_{r(x)} = \frac{n!}{(r-1)!(n-r)!} f(x) \left[F(x)\right]^{r-1} \left[1 - F(x)\right]^{n-r}$$

$$g_{r(x)} = \frac{2\alpha n!}{\beta(r-1)!(n-r)!} \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}} \left(1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)^{2r-1} \left(2e^{-\left(\frac{x}{\beta}\right)^{\alpha}} - e^{-2\left(\frac{x}{\beta}\right)^{\alpha}}\right) \quad (13)$$

The expression of the PDF of the sample minimum X_1 of order statistics is defined as follows in Equation (14). $fX_{(1)}(x) = n\left(1 - F(x)\right)^{n-1} f(x)$

$$fX_{(1)}(x) = \frac{2\alpha n}{\beta} \left(\frac{x}{\beta}\right)^{\alpha} e^{-\left(\frac{x}{\beta}\right)^{\alpha}} \left(1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right) \left(e^{-2\left(\frac{x}{\beta}\right)^{\alpha}} - 2e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)^{n-1} \quad (14)$$

Similarly, the expression of the density of sample maximum X_n is defined as follows in Equation (15).

$$fX_{(n)}(x) = n\left(F(x)\right)^{n-1} f(x)$$

$$fX_{(n)}(x) = \frac{2\alpha n}{\beta} \left(\frac{x}{\beta}\right)^{\alpha} e^{-\left(\frac{x}{\beta}\right)^{\alpha}} \left(1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)^{2n-1} \quad (15)$$

Finally, the PDF of the joint density of the sample maximum and sample minimum is defined as follows in Equation (16).

$$fX_{(1)X_{(n)}}(x, y) = n(n-1)\left[F(y) - F(x)\right]^{n-2} f(x) f(y) \quad (16)$$

This holds for $x < y$ and both variables are in the support of the MVW distribution.

Mean Deviation

Consider a random variable X follows the MVW distribution with average (μ) and median (M). The expression of mean deviation taken from the mean and median is formulated as:

$$MD(\mu) = \int_0^\infty |x - \mu| f(x)\, dx = \mu - 2\int_0^M x f(x)\, dx$$

$$MD(M) = \int_0^\infty |x - M| f(x)\, dx = 2\mu F(\mu) - 2\int_0^\mu x f(x)\, dx$$

Therefore, the equation of mean deviation for MVW distribution taken respectively from mean and median are:

$$MD(\mu) = 2\mu\left(1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right)^2 - 2m_1(\mu)$$

$$MD(M) = \mu - 2m_1(M)$$

Where $\mu = \frac{\beta}{\alpha\alpha}\Gamma\left(\frac{1}{\alpha}\right)\left(2 - 2^{-\frac{1}{\alpha}}\right)$, $M = \beta\left[-ln\left(1 - \sqrt{0.5}\right)\right]^{\frac{1}{\alpha}}$. And the value of m_1 (.) can be obtained from Equation (8).

Estimation and Inferences

The maximum likelihood technique is applied to formulate the expression for estimating the associated constants of the MVW distribution. Suppose $X_1, X_2, \ldots, X_n$ consists of n samples following MVW distribution. To formulate the parameter, the ″likelihood function" L is defined in Equation (17) as:

$$L = \left(\frac{2\alpha}{\beta}\right)^n \prod_{i=1}^n \left(\frac{x_i}{\beta}\right)^{\alpha-1} \prod_{i=1}^n \left(e^{-\left(\frac{x_i}{\beta}\right)^{\alpha}} - e^{-2\left(\frac{x_i}{\beta}\right)^{\alpha}}\right) \quad (17)$$

The log-likelihood function (lnL) of the MVW distribution is derived in Equation (18) as:

$$lnL = \mathrm{n}ln\left(\frac{2\alpha}{\beta}\right) + (\alpha - 1)\sum_{i=1}^n ln\left(\frac{x_i}{\beta}\right) + \sum_{i=1}^n ln\left(e^{-\left(\frac{x_i}{\beta}\right)^{\alpha}} - e^{-2\left(\frac{x_i}{\beta}\right)^{\alpha}}\right) \quad (18)$$

