Measurable Functional Calculi and Spectral Theory

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Abstract: In this article, the spectral theory is considered, we study the spectral families and their correspondence to the operators on the reflexive Banach spaces; assume $A$ is a well-bounded operator on reflexive Lebesgue spaces then the operator $A$ is a scalar type spectral operator. The main goals are to obtain the characterization of the well-bounded operators in the terms of the associated spectral family in the topology of dual pairing and to construct the continuous functional calculus for well-bounded operators on the Lebesgue space. It is proven that if a weak spectral family $E(\lambda)$ is concentrated on $[a, b]$ then there is a linear well-bounded operator $A \in L(X)$ on the reflexive Banach space $X$ such that $\langle A(x), y^* \rangle = b \langle x, y^* \rangle - \int_{[a, b]} \langle x, E(\lambda) y^* \rangle d\lambda$ holds for all $x \in X$ and $y^* \in X^*$.

Keywords: Functional Calculus, Banach Space, Spectral Theorem, C*-Algebra, Measurable Space, Spectral Integral, Well-Bounded Operator

Introduction

This article is dedicated to the spectral theory of the operators that are defined on the subset of the reflexive Banach space $X$. An important example of such operators is a class of well-bounded operators, which have spectral decomposition with special properties. Let us presume that the functional calculus defined on the Banach algebra of the continuous functions $AC([a, b])$ on a compact interval $[a, b]$ then its operator is well-bounded. Assuming that the functional calculus of the weak-bounded operator on $L^p$, $1 < p < \infty$ space is contractive then this operator has a scalar-type spectral. The last statement is not true in the cases when the Banach spaces are not reflexive, for example, on $L^\infty$ (Budde and Landsman, 2016; Colombo et al., 2007; Haase, 2014).

Let us consider a simpler example of the theory in Banach space, the structure of the projection measure in the Hilbert space $H$. Let $(Z, \Sigma, \eta)$ be a measurable Borel space and $(H_\xi)_{\xi \in \eta}$ be an $\eta$- measurable set of separable Hilbert spaces (Schmidgen, 2012). The projection-valued measure $E$ on $(Z, \Sigma)$ can be defined as a mapping from $\Sigma$ to the set of self-adjoint orthogonal projections on $H$ that satisfies $E(Z) = Id_H$ and the mapping from $\sigma$- algebra $\Sigma$ into the field $\phi \mapsto \langle E(\phi) x, y \rangle$ is a complex measure on $\Sigma$. In terms of the functional calculus this definition can be reformulated in the following form: Let $(\Phi, H)$ be functional calculus on a measurable space $(Z, \Sigma)$, the projection-valued measure is a mapping:

$$E : \Sigma \to L(H), \quad E(\phi) = \Phi(\chi_\phi) \in L(H)$$

for any $\phi \in \Sigma$. The main result of the theory for separable Hilbert spaces is the statement that for each projection-valued measure on the measurable space there is a unique measurable functional calculus that generates this projection-valued measure, and conversely, for each measurable functional calculus on a measurable space, there is a uniquely defined projection-valued measure (Haase, 2014).

In the present article, these results are developed and extended in the case of the reflexive Banach spaces. We show that presuming $(\Phi, X)$ is a functional calculus on the measurable space $(Z, \Sigma)$ then there is a semi-finite measure space $(\Omega, F, \mu)$ and operator $U : X \to L^p(\Omega, F, \mu)$ and an injective pointwise continuous $*$-homomorphism:

$$F : M(Z, \Sigma) \to M(\Omega, F),$$

such that $\Phi(f) = UM_f U^\dagger$, where $M_f$ is the operator of the multiplication by function $f$.

An important result of the representation theory is the following statement that if the $AC$ functional calculus of the operator is contractive then the operator can be represented as the integral concerning a spectral measure.

There is extensive literature on the subject, for clarification of definitions and basic concepts, the reader can consult the list of references.

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The Spectral Decomposition for the Operator in Reflexive Banach Spaces

Some Definitions and Notations

The letter $P$ denotes the scalar field usually real or complex numbers, the letters $X$, $Y$, $Z$ denote reflexive Banach spaces; $L(X)$ denotes the Banach algebra of all bounded linear operators on $X$, for any real compact interval $AC([a, b])$ denotes the Banach algebra of all continuous functions with its natural norm and $BV([a, b])$ denotes the Banach algebra of all functions of bounded variation with its natural norm. It is easy to show that if a function $f$ belongs to $AC([a, b])$ then this function $f$ necessarily belongs to $BV([a, b])$, however not reciprocally, there is such function $g \in BV([a, b])$, which $g \notin BV([a, b])$, in other words, the algebra $AC([a, b])$ is a proper subalgebra of the algebra $BV([a, b])$. Indeed, let $f \in AC([a, b])$ then for any $\epsilon > 0$ there is $\delta > 0$ such that for any sequence of disjointed intervals $(a_i, b_i)$, the property: That from $\sum_{i=1}^{n} b_i - a_i < \delta$ follows $\sum_{i=1}^{n} \|f(b_i) - f(a_i)\| < \epsilon$ is satisfied. Let us divide the interval $[a, b]$ by points $a = \lambda_1 < \lambda_2 < \ldots < \lambda_n = b$ into parts in such a way that $\lambda_{i+1} - \lambda_i < \delta$ for $i = 1, \ldots, n-1$. Then for any division $\{[\sigma_j, \sigma_t]\}_{j=1}^{n+1}$ of the interval $[\lambda_{i+1}, \lambda_i]$, on these parts, the sum $\sum_{j=1}^{n+1} \|f(\sigma_{j+1}) - f(\sigma_j)\|$ is the variation of the function $f$ on the interval $[\sigma_{j+1}, \lambda_i]$, is necessarily less than $\epsilon$, thus the variation of the function $f$ on the interval $[a, b]$ is less than $\epsilon\sigma$, so the function $f \in BV([a, b])$.

