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Mathematical Approach to the Ruin Problem with Compounding Assets

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Abstract: This study considered the Ruin problem with an income process with stationary independent increments. The characterization is obtained which is general for the probability of $r(y)$, that the asset of a firm will never be zero whenever the initial asset level of the firm is y. The aim of this study is also to determine $r(\hat{y}) = P\{T < \infty \mid Y(0) = y\}$, If we let $T = in\,f\,\{t \geq 0; Y(t) < 0\}$, A condition that is necessary and sufficient is studied for a distribution that is one – dimensional of X_N which coverages to X_*. The result that is obtained concerning the probability, is of ruin before time t. Riemann-Stieltjes integral, two functions f and α with symbol as $\int_a^b f(x)d\alpha(x)$ was used and is a special case in which $\alpha() = x$, where α has a continuous derivative. It is defined such that the Stieltjes integral $\int_a^b f(x)d\alpha(x)$ becomes the Riemann integral $\int_a^b f(x)\alpha^{|}d(x)$.

Keywords: Ruin, Compounding Asset, Income, Stationary, Independent

Introduction

This research considers the generation of the collective risk theory of the classical model, where the assumption is that the cumulative income of the firm is given by a process X_*. The process has stationary independent increments and the interest is continuously earned on the firm's assets. The assets of the firm of the first-time t, which is $Y(t)$ are represented by a simple path-wise integral concerning the income process X (Harrison, 1977). A general characterization can be obtained for the probability $r(y)$, which implies that assets will not fall to zero when the initial asset level is y. By the action of the force of interest, a classical surplus process is modified, which can be considered as recursive algorithms for the calculation of the probability of ruin in finite time. This is in line with (Dickson and Waters, 1999) who worked on two numerical algorithms, such as the stable recursive algorithm and the method of product integration that was used to obtain numerically an approximation to this probability of ultimate ruin. Also, the survey of the problem of ruin in a risk model when assets earn investment income is treated (Paulsen, 2008), but the exact recursive expressions for ruin probabilities will also be looked into.

Income Process

Let $X = \{X(t),\ t \geq 0\}$ represent an independent increment of a stochastic process with stationary and, finite variance where $X(0) = 0$, which is called the income process. Let y also represent a positive level of initial asset and a positive interest rate.

Therefore, the asset process:

$$Y = \beta y + \int_0^1 \beta(s)dX(s), t \geq 0 \tag{1}$$

An investment during a period t becomes $e^{\beta(t)}$ at the end, then, Eq. (1) becomes a compounding asset (De Kok, 2003):

$$Y(t) = e^{\beta(t)}y + \int_0^1 \beta^{(t-s)}dX(s), t \geq 0 \tag{2}$$

If $t \geq 0$, we define:

$$\int_0^t e^{\beta(t-s)}dX(s) = -\int_0^t e^{\beta(t-s)}dX(s) \tag{3}$$

Since $\int_0^t e^{\beta(t-s)}dX(s)$ exist and is finite for all $t \geq 0$.

Riemann-Stieltjes integral on Eq. (2) is given as:

$$Y(t) = e^{\beta(t)}y + \int_0^t \beta^{(t-s)}dX(s), t \geq 0 \tag{4}$$

will exist and is finite for all $t \geq 0$ and almost all sample paths of X will be a well-defined path-wise functional of the income process (Feller, 2008; Eisen, 1969):

Proposition

The Riemann-Stieltjes integral:

$$Z(t) = \int_0^t e^{-\beta s} dX(s), t \geq 0 \tag{5}$$

Is finite, exists, and satisfies:

$$Z(t) = e^{-\beta t} X(t) + \beta \int_0^1 e^{-\beta s} X(s) ds, t \geq 0 \tag{6}$$

Furthermore, Z is almost surely in $D[0, \infty]$.

Example

Let the income process of a compound Poisson form i.e., $X = \sum_{i=1}^{N} W_i$ then the exponent function is:

$$V(u) = icu - \lambda \int_{\mathbb{R}} \left(1 - e^{-iu\lambda}\right) F(dx), u \in \mathbb{R} \tag{7}$$