The following equations are formulated to estimate the parameters of the MVW distribution in Equations (19) and (20) respectively as $\theta = (\alpha, \beta)$:

$$\frac{\partial lnL}{\partial \alpha} = \frac{\mathrm{n}}{\alpha} + \sum_{i=1}^n ln\left(\frac{x_i}{\beta}\right) - \sum_{i=1}^n \frac{\left(\frac{x_i}{\beta}\right)^{\alpha} ln\left(\frac{x_i}{\beta}\right)\left(1 + 2e^{-\left(\frac{x_i}{\beta}\right)^{\alpha}}\right)}{\left(1 - e^{-\left(\frac{x_i}{\beta}\right)^{\alpha}}\right)} \quad (19)$$

$$\frac{\partial lnL}{\partial \beta} = -\frac{\mathrm{n}}{\beta} - \frac{n(\alpha - 1)}{\beta} + \frac{\alpha}{\beta^2}\sum_{i=1}^n \frac{\left(\frac{x_i}{\beta}\right)^{\alpha-1}\left(1 + 2e^{-\left(\frac{x_i}{\beta}\right)^{\alpha}}\right)}{\left(1 - e^{-\left(\frac{x_i}{\beta}\right)^{\alpha}}\right)} \quad (20)$$

By solving these nonlinear systems of equations and setting the score vector to zero, we obtain the maximum likelihood estimators (MLEs) for the unknown values of parameters $\theta = (\alpha, \beta)$ of the MVW distribution. Since the maximum likelihood equations derived for the MVW distribution are nonlinear and do not admit closed-form solutions, we can use numerical optimization techniques to estimate the parameters. Specifically, the Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm can be implemented using the optim function available in different software (R language, Python, or SAS). Initial values for the parameters can be selected based on the method of moments, and convergence was assessed using a relative tolerance of limits. The log-likelihood function can be maximized iteratively, ensuring that the parameter estimates remain within the valid parameter space throughout the optimization process. For statistical inference and interval estimation, we require the observed information matrix as follows:

$$J_n(\theta) = \begin{bmatrix} J_{\alpha\alpha} & J_{\alpha\beta} \\ J_{\beta\alpha} & J_{\beta\beta} \end{bmatrix}$$

Every components are the ″second-order derivative of the log-likelihood function" concerning parameters in the subscripts of the ″Information Matrix" Jn(θ). By using this information matrix Jn(θ), we can obtain the Fisher

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Information matrix, which can be used for the inference of parameters.

Numerical Application

This section incorporates the simulation study of the MVW and compares the result with Weibull and Gamma distributions. Further, the MVW model is applied to real data sets to test the performance analysis of the proposed model.

A Simulation Study

To compare the performance of the Maximum likelihood estimate method presented previously, a simulation study was performed. We generate four sets of samples of size n= fifty, hundred, two hundred and five hundred each random sample for three sets of parameters as $I: \alpha = 1$ and $\beta = 3$; $II: \alpha = 2$ and $\beta = 2$, and $III: \alpha = 3$ and $\beta = 1$ by using the random number generator of the MVW distribution. The simulation study is repeated $N = 1000$ times for each pair of parameter sets.

Table 2: ML Estimates of parameters with MSEs, and SDs for MVW

Sample Size (n)	Actual Values		Estimated		MSEs		SDs	
	α	β	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
50	1	3	1.0322	3.0263	0.0137	0.1282	0.1099	0.3548
	2	2	2.0483	1.9976	0.0538	0.0147	0.2232	0.1196
	3	1	3.0838	1.0002	0.1229	0.0016	0.3447	0.0405
100	1	3	1.0117	3.0114	0.0061	0.0666	0.0792	0.2638
	2	2	2.0221	1.9989	0.0240	0.0073	0.1596	0.0861
	3	1	3.0500	1.0019	0.0599	0.0007	0.2294	0.0288
200	1	3	1.0079	3.0089	0.0028	0.0305	0.0553	0.1820
	2	2	2.0152	2.0012	0.0122	0.0036	0.1069	0.0620
	3	1	3.0133	1.0014	0.0228	0.0004	0.1631	0.0210
500	1	3	1.0036	3.0064	0.0012	0.0137	0.0341	0.1168
2	2	2	2.0096	2.0012	0.0046	0.0015	0.0676	0.0384
	3	1	3.0151	1.0004	0.0100	0.0002	0.1006	0.0129