Definition 1.

Let $A: X \to Y$ be an operator defined on Banach spaces $X$ then the operator $A*: Y \to X^*$ is called the adjoint operator to $A: X \to Y$, namely, $(A^*(f))(\chi) = f(A(\chi))$ for all $f \in Y^*$ and all $x \in X$.

In particular, assuming $X$ is a reflexive Banach space then if operator $A: X \to X$ then the adjoint operator is $A*: X^* \to X$ if operator $A: X \to X^*$ then the adjoint operator is $A: X^* \to X^*$

Definition 2.

Let operator $A: X \to X$ then the set $\rho(A)$ of all complex numbers such that: $\rho(A) = \{ \lambda \in \mathbb{C} : \lambda I - A \text{ has inverse} \}$ is called the resolvent set.

The complement $\sigma(A)$ to the resolvent set is a spectrum of the operator $A: X \to X$.

The operator $R(\lambda, A) = (\lambda I - A)^{-1}$ is called a resolvent of operator $A$.

Definition 3.

The set $\{E(\lambda), \lambda \in \mathbb{R}\}$ of projection operators that satisfies the following conditions:

$E(\lambda) E(\mu) = E(\mu) E(\lambda)$ for $\lambda \leq \mu$, and

$\sup_{\lambda} \|E(\lambda)\| < \infty$:

1. $E(\lambda) = \text{strong} \lim_{\mu \to \lambda} E(\mu)$

and:

2. $\text{strong} \lim_{\lambda \to \infty} E(\lambda) = O$

3. $\text{strong} \lim_{\lambda \to -\infty} E(\lambda) = I$

$A = \int \lambda \, dE(\lambda) = \text{strong} \lim_{\mu \to \infty} \int_{[\mu, \infty]} \lambda \, dE(\lambda)$

is called the spectral family of operator $A$.

Condition 1 is a definition of the projection, which means the operator $E(\lambda)$ is a projection onto the subspace of $X$ created by all eigenvectors corresponding to all eigenvalues that are no larger than $\lambda$.

An operator $A$ can be written as:

$$A = \int_{\sigma(A)} \lambda \, dE(\lambda)$$

where, $E$ is a spectral family of $A$, all limits are understood as limits concerning the natural topologies. This integral is an operator-valued Riemann-Stieltjes integral in the topology of the operator norm $L$.

Let us consider the integral $\int_{[a, b]} f(\lambda) \, dE(\lambda)$ as an operator-valued Riemann-Stieltjes integral. We can build a partition $P$ of the compact interval $[a, b]$ as $a = \lambda_0 < \lambda_1 < \ldots < \lambda_n$ and the direction of the partition $|P| = \max_{i=0}^{n-1} |\lambda_i - \lambda_{i-1}|$ then if for any chosen set $\{\xi_i\}_{i=1}^{n}$ of points $(\xi_i, \lambda_i, \lambda_{i+1})$ there is a limit:

$$\lim_{|P| \to 0} \sum_{i=1}^{n} f(\xi_i)(E(\lambda_i) - E(\lambda_{i-1}))$$

and this limit is independent of the specifics of the partitions, this limit is called the Riemann-Stieltjes integral of the continuous function $f$ and can be written as:
Proof of the existence of both integrals is obvious. By definition, the Stieltjes integral is the limit of the following integral sums:

$$\sum_{i=1}^{n} \varphi(\xi_i)(f(t_i) - f(t_{i-1}))$$

Since:

$$f(t_i) - f(t_{i-1}) = \int_{[t_{i-1}, t_i]} f'(t)\,dt$$

we have:

$$\sum_{i=1}^{n} \varphi(\xi_i)(f(t_i) - f(t_{i-1})) - \int_{[t_{i-1}, t_i]} \varphi(t)f'(t)\,dt =$$

$$= \sum_{i=1}^{n} - \int_{[\xi_i, \xi_{i+1}]} \varphi(\xi)\varphi(f')\,dt$$

and:

$$\left| \sum_{i=1}^{n} \varphi(\xi_i)(f(t_i) - f(t_{i-1})) - \int_{[t_{i-1}, t_i]} \varphi(t)f'(t)\,dt \right| \leq$$

$$\leq \sum_{\xi_i \in [a,b]} \left(\sup_{\xi_0 \in [\xi_i, \xi_{i+1}]} \varphi(\xi) - \inf_{\xi_0 \in [\xi_i, \xi_{i+1}]} \varphi(\xi)\right) \int_{[\xi_i]} |f'(t)|\,dt.$$  

Next, we have that $$\sup_{i=1}^{\infty} \left(\sup_{\xi_0 \in [\xi_i, \xi_{i+1}]} \varphi(\xi) - \inf_{\xi_0 \in [\xi_i, \xi_{i+1}]} \varphi(\xi)\right)$$ converge to zero when the maximal longitude of the segments of the partitions converges to zero. The lemma has been proven.