Let c in Eq. (7) be positive and that:

$$F(x) = 1 - e^{-\frac{x}{m}}, x \geq 0 (m > 0)$$

with, $F(x) = 0$ for $x = 0$ (Feller, 1971). Thus, for the parts of the income process slope, there is an absolute jump size with an exponential distribution of mean m. The exponent function is given as:

$$V(u) = icu - \lambda_{imc} / (1 + imu),$$

Now if:

$$\begin{aligned}
\psi(u) &= \int_0^\infty v\left(ue^{-\beta t}\right) dt \\
\psi(u) &= \int_0^\infty V\left(ue^{-\beta t}\right) dt \\
&= \int_0^\infty e^{-\beta t} v(u) dt = \int_0^\infty e^{-\beta t} \left(icu - \lambda ime / (1 + imu) dt\right) \\
&= \int_0^\infty e^{-\beta t} icudt - \int_0^\infty e^{-\beta t} \lambda_{ime} / (1 + imu) dt \\
&= \frac{icu}{\beta} - (\lambda / \beta) \ln(1 + imu)
\end{aligned} \tag{8}$$

From Eq. (8) the c.f. of H is then:

$$e^{\psi(u)} = e^{icu/\beta (1+imu)^{-\lambda/\beta}} \tag{9}$$

By c.f. of gamma distribution:

$$H(z) = \left[\int_{\frac{\in}{\beta}-2}^{\infty} x^{\frac{\lambda}{\beta}-1} e^{-\frac{x}{m}} dx\right] / M^{\lambda/\beta} T(\lambda / \beta), x \in \mathbb{R} \tag{10}$$

It was observed that the jumps of Y and X are the same, which implies that the property of the exponential distribution gives the amount the assets will fall below zero upon ruin.

Ruin will occur if a general jump of the same income process has the same exponential distribution. Therefore:

$$E\left[e^{iuY(t)} \middle| T < \infty\right] = (1 + imu)^{-1} \tag{11}$$

Combining (9) and (11), with $T < \infty$ a gamma distribution there is the convolution of H with the conditional distribution of $Y(T)$, then:

$$E\left[H(-Y(T)) \middle| T < \infty\right] = \left[\int_{c/\beta}^{\infty} x^{\lambda/\beta} e^{-x/m} dx\right] / M^{\lambda/\beta+1} T(\lambda / \beta + 1) \tag{12}$$

Now suppose in Eq. (7) that c is negative and that:

$$F(x) = e^{x/m}, x \leq 0 (m > 0).$$

Eq. (9) i.e.:

$$V(x) = icu - \lambda \int_{\mathbb{R}} 1 - e^{iux} F(dx), u \in \mathbb{R}.$$

for, $f(x) = 0$ and $x = 0$. Thus, there is an absolute jump size with an exponential distribution of mean m and the exponent function is given as the paths of the income process.

Then we obtain:

$$\begin{aligned}
V(u) &= \lambda imu \left| (1 - imu) - i \,|\, c \,|\, u, \right. \\
e^{\psi(u)} &= e^{-i|c|u/\beta} (1 - imu)^{-\lambda/\beta}
\end{aligned}$$

The gamma distribution:

$$H(z) = \begin{cases} 0, & \text{if } z \leq -c/\beta \\ \left[\int_0^{|c|/\beta+2} x^{\gamma/\beta} e^{-x/m} dx\right] \Big/ m^{\gamma/\beta} \Gamma\left(\gamma/\beta\right), & \text{if } z > -c/\beta \end{cases}$$

There are no negative jumps in the income process.

The Ruin Problem

Suppose that the asset of a firm will not fall to zero when the initial asset level is taken as y.

We assume that an insurance company received a $N(t)$, number of claims at time t and this is a Poisson Process with parameter λ. If the insurance company received cash at a constant rate of 1 per unit time *t*. Then, the cash balance at time *t* can be expressed as:

$$t = x + t - \sum_{i=1}^{N(t)} Y_1$$

where the initial capital of the company is *x* and Y_i, ≥ 1, are the claims.

Suppose that the company invested the risk reserve of the company with a continuous interest rate of β (Davidson, 1969) and if the account of the company ever falls to zero, money can be borrowed from the bank at the same interest rate β. Then, the content of the account at time *t* is given as:

$$Y(t) = e^{\beta t} y + \int_0^t Ce^{\beta(t-s)} ds - \sum_{i=1}^{N(t)} e^{\beta(t-ti)} W_i \quad (13)$$

which is gotten from:

$$Y(t) = e^{\beta t} y + \int_0^1 e^{\beta(t-s)} dx(s)$$

where:

$$X(t) = ct - \sum_{i=1}^{N(t)} W_1$$

Substituting, we then have:

$$Y(t) = e^{\beta t} y + \int_0^t e^{\beta(t-s)} \left(ct - \sum_{i=1}^{N(t)} W_1 \right)$$

where, *y* and β are already defined as an initial asset and interest rate through the investment of a firm's assets respectively (Harrison *et al*., 1975)

X(*t*) is the firm's net operating profit during the interval [0, *t*].