Table 3: ML Estimates, MSEs, SDs for two-parameter Gamma Distribution

Sample Size (n)	Actual Values		Estimated		MSEs		SDs	
	α	β	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
50	1	3	1.0591	2.9431	0.0434	0.4697	0.2052	0.6667
2	2	2	2.1123	1.9601	0.1749	0.1570	0.4306	0.4387
3	3	1	3.1746	0.9843	0.4532	0.0455	0.6372	0.2135
100	1	3	1.0244	2.9895	0.0179	0.2571	0.1292	0.4786
2	2	2	2.0595	1.9783	0.0837	0.0882	0.2907	0.3029
3	3	1	3.0832	0.9902	0.1847	0.0314	0.4382	0.1458
200	1	3	1.0120	2.9910	0.0088	0.1190	0.0900	0.3282
2	2	2	2.0240	1.9922	0.0373	0.0443	0.1922	0.2127
3	3	1	3.0730	0.9857	0.0997	0.0111	0.3004	0.1018
500	1	3	1.0097	3.0185	0.0026	0.0470	0.0556	0.2141
2	2	2	2.0120	1.9917	0.0145	0.0174	0.1159	0.1295
3	3	1	3.0184	0.9977	0.0351	0.0044	0.1887	0.0663

The sets of parameters assigned, the estimated value, Mean Square of Errors (MSEs), and Standard Deviations (SDs) have been presented in Table 2. As the sample size grows, a consistent decrease in the MSEs and SDs for the estimates is observed across all cases. For comparison of performing the MVW distribution, a similar simulation analysis is performed for the Gamma and Weibull distributions. The simulation results of the Gamma distribution are presented in Table 3. The results for the Weibull distribution are presented in Table 4. The simulation study highlighted the superior performance of the MVW distribution compared to both the Gamma and Weibull distributions.

Table 4: ML Estimates, MSEs, SDs for two-parameter Weibull Distribution

Sample Size (n)	Actual Values		Estimated		MSEs		SDs	
	α	β	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
50	1	3	1.0330	2.9991	0.0158	0.1867	0.1194	0.4327
	2	2	2.0514	1.9999	0.0581	0.0209	0.2342	0.1429
	3	1	3.0961	0.9967	0.1331	0.0024	0.3463	0.0502
100	1	3	1.0198	3.0254	0.0067	0.1041	0.0836	0.3156
	2	2	2.0197	1.9988	0.0275	0.0117	0.1636	0.1044
	3	1	3.0357	0.9992	0.0619	0.0013	0.2331	0.0347
200	1	3	1.0055	3.0036	0.0032	0.0496	0.0554	0.2159
	2	2	2.0095	1.9945	0.0155	0.0051	0.1139	0.0763
	3	1	3.0167	0.9990	0.0273	0.0006	0.1646	0.0245
500	1	3	1.0015	2.9962	0.0019	0.0200	0.0350	0.1414
	2	2	2.004	1.999	0.0047	0.0024	0.0717	0.0465
	3	1	3.0125	0.9995	0.0118	0.0002	0.1031	0.0159

Steps of Random Number Generation, Parameter Estimation, and Simulation Process for MVW Distribution

In this section, the step-by-step summary of the simulation study procedure for estimating parameters of the MVW distribution using the Maximum Likelihood Estimation (MLE) method, random number generation, and the simulation process is discussed. The simulation aims to assess how well the MLE procedure recovers true parameter values under repeated sampling. The fundamental steps are presented as follows.

a. For Random Number Generation

- Phase 1: Understand the Inverse Transform Method
- Phase 2: Derive the Inversion formula
- Phase 3: Generate and validate random numbers

b. For Maximum Likelihood Estimation

- Phase 1: Initialization and Model Setup
- Phase 2: Differentiation and Equation Formation
- Phase 3: Numerical Optimization and Output

c. For the Simulation Study

- Phase 1: Preparation and Setup
- Phase 2: Data Generation and Parameter Estimation
- Phase 3: Analyze Simulation Results

The algorithms for random number generation, MLE, the simulation study, and the pseudo code (R code) for each process are presented in the Appendix.