**Theorem 4.**

Let, X be a reflexive Banach space and let the operator $$A \in L(X)$$ be well-bounded then there is a unique spectral family $$E(\cdot)$$ in X such that:

$$A = a E(a) + \int_a^b \lambda \, dE(\lambda).$$

Remarks the spectral family $$E(\cdot)$$ is concentrated on a compact interval.

**Proof.**

Let us define a functional calculus $$\gamma : AC([a, b]) \rightarrow LB(X)$$. We define a set $$F(\lambda, \eta)$$ of all real-valued continuous functions $$f \in AC([a, b])$$ such that:

$$f = \begin{cases} 1 & \text{on } [a, \lambda] \\ \text{decreasing} & \text{on } [\lambda, \lambda + \eta] \\ 0 & \text{on } [\lambda + \eta, b] \end{cases}$$
for all $\lambda \in [a, b)$ and $0 < \eta < (b - \lambda)$. Next, we have $\|f\|_{\text{bound}} \leq 1$ for any $f \in F(\lambda, \eta)$. The class $K(\lambda, \eta)$ can be defined as a closure in the weak topology:

$$K(\lambda, \eta) = \text{weak cl} \{Y(f): f \in F(\lambda, \eta)\} \subset LB(X^*)$$

For $\eta_1 < \eta_2$ we obtain $K(\lambda, \eta_1) \subset K(\lambda, \eta_2)$ and it can be deduced that set $K(\lambda) = \bigcap_{\eta \in \mathbb{Q}} K(\lambda, \eta)$ is a weakly compact uniformly bounded set.

The set $Z$ is a subset of the reflexive Banach space defined by the formula:

$$Z(\lambda) = \{x \in X : Y(f)x = 0, \quad f \in \bigcup_{\eta \in \mathbb{Q}} \{1 - F(\lambda, \eta)\}\},$$

$$y \in Z(\lambda) \subset K(\lambda) = \bigcap_{\eta \in \mathbb{Q}} K(\lambda, \eta)$$

Let then there is a net $\{g_{a,} \}_{a \in A} \subset K(\lambda, \eta)$ with the following property:

$$\langle Ex, y' \rangle = \lim_{a \in A} \langle Y(g_{a})x, y' \rangle = \lim_{a \in A} \langle 1 - Y(1 - f_{a})x, y' \rangle$$

for all $x \in X$. Since $\langle Ex, y' \rangle = \langle x, y' \rangle$ we have $x \in \text{Rang}(E)$ thus set $Z(\lambda)$ is the range of each $K(\lambda) = \bigcap_{\eta \in \mathbb{Q}} K(\lambda, \eta)$.

For any $\theta > 0$, there is $\eta_0 > 0$ such that $0 \leq f(t) \leq \theta/2$ for all $t \in [\lambda, \lambda + \eta_0]$, so for $E \in K(\lambda, \eta_0)$ there is a net $\{g_{a,} \}_{a \in A} \subset K(\lambda, \eta_0)$ with the property $\text{weak } - \lim_{a \in A} Y(g_{a}) = E$.

Now, we are going to apply the fourth condition of the definition:

$$\int_{[\theta]} \left| f_{g_{a}} \right|^2 \int_{[\theta]} f_{g_{a}} f_{g_{a}} \leq \int_{[\theta]} \int_{[\theta]} f_{g_{a}} f_{g_{a}} \leq \frac{\theta^2 + \theta^2}{2} = \theta,$$

so:

$$\left| \langle Y(f)x, y' \rangle \right| \leq \left| \langle Y(f)x, y' \rangle \right| = \left| \langle Ex, y' \rangle \right| = \left| \langle Y(f)x, y' \rangle \right| = \lim_{a \in A} \left| \langle Y(g_{a})x, y' \rangle \right| = \lim_{a \in A} \left| \langle Y(f g_{a})x, y' \rangle \right| \leq \text{sub} \left| \langle Y(f g_{a})x, y' \rangle \right|$$

for all $y' \in X^*$ so $E \in K(\lambda) = \bigcap_{\eta \in \mathbb{Q}} K(\lambda, \eta)$. Thus, from the inequality $|\langle Y(f)x, y' \rangle| \leq \theta ||f|| |x| |y'||$ follows $Y(f)x = 0$, so the range of $E$ coincides with $Z(\lambda)$; the set $E$ is a projection.