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Numerical Application to Waiting Time and Tensile Strength Data

The adequacy and validity of the proposed model were assessed by using two real datasets that have been employed for fitting the distributional pattern. The initial data sets consist of 100 customers' queue time (minutes) at the bank for the service (Ghitany *et al.*, 2008). This data set was also utilized by Gaire (2023) and Gaire and Gurung (2024). The second data set has 69 data points on the ″tensile strength of carbon fibers tested under tension at gauge lengths of 20 mm, measured in GPa units" (Bader and Priest, 1982).

For assessing the goodness of fit, we employed likelihood-based criteria, including Negative Log-Likelihood (NLL), $AIC = 2k - 2ln\left(\hat{L}\right)$, and $BIC = ln\left(n\right)k - 2ln\left(\hat{L}\right)$. In addition, goodness of fit was evaluated using the tests based on the empirical distribution functions, such as the K-S test, A-D test, and C-M criterion. Here, k represents the total number of model parameters, n is the sample size, and $\left(\hat{L}\right)$, signifies the maximum likelihood values corresponding to the distribution. Parameter estimation was conducted using the ″standard likelihood method" available tools $nlmixed$ procedure in SAS software.

Values of α and β were selected using MLE, guided by empirical characteristics of the data. Initial values were derived from exploratory analysis to ensure convergence. Hyper-parameter tuning involved the use of the BFGS optimization tools adaptive step sizes and convergence tolerance set at 10−6, ensuring stable and accurate parameter estimation. Simulated data were used to assess the sensitivity and robustness of the parameter estimates under different initialization scenarios.

Results and Discussion

The fitted results of both data sets for the MVW distribution were compared with two-parameters Weibull, two-parameter Gamma, new exponential Weibull (EWD) (Tashkandy and Emam, 2023), new cosine Weibull (NC-Weibull) (Wu *et al.*, 2003), new flexible Weibull (NF-Weibull) (Bebbington *et al.*, 2007), SLLog (Gaire *et al.*, 2019; Gaire and Gurung, 2024). All the distributions chosen for comparison are in the positive realm. The distribution fitting results for both datasets and the associated test statistics are presented in Tables 5 and 6. The best comparative results of the proposed model MVW are presented in boldfaces appeared in both tables.

Table 5: Parameter estimation and different test statistics of waiting time data

PDF	Parameter Estimates (standard error)	NLL	AIC	BIC	K-S (p- value)	A-D (p-value)	C-M (p-value)
MVW	α = 1.0345(0.0766)	317.155	638.3	643.5	0.038	0.151	0.022
	β = 6.7227(0.5614)				(0.998)	(0.999)	(0.995)
Gamma	α = 2.0088(0.2639)	317.300	638.6	643.8	0.043	0.186	0.029
	θ = 4.9168(0.7332)				(0.994)	(0.994)	(0.980)
Weibull	α = 1.4585(0.1098)	318.731	641.5	646.7	0.058	0.406	0.061
	β = 10.955(0.7942)				(0.892)	(0.843)	(0.809)
EWD	λ = 0.0032(0.0018)	317.903	641.8	649.6	0.045	0.221	0.034
	θ = 1.6221(0.1625)				(0.988)	(0.984)	(0.963)
	ψ = 7.3945(4.4730)						
NC-Weibull	θ = 13.924(0.9601)	319.649	643.3	648.5	0.060	0.503	0.069
	τ = 1.2103(0.0963)				(0.869)	(0.743)	(0.756)
NF-Weibull	α = 0.0535(0.0047)	321.268	646.5	651.7	0.085	0.771	0.112
	β = 5.9415(0.6622)				(0.472)	(0.502)	(0.532)
SLLog	α = 1.7878(0.1432)	322.805	649.6	654.8	0.063	0.733	0.073
	β = 4.5755(0.3855)				(0.825)	(0.531)	(0.734)

Table 6: Parameter estimation and different test statistics for tensile strength data

PDF	Parameter Estimates (standard error)	NLL	AIC	BIC	K-S (p- value)	A-D (p-value)	C-M (p-value)
MVW	α = 3.8880(0.3488)	48.860	101.7	106.2	0.040	0.144	0.015
	β = 2.3286(0.0623)				(1.000)	(0.999)	(1.000)
NF-Weibull	α = 1.0197(0.0884)	49.382	102.8	107.2	0.059	0.273	0.035
	β = 7.1683(0.7204)				(0.972)	(0.957)	(0.957)
Weibull	α = 5.5049(0.5005)	49.596	103.2	107.7	0.056	0.274	0.034
	β = 2.6509(0.0612)				(0.981)	(0.956)	(0.960)
NC-Weibull	θ = 2.8246(0.0617)	50.009	104.0	108.5	0.054	0.308	0.036
	τ = 4.5845(0.4394)				(0.988)	(0.932)	(0.953)
Gamma	α = 23.382(3.9618)	50.037	104.1	108.5	0.059	0.338	0.046
	θ = 0.1048(0.0180)				(0.970)	(0.907)	(0.901)
EWD	λ = 0.0004(0.0004)	49.102	104.2	110.9	0.039	0.142	0.016
	θ = 6.1026(0.7721)				(1.000)	(0.999)	(0.999)
	ψ =7.6247(6.6542)						
SLLog	α = 6.5694(0.6364)	54.048	112.1	116.6	0.074	0.765	0.082
	β = 2.1017(0.0583)				(0.844)	(0.507)	(0.680)

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For both datasets, among the comparative models, the MVW has the lowest NLL, which indicates it captures the given data sets better than the others in terms of log likelihood. The MVW model also exhibits the lowest AIC and lowest BIC for both data sets. These criteria penalize model complexity, so a lower value means a better balance of fit and simplicity. This suggests that MVW is both accurate and parsimonious in describing the datasets. Figure 8 depicts the histogram of observed frequencies and curves of fitted frequency distribution by different models to the waiting times of customers across various models, while Figure 9 illustrates the histogram of observed frequencies and curves of fitted frequency distributions by different models to the tensile strength data. These graphs present the visualizations that underscore the superior fit and enhanced flexibility of the proposed MVW model compared to alternative models.

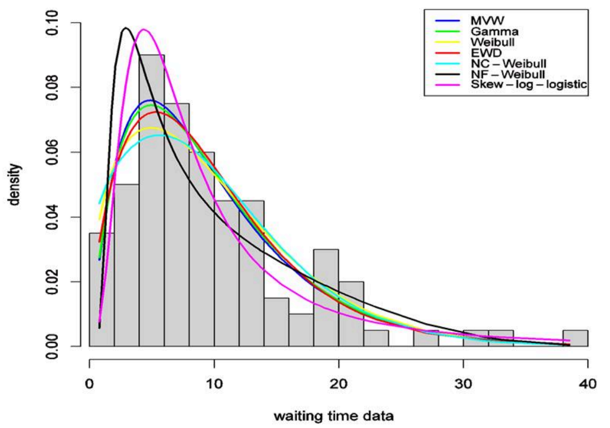

Fig. 8: Observed and fitted values of the waiting time data

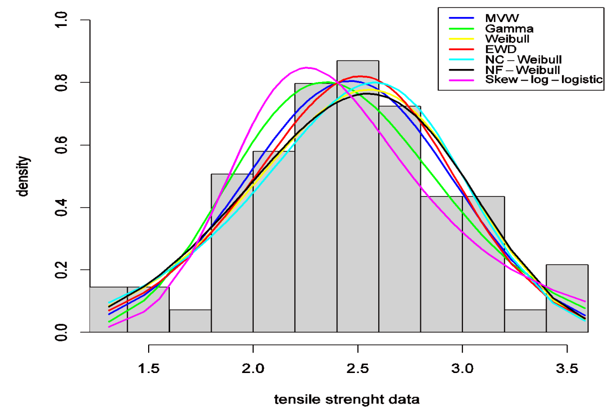

Fig. 9: Empirical and fitted values of the tensile strength data

Similarly, the MVW model shows a strong goodness of fit performance. The K-S, A-D, as well as the C-M values with p values, are the highest for this model for both datasets, showing superior performance to comparable models. The MVW has only two parameters (α, and β), like many competing models (e.g., Weibull, NF-Weibull), but still performs better. Further, Standard errors are relatively small, indicating stable parameter estimates. This model shows robustness across all measures and consistently ranks at or near the top across all evaluation metrics. The proposed MVW distribution balances model simplicity with excellent fit, indicating that the model effectively represents the underlying data pattern. The low AIC/BIC confirms MVW's efficiency and generalizability, while acceptable goodness-of-fit values show it's not just tailored to this specific dataset.