Let us establish that $K(\lambda, \eta)$ is a commutative multiplicative semigroup. Let $K, K \in K(\lambda, \eta)$, us have that there are nets $\{g_{a,} \}_{a \in A}, \{h_{a,} \}_{a \in B} \in F(\lambda, \eta)$ such that:

$$\bar{K} = \text{weak } - \lim_{a \in A} Y(g_{a})$$

and:

$$K = \text{weak } - \lim_{a \in B} Y(h_{a})$$

For all $x \in X$, we have:

$$\langle \bar{K} Kx, y' \rangle = \lim_{a \in A} \langle Y(g_{a})Kx, y' \rangle = \lim_{a \in A} \langle Y(g_{a})h_{a}x, y' \rangle = \lim_{a \in A} \langle Y(h_{a})y', y' \rangle = \lim_{a \in A} \langle Y(h_{a})y', y' \rangle = \lim_{a \in A} \langle Y(h_{a})y', y' \rangle = \langle \bar{K} Kx, y' \rangle$$

so $K = K\bar{K}$, thus:

$$E(\lambda) \in K(\lambda) = \bigcap_{\eta \in \mathbb{Q}} K(\lambda, \eta)$$

uniqueness follows from the properties of the projections. We define the set of the projection $\{E(\lambda)\}_{\lambda \in [a,b]}$ on $X$ by presuming $E(\lambda) = O$ for $\lambda < a$ and $E(\lambda) = I$ for $\lambda > b$.

Now, let us establish the properties $\{E(\lambda)\}_{\lambda \in [a,b]}$.

Assuming that $a \leq \lambda < \mu < b$ and assuming $\eta$ is large enough, we are going to obtain that from $E(\lambda), E(\mu) \in K(\lambda, \eta)$ follows $E(\lambda), E(\mu) = E(\lambda)$ $E(\mu) = E(\lambda)$. If $\eta > \mu - \lambda$, then from $E(\lambda) \in K(\lambda, \eta)$ follows the existence of the nets $\{g_{a,} \}_{a \in A} \in F(\lambda, \eta)$ $\{h_{a,} \}_{a \in B} \in F(\lambda, \eta)$ and with the properties $\text{weak } - \lim_{a \in A} Y(g_{a}) = E(\lambda)$ and $\text{weak } - \lim_{a \in B} Y(h_{a}) = E(\mu)$.

Next, since $g_{a,} h_{a} = g_{a,}$ we have:
\[
\langle E(\lambda) E(\mu)x, y' \rangle = \\
= \lim_{\alpha \to +\infty} \langle \gamma(g_{\alpha}) E(\mu)x, y' \rangle = \\
= \lim_{\alpha \to +\infty} \langle E(\mu)x, (\gamma(g_{\alpha}))' y' \rangle = \\
= \lim_{\alpha \to +\infty} \lim_{\alpha \to B} \langle \gamma(h_{\alpha}) x, (\gamma(g_{\alpha}))' y' \rangle = \\
= \lim_{\alpha \to +\infty} \lim_{\alpha \to B} \langle \gamma(g_{\alpha}) h_{\alpha} x, y' \rangle = \\
= \lim_{\alpha \to +\infty} \lim_{\alpha \to B} \langle \gamma(g_{\alpha}) x, y' \rangle
\]
for all \(x \in X, y' \in X'\). Thus, it has been obtained \(\langle E(\lambda) E(\mu)x, y' \rangle = \langle E(\lambda)x, y' \rangle\) and so equality of projection:

\[
E(\lambda) E(\mu) = E(\mu) E(\lambda) = E(\lambda)
\]
holds for all \(a \leq \lambda < \mu < b\).

Since strong \(-\lim_{\alpha \to +\infty} E(\mu) = E(\lambda = 0 + 0)\) we have \(E(\lambda + 0) \in K(\lambda)\).

For any pair \(x \in X, y' \in X'\) and for any function \(f \in AC[a, b]\) the morphism \(f \to \gamma(f)(x, y')\) is an element of the dual space to \(AC[a, b]\) and since \(AC[a, b]\) is isometric to \(L^1[a, b] \otimes C\), from the duality argument, we have that there are \(\gamma(x, y') \in L^\infty([a, b])\) satisfies the following equality:

\[
\langle \gamma(f) x, y' \rangle = \\
= \tilde{e}(x, y') f(b) + \int_{[a,b]} f'(t) \gamma(x, y') (t) dt
\]
for all \(f \in AC([a, b])\).

For any \(\lambda \in [a, b]\), we assume \(0 < \lambda + \eta < b\) then the function:

\[
g(\lambda, \eta)(t) = \begin{cases} 
1 & \text{on \([a, \lambda]\)} \\
\text{increasing on \([\lambda, \lambda + \eta]\)} \\
0 & \text{ on \([\lambda + \eta, b]\)}
\end{cases}
\]
belongs to \(F(\lambda, \eta)\) and:

\[
\langle \gamma\left(g(\lambda, \eta)\right) x, y' \rangle = \frac{-1}{\eta} \int_{[\lambda, \lambda + \eta]} \gamma(x, y') (t) dt
\]
Thus, there is a weak limit \(g(\lambda, \eta) \xrightarrow{\text{weak-*}} E(\lambda)\).

So, \(\lambda\) - almost everywhere, we obtain \(\gamma(x, y')(\lambda) = -\langle E(\lambda)x, y' \rangle\), and for arbitrary \(x \in X, y' \in X'\), the integral equality:

\[
\langle \gamma(f) x, y' \rangle = \langle x, y' \rangle f(b) - \int_{[a,b]} f'(t) \langle E(\lambda)x, y' \rangle dt
\]
holds for all \(f \in AC[a, b]\).