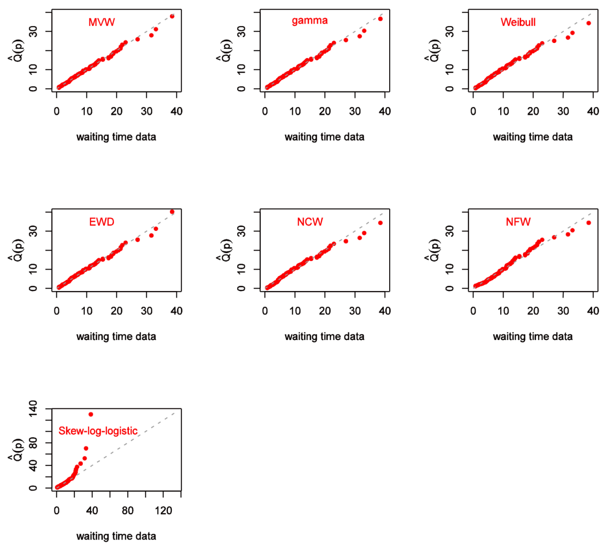

Fig. 10: Q-Q plot of waiting time data

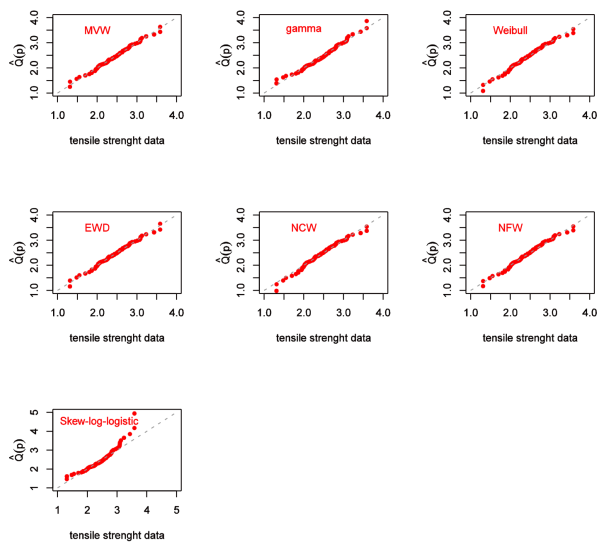

Fig. 11: Q-Q plot of tensile strength data

After utilizing the parameters result obtained by maximizing the NLL value for MVW distribution and other comparative models, we generated Q-Q plot for both datasets: waiting time (Figure 10) and strength data (Figure 11), the Q-Q plots, revealing the alignment between observed and fitted values for waiting time value and tensile strength data, respectively. The graphical illustration demonstrates the closest alignment with the 45-degree reference line and remains well within the confidence bands, indicating a superior fit to both datasets compared to alternative distributions. All the test statistics, both sets of figures, along with corresponding analysis results, affirm the MVW distributions' superior fit to the datasets.

Conclusion

Formulation of a newly proposed two-parameter probability model termed as a modified version of Weibull has been presented. The statistical features of the distribution have been examined in detail, and the rule of estimation utilizing MLE has been formulated.

A simulation was performed for the flexibility test of the MVW distribution, and various goodness-of-fit criteria were employed to evaluate its suitability. These criteria included NLL, AIC and BIC criterions. Empirical distribution criteria, K-S, A-D and C-M criteria. The model's validity was tested using real datasets on customer waiting time and the tensile strength of carbon fibers. Results from test statistics, Q-Q plots, and fitted data graphs affirm the flexibility of the proposed distribution than the Weibull distribution and other two-parameter models. The CDF of the models used for comparison are expressed in Table 7.

Table 7: Distribution models and their CDF used of comparison

Distributions	CDF
MVW	$F(x) = \left[1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^2 \; for\ x, \alpha, \beta > 0$
Gamma	$F(x) = \left[1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^2 \; for\ x, \alpha, \theta > 0$
Weibull	$F(x) = 1 - e^{-\left(\frac{x}{\beta}\right)^{\alpha}} \; for\ x, \alpha, \beta > 0$
EWD	$F(x) = 1 - e^{-\psi\left(1-e^{-\lambda x^{\theta}}\right)} \; for\ x, \lambda, \theta, \psi > 0$
NC-Weibull	$F(x) = \frac{\exp\left[\cos\left(\frac{\pi}{2} e^{-\left(\frac{x}{\theta}\right)^{\tau}}\right)\right]-1}{e-1}; \; for\ x, \theta, \tau > 0$
NF-Weibull	$F(x) = 1 - \exp\left[-e^{-\alpha x - \frac{\beta}{x}}\right]; \; for\ x, \alpha, \beta > 0$
SLLog	$F(x) = \left[\frac{1}{1+\left(\frac{x}{\beta}\right)^{-\alpha}}\right]^2 \; for\ x, \alpha, \beta > 0$

Further, suggests the applicability of the MVW distribution to fit distributional patterns in diverse world problems. Future research avenues include applying this model to other data sets and formulating regression models based on the MVW distribution as well as different methods such as the Bayesian technique were suggested to test the performance on real-world problems.

Conflicts of Interest

The corresponding author, on behalf of all co-authors, affirms that there are no conflicts of interest to declare. Also declare no ethical issues arises after publication.

Author's Contributions

Arjun Kumar Gaire: Contributed to conceptualization, methodology, validation, formal analysis, investigation, writing - original draft preparation, and visualization.

Shahid Mohammad: Contributed to conceptualization, methodology, software, validation, formal analysis, investigation, data curation, writing - review and editing, and visualization.

Yogendra Bahadur Gurung: Contributed to validation, writing - review and editing, and supervision.

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Appendix

Algorithm of Random Number Generation

Input: Parameters $\alpha > 0, \beta > 0$

Generate $p \sim Uniform(0,1)$

Compute $x = \beta\left(-log\left(1 - \sqrt{p}\right)\right)^{\frac{1}{\alpha}}$

Return: x

R Code of Random Number Generation

```
rmvw <− function (n , alpha , beta ) {
    p <− runif ( n )
    # Generate n uniform (0, 1) random numbers
    x <− \beta * (− log (1 − \sqrt {p }))^{1 / \ alpha }
    return ( x )
}
```

Algorithm of ML Estimation for MVW Distribution

Input: Sample data x = $(x_1, x_2, ..., x_n)$

Define log-likelihood function l(α,β) for MVW

Use numerical optimizer (e.g., BFGS) to maximize l(α,β)

Return: Estimated parameters $\hat{\alpha}, \hat{\beta}$

R Code for ML Estimation

```
loglik_mvw <− function (params , data ) {
    alpha <− params [1]
    beta <− params [2]
    if (alpha <= 0 | | beta <= 0) return (− Inf )
    # Constraints
    x <− data
    n <− length ( x )
    loglik <− n * log (2 * alpha \ beta ) + ( alpha − 1) * sum( log ( x / beta ) ) + sum( exp(−x / beta ) − exp(−2x / beta ) )
    return (− loglik )
    # Negative log−Likelihood for minimization
}
# Exaple Usage:
# Optim (par = c(1, 1), fn =Loglik_mvw, data = your_data)
```

Algorithm of Simulation Study for MLE

Input: True parameters (α_0, β_0), sample size n, number of simulations M

for i = 1 to M do

  Generate sample of size n from MVW(α_0, β_0)

  Estimated parameters , $\left(\hat{\alpha}_i, \hat{\beta}_i\right)$ using MLE

  Store Estimates

end for

Compute average, bias, and MSE for $\left(\hat{\alpha}, \hat{\beta}\right)$

Return: Simulation results

R Code for Simulation Study

```
Simulate_mvw <- function (nsim = 1000, n = 100, alpha = 1.5, beta = 2.0) {
    results <- matrix (NA, nrow = nsim, ncol = 2)
    for (I in 1:nsim) {
        data <- rmvw(n, alpha, beta)
        est <- tryCatch (
            optim(par = c(1, 1), fn = loglik_mvw, data = data),
            error = function(e) list(par = c(NA, NA))
        )
        results[i, ] <- est$par
    }
    colnames(results) <- c("alpha_hat", "beta_hat")
    return(results)
}
# Simulate and analyze:
# res <- simulate_mvw()
# summary(res)
```