Next, we have:

\[
\langle \left( \int_{[a,b]} f \right) x, y' \rangle = \\
= \lim_{\lambda \to +\infty} \left( \sum_{\lambda, \eta} \langle f(\lambda)(x, y') \rangle f(b) - \langle (f(\lambda) - f(\lambda_0))(x, y') \rangle f(b) \right) = \\
= \langle x, y' \rangle f(b) - \int_{[a,b]} f'(t) \langle E(\lambda)x, y' \rangle dt
\]

Thus, by taking \(f(\lambda) = \lambda\), we have:

\[
\langle Ax, y' \rangle = b \langle x, y' \rangle - \int_{[a,b]} \langle E(\lambda)x, y' \rangle d\lambda
\]

**The Characteristics of Well-Bounded Operators in Terms of the Weak Spectral Family**

**Definition 4.**

The set \(\{E(\lambda) \in L(X') \mid \lambda \in \mathbb{R}\}\) of projection operators that satisfies the following conditions:

1. \(E(\cdot)\) is concentrated on a compact interval \([a, b]\)
2. \(E(\lambda), E(\mu) = E(\mu) E(\lambda) = E(\lambda)\) for \(\lambda \leq \mu\), and \(\sup_{\lambda} \|E(\lambda)\| < \infty\)
3. \(E(\lambda) = 0\) for all \(\lambda < a\) and \(E(\lambda) = I\) for all \(b < \lambda\)
4. there is \(\lim_{t \to E,[a,b]} \int_{[a,b]} \langle x, E(\lambda)y' \rangle d\lambda = \langle x, E(t)y' \rangle\)

for all \(x \in X, y' \in X'\) and for all \(t \in (a, b)\) is called a weak spectral family.

**Theorem 5.** Let \(A \in L(X)\) be a linear well-bounded operator then there is a unique weak spectral family \(\{E(\lambda) \in L(X') \mid \lambda \in \mathbb{R}\}\) concentrated on \([a, b]\) such that the equality:
\[
\{ A(x), y^* \} = b(x, y^*) + \int [x, E(\lambda) y^*] d\lambda
\]
holds for all \( x \in X, y^*, X^* \).

Proof. Let \( \Phi \) denotes a functional calculus \( \Phi: AC([a, b]) \to LB(X) \) then we define a functional calculus \( \hat{\Phi}: AC([a, b]) \to LB(X^*) \) by the formula
\[
\hat{\Phi}(f) = (\Phi(f))^*.
\]
The \( \hat{\Phi} \) is compact functional calculus in the weak topology of \( AC([a, b]) \), for the operator \( A^* \in LB(X^*) \).
Let us define a set \( F(\lambda, \eta) \) of all real-valued functions \( f \in AC([a, b]) \) such that:
\[
f = \begin{cases}
1 & \text{on } [a, \lambda], \\
decreasing & \text{on } [\lambda, \lambda + \eta], \\
0 & \text{on } [\lambda + \eta, b]
\end{cases}
\]
for all \( \lambda \in [a, b] \) and \( 0 < \eta < (b - \lambda) \). The class \( K(\lambda, \eta) \) can be defined as a closure in the weak topology:
\[
K(\lambda, \eta) = \text{weak cl} \{ Y(f): f \in F(\lambda, \eta) \} \subseteq LB(X^*)
\]
From \( \eta_1 < \eta_2 \) follows \( K(\lambda, \eta_1) \subseteq K(\lambda, \eta_2) \) and we can deduce that set \( K(\lambda) = \bigcap_{\eta \in \Theta} K(\lambda, \eta) \) is a weakly compact uniformly bounded set.
We define the subset \( Z \) of the reflexive Banach space by the formula:
\[
Z(\lambda) = 
\left\{ x^* \in X^*: Y(f)x^* = 0, f \in \bigcup_{\eta \in \Theta} \{ 1 - F(\lambda, \eta) \} \right\}
\]
Let \( y^* \in Z(\lambda) \in K(\lambda) = \bigcap_{\eta \in \Theta} K(\lambda, \eta) \) then there is a net \( \{ g_\alpha \}_{\alpha \in \Lambda} \subseteq K(\lambda, \eta) \) with the following property:
\[
\lim_{\alpha \to \Lambda} \{ x, Y(g_\alpha)x^* \} = \lim_{\alpha \to \Lambda} \{ x, (1 - Y(1 - f_\alpha))x^* \}
\]
for all \( x \in X \). Since \( \langle x, Ey^* \rangle = \langle x, y^* \rangle \) we have \( y^* \in Rang(E) \).
For any \( \theta > 0 \), there is \( \eta_0 > 0 \) such that \( 0 \leq f(t) \leq \frac{\theta}{2} \) for all \( t \in [\lambda, \lambda + \eta_0] \), so for \( E \in K(\lambda, \eta_0) \) there is a net \( \{ g_\alpha \}_{\alpha \in \Lambda} \subseteq F(\lambda, \eta_0) \) with the property:
\[
\text{weak} = \lim_{\alpha \to \Lambda} Y(g_\alpha) = E.
\]
Now, we are going to apply the fourth condition of the definition:
\[
\int_{[\lambda, \lambda + \eta]} |f(x)|^2 dx \leq \int_{[\lambda, \lambda + \eta]} |f(x)| dx + \int_{[\lambda, \lambda + \eta]} \left| f(x) + f(x) \right|^2 dx \\
\leq \frac{\theta}{2} + \frac{\theta}{2} = \theta,
\]
so:
\[
\| Y(f)x^* \| \leq \| Y(f)\| \| x \| \| y^* \|
\]
for all \( y^* \in X^* \) so \( E \in K(\lambda) = \bigcap_{\eta \in \Theta} K(\lambda, \eta) \). Thus, from the inequality \( \| Y(f)x^* \| \leq \| Y(f)\| \| x \| \| y^* \| \) follows \( Y(f)x^* = 0 \), so the range of \( E \) coincides with \( Z(\lambda) \); the set \( E \) is a projection.
Let us establish that \( K(\lambda, \eta) \) is a commutative multiplicative semigroup. Let \( K, K \in K(\lambda, \eta) \), we have that there are nets \( \{ g_\alpha \}_{\alpha \in \Lambda}, \{ h_\beta \}_{\beta \in \Lambda} \in F(\lambda, \eta) \) such that:
\[
\hat{K} = \text{weak} - \lim_{\alpha \to \Lambda} Y(g_\alpha)
\]
and:
\[
\hat{K} = \text{weak} - \lim_{\alpha \to \Lambda} Y(h_\beta).
\]
For all \( x \in X \), we have:
\[
\{ x, K Ky^* \} = \lim_{\alpha \to \Lambda} \{ x, Y(g_\alpha)K y^* \} = \\
= \lim_{\alpha \to \Lambda} \{ \{ Y(g_\alpha) \} x, K y^* \} = \\
= \lim_{\alpha \to \Lambda} \{ \{ Y(g_\alpha) \} x, Y(h_\beta) y^* \} = \\
= \lim_{\alpha \to \Lambda} \{ \{ K \} x, Y(h_\beta) y^* \} = \\
= \{ \{ K \} x, \hat{K} y^* \} = \{ x, KK y^* \}.
\]
so \( \tilde{K} \tilde{K} = K \tilde{K} \), thus:

\[
E(\lambda) \in K(\lambda) = \bigcap_{\eta > 0} K(\lambda, \eta),
\]

uniqueness is follows from the properties of the projections. We define the set of the projection \( \{E(\lambda)\}_{\lambda \in [a, b]} \) on \( X \) by presuming \( E(\lambda) = O \) for \( \lambda < a \) and \( E(\lambda) = I \) for \( \lambda > b \).

Now, let us establish the properties \( \{E(\lambda)\}_{\lambda \in [a, b]} \).

Assuming that \( a \leq \lambda < \mu < b \), and assuming \( \eta \) is large enough, we are going to obtain that from \( E(\lambda), E(\mu) \in K(\lambda, \eta) \) follows \( E(\lambda), E(\mu) = E(\mu) E(\lambda) = E(\lambda) \).

If \( \eta = \mu - \lambda \), then from \( E(\lambda) \in K(\lambda, \eta) \) follows the existence of the nets \( \{g_{\lambda}\}_{\lambda \in [a, b]} \) and \( \{h_{\lambda}\}_{\lambda \in [a, b]} \) with the properties:

\[
\text{weak} - \lim_{\lambda \to \lambda} \gamma(g_{\lambda}) = E(\lambda)
\]

and:

\[
\text{weak} - \lim_{\lambda \to \lambda} \gamma(h_{\lambda}) = E(\mu).
\]

Next, since \( g_{\lambda} h_{\lambda} = g_{\lambda} \), we have:

\[
\begin{align*}
\langle x, E(\lambda) E(\mu) y' \rangle &= \lim_{\lambda \to \lambda} \gamma(g_{\lambda}) E(\mu) y' = \lim_{\lambda \to \lambda} \gamma(g_{\lambda}) x, E(\mu) y' = \lim_{\lambda \to \lambda} \Phi(g_{\lambda}) x, E(\mu) y' = \lim_{\lambda \to \lambda} \Phi(h_{\lambda}) E(\mu) x, y' = \lim_{\lambda \to \lambda} \Phi(h_{\lambda}) g_{\lambda} x, y'
\end{align*}
\]

for all \( x \in X, y' \in X' \). So, we have obtained \( \langle x, E(\lambda) E(\mu) y' \rangle = \langle x, E(\lambda) y' \rangle \) and thus equality:

\[
E(\lambda) E(\mu) = E(\mu) E(\lambda) = E(\lambda)
\]

holds for all \( a \leq \lambda < \mu < b \).

Since:

\[
\text{strong} - \lim_{\mu \to \lambda} E(\mu) = E(\lambda + 0)
\]

we have \( E(\lambda + 0) \in K(\lambda) \).

For any pair \( x \in X, y' \in X' \) and any function \( f \in AC([a, b]) \) the morphism \( f \mapsto \gamma(f, y') \) is an element of the dual space to \( AC([a, b]) \) and since \( AC([a, b]) \) is isometric to \( E'([a, b]) \oplus C \), from the duality argument, we have that there are \( \gamma(\lambda, y' \in E'(\lambda) + C \) which satisfy the following equality:

\[
\begin{align*}
\{ x, \gamma(f, y') \} &= \gamma(x, y') f(b) + \int_{[a, b]} f'(t) \gamma(x, y')(t) dt.
\end{align*}
\]

For any \( \lambda \in [a, b] \), we assume \( 0 < \lambda - \mu < \delta \) then the function:

\[
g(\lambda, \mu)(t) = \begin{cases} 1 & \text{on } [a, \lambda] \\ \text{increasing on } [\lambda, \lambda + \eta] \\ 0 & \text{on } [\lambda + \eta, b] \end{cases}
\]

belongs to \( F(\lambda, \eta) \) and:

\[
\langle x, \gamma(g(\lambda, \mu)) y' \rangle = \frac{1}{\eta} \int [x, \lambda + \delta] \gamma(x, y')(t) dt.
\]

Thus, there is a weak limit \( g(\lambda, \eta) \to \int_{[a, b]} E(\lambda) \).

So, \( \lambda \)-almost everywhere, we obtain \( \gamma(x, y')(\lambda) = -\langle x, E(\lambda) y' \rangle \) and for arbitrary \( x \in X, y' \in X' \), the integral equality:

\[
\begin{align*}
\langle x, \gamma(f, y') \rangle &= \langle x, y' \rangle f(b) - \int_{[a, b]} f'(t) \langle x, E(\lambda) y' \rangle dt + \langle Ax, y' \rangle = b(x, y') - \int [a, b] \langle x, E(\lambda) y' \rangle d\lambda.
\end{align*}
\]

Let function \( \phi \in E'([a, b]) \) then we can define:

\[
f(\phi) = \int_{[a, b]} \phi(t) dt
\]

thus \( f(\phi) \in AC([a, b]) \) and almost everywhere \( f'(\phi)(\lambda) = -\phi'(\lambda) \). For any fixed \( x \in X \), the mapping \( A(x)(\phi) = \Phi(f(\phi))(x) \) is continuous as the mapping \( E'([a, b]) \to X \). So, we have:

\[
\begin{align*}
(A(x)(y'))(x, y') &= \langle x, E(\lambda) y' \rangle d\lambda
\end{align*}
\]

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and the mapping $A'(x) : X \to L^r([a, b])$ is such that:

$$\langle \phi, A'(x)y' \rangle = \int_{[a, b]} \phi(\lambda) \langle x, E(\lambda)y' \rangle d\lambda.$$ 

Theorem 6. Let $\{E(\lambda) \in L(X') : \lambda \in \mathbb{R}\}$ be a weak spectral family concentrated on $[a, b]$ then there is a linear well-bounded operator $A \in L(X)$ on the reflexive Banach space $X$ such that:

$$\langle A(x), y' \rangle = b'(x, y') - \int_{[a, b]} \langle x, E(\lambda)y' \rangle d\lambda,$$

holds for all $x \in X, y' \in X'$.

Proof. Assuming $\{E(\lambda) \in L(X') : \lambda \in \mathbb{R}\}$ is a weak spectral family concentrated on $[a, b]$, the linear operator $A \in L(X)$ can be defined by the following formula:

$$\langle A(x), y' \rangle = b'(x, y') - \int_{[a, b]} \langle x, E(\lambda)y' \rangle d\lambda,$$

it is easy to see that this operator is linear and the only property of it that has to be established is well-boundedness.

By the induction and the Fubini theorem, we have:

$$\langle (A(x))^n, y' \rangle = b^n(x, y') - \int_{[a, b]} n! \lambda^{n-1} \langle x, E(\lambda)y' \rangle d\lambda,$$

thus:

$$\| (A(x))^n \| \leq b^n + \sup_{\lambda \in [a, b]} \| E(\lambda) \| \int_{[a, b]} \lambda^{n-1}d\lambda,$$

and operator $A$ is well-bounded.

**Continuous Functional Calculus on Lebesgue Spaces**

Theorem 7. Let $A$ be a well-bounded linear operator on Lebesgue spaces $L^q(\Omega, \Sigma, \mu), p \in (1, \infty)$. Then the operator $A$ is a scalar type spectral operator.

Proof. The spectral family $\{E(\lambda)\}$ of operator $A$ is concentrated on the interval $[a, b] \subset \mathbb{R}$.

Let us assume that $u \in L^p(\Omega, \Sigma, \mu), p \in (1, \infty)$ and $v \in L^q(\Omega, \Sigma, \mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$. We have to show that the variation of the function $\langle E(\lambda)u, v \rangle$ is bounded as the function of $\lambda$. Assume that $a = \lambda_0 < \lambda_1 < \ldots < \lambda_n = b$ is a partition of the interval $[a, b]$. For arbitrary elements $u \in L^p(\Omega, \Sigma, \mu), p \in (1, \infty)$ and $v \in L^q(\Omega, \Sigma, \mu)$, the variation of the function $\langle E(\lambda)u, v \rangle$ equals:

$$\text{var} \langle E(\lambda)u, v \rangle = \sum_{\lambda_{i-1} < \lambda_i} \| [E(\lambda_{i-1}) - E(\lambda_i)]u, v \| \leq \sum_{\lambda_{i-1} < \lambda_i} \| E(\lambda_{i-1}) - E(\lambda_i) \| \| u \| \| v \|.$$ 

Let $m$ be an integer such that $\lambda_{m-1} < c < \lambda_m$, so we have:

$$\sum_{\lambda_{i-1} < \lambda_i} \| E(\lambda_i) - E(\lambda_{i-1}) \| \leq \sum_{\lambda_{i-1} < \lambda_i} \| E(\lambda_{i-1}) - E(\lambda_m) \| + \| E(\lambda_i) - E(\lambda_m) \| + \| E(\lambda_{i-1}) - E(\lambda_i) \|,$$

thus for $\lambda < c$ we have $\| E(\lambda) \| \leq 1$, and for $\lambda \geq C$ we have $\| I - E(\lambda) \| \leq 1$. So $\| E(\lambda_0) \| \leq 2$ and $\| E(\lambda_{m-1}) \| \leq 1$. Since $\{E(\lambda_i)\}_{i=1, \ldots, m}$ and $\{I - E(\lambda_i)\}_{i=1, \ldots, m}$ are the increasing sequences of contractive projections, we have:

$$\sum_{\lambda_{i-1} < \lambda_i} \| E(\lambda_i) - E(\lambda_{i-1}) \| \leq 2(q-1)$$

and:

$$\sum_{\lambda_{i-1} < \lambda_i} \| E(\lambda_i) - E(\lambda_{i-1}) \| \leq 2(q-1).$$

In the final conclusion, we obtain:

$$\sum_{\lambda_{i-1} < \lambda_i} \| E(\lambda_i) - E(\lambda_{i-1}) \| \leq 4(q-1) + 3,$$

thus, the variation of $\langle E(\lambda)u, v \rangle$ cannot exceed the value $(4(q-1) + 3)\| u \| \| v \|$. The theorem is proven.

Definition 5. A solitary operator is a bounded linear surjective operator $U : X \to X$ on a Banach space that for all $x \in X$ and $y \in X^*$ satisfies the following equality:

$$\langle Ux, U^*y \rangle = \langle x, y \rangle.$$
where, \( U^* : X^* \to X^* \).

Theorem 8. Assuming \((\Phi, X)\) is a functional calculus on the measurable space \((Z, \Sigma)\). Then there are a semi-finite measure space \((\Omega, F, \mu)\) and solitary operator \(U : X \to L^r(\Omega,F,\mu)\) and an injective pointwise continuous \(*\)-homomorphism \(F : M(Z, \Sigma) \to M(\Omega,F)\), such that \(\Phi(f) = UM_fU^*\), where \(M_f\) is the operator of the multiplication by \(f\).

Proof. For every set \(A \in \Sigma\), we define measure \(\mu(A) = \langle \Phi(x), x^* \rangle\) as a function of \(x \in X\), so \(\langle \Phi(f)x, x^* \rangle = \langle \Phi(f) \rangle_{\mu} \) for every bounded \(f\). Now, for every bounded \(f\), we define the space \(B_\mu = [\langle \Phi(f)x, f \in M_\mu(Z, \Sigma) \rangle]\), thus there is a solitary operator \(W_f : L^r(Z, \Sigma, \mu) \to B_\mu\) as an extension of mappings \(M_\mu(Z, \Sigma) \to B\) and \(f \to \Phi(f)x\).

Let \(\{x_i\}\) and \(\{x_i^*\}\) be two sets of unit vectors in \(X\) and \(X^*\) spaces, respectively, with properties:

\[
\langle x_i, x_j^* \rangle = \|x_i\| \cdot \|x_j^*\| = 1 \forall k \in N
\]

and:

\[
\langle x_i, x_j^* \rangle = 0
\]

for every \(i \neq k\).

For every \(k\), we can define the set \(Z_k = Z \times \{k\}\) as an exemplar of \(Z\) then the set \(\Omega\) can be represented as the disjoint union \(\bigsqcup_i Z_i\). Let

Let us define an additive set function \(\mu\) by the following formula:

\[
\mu(A) = \sum_i \mu_{Z_i}(A \cap Z_i) \quad \forall A \in F.
\]

The additive set function \(\mu\) is the measure on the maximal sigma-algebra \(F\) on \(\Omega\), which includes all measurable mapping \(Z_i = Z \times \{k\}\) into \(\Omega\).

The operator \(W_\mu\) is correctly defined on \(L^r(Z_i, \Sigma, \mu_{Z_i})\) and \(W_\mu : L^r(Z_i, \Sigma, \mu_{Z_i}) \to B_\mu\) so we define the operator \(U : X \to L^r(\Omega,F,\mu)\) by the condition \(U^{-1}W_\mu = \Phi(m)\) on \(L^r(Z_i, \Sigma, \mu_{Z_i}) \subseteq L^r(\Omega,F,\mu)\).

Then the \(*\)-homomorphism \(F : M(Z, \Sigma) \to M(\Omega,F)\), we introduce by the formula:

\[
(Ff)(x,k) = f(x), \quad x \in X
\]

For all \(f \in (Z, \Sigma)\), we define the multiplication operator calculus as \(M_f = U\Phi(f)U^*\), so the theorem has been proven.

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Ethics

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and no ethical issues involved.

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