Let:

$$T = \inf\{t \geq 0 : Y(t) < 0\}$$

We want to determine:

$$r(y) = P\{T < \infty | Y(0) = y\}$$

With the time of ruin and ruin function as *T* and *r*(.) respectively.

The ruin function:

Let *X* be defined on (Ω, $\mathcal{F}$, *P*) probability space and has a stationary, independent increment with $E[X(t)] = \mu t$ and $Var[X(t)] = \delta^2 t$ where $-\infty < \mu < \infty$ and $0 < \delta^2 < \infty$. We assume that *X* has a continuous probability distribution. The characteristic function (c.f.) of *X*(*t*) we then have:

$$E\left[e^{iut}\right] = e^{vcwt} \, for \, t \geq 0, u \in \mathbb{R}$$

where the exponential is given by Levy-Khintchine representation.

If *V*(.) is a Levy-Khintchine distribution of finite variance. Then:

$$\begin{aligned} V(u) &= i\mu u + \delta^2 \int_{\mathbb{R}} \left[e^{iux} - 1 - \frac{iux}{1+x^2} \right] \frac{1+x^2}{x^2} G(dx) \\ &= i\mu u + \delta^2 \int_{\mathbb{R}} x^{-2} \left[e^{iux} - 1 - iux \right] G(dx) \end{aligned}$$

where, *G* is a probability distribution on ℝ. *X* is strong Markov (Apostol, 1974).

From Eq. (2), we have:

$$Y(t) = e^{\beta(t)} \left[y + z(t) \right], t \geq 0 \quad (14)$$

where:

$$Z(t) = \int_0^t e^{-\beta s} dX(s), t \geq 0 \quad (15)$$

That is:

$$\begin{aligned} Y(t) &= e^{\beta(t)} y + \int_0^t e^{\beta(t-s)} dX(s), t \geq 0 \\ &= e^{\beta t} y + Z(t) \int_0^t e^{\beta(t)} dX(s), t \geq 0 \\ &= e^{\beta(t)} y + Z(t) e^{\beta(t)}, t \geq 0 \\ &= e^{\beta(t)} \left[y + z(t), t \geq 0 \right] \end{aligned} \quad (16)$$

the random variance *Z*(*t*) is the value of income earned during the period [0, t].

The Compound Brownian Approximation

The Brownian motion with drift approximates the income process, then the assets process Y_* and *r*(*y*) is also approximated by a corresponding probability $r_*(t)$ (Karlin and Taylor, 1981). The diffusion Y_* is called compounding Brownian motion.

Let *y* be the initial asset level and β the interest rate be fixed, then.

Let $X_1, X_2, \ldots$, be income process defined on a common probability space Ω, $\mathcal{F}$, *P*. and.

Let X_* be a Brownian motion with mean μ_* and variance σ^2, also defined on (Ω, $\mathcal{F}$, *P*), we assume that:

$$\mu_1 = \mu_2 = \cdots = \mu_*, \delta_1^2 =, \delta_2^2 = \cdots =, \delta_*^2 \quad (18)$$

Thus, X_* be each of the natural diffusion approximation for the income process X_n.

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Let $\Rightarrow$ denote weak convergence for a sequence of a probability measure on a metric space. In the case of a random variable (random elements of $\mathbb{R}$), weak convergence is equivalent to the convergence of distribution function at continuity points of the limit (Bartle, 1974) The restriction to [0, t] at any process above is a random element of D[0, t].

Example

We assume now that $X(t) = \delta W(t) + \mu t$.
where, W is the standard (zero drift and unit variance) Weiner process.

Our basic representation is given by:

$$Y(t) = e^{\beta t} y + \int_0^t e^{\beta(t-s)} dX(s), t \geq 0$$

The assets process yields:

$$\begin{aligned} Y(t) &= e^{\beta t}\left[y + \delta \int_0^t e^{-\beta s} dW(s) + (\mu/\beta)(1 - e^{\beta t}) \right] \\ &= e^{\beta t} y + \delta W(t) + \delta\beta \int_0^t e^{\beta(t-s)} W(s) ds + (\mu/\beta)(e^{\beta t} - 1) \end{aligned}$$

The present value process Z(.) is Gaussian and has independent increments. Thus, the representation of the present value process which is a rescaling of Brownian motion (Gerber, 1972):

$$Z(t) = \left(\frac{\sigma^2}{2\beta} \right)^{\frac{1}{2}} W\left(1 - e^{-2\beta t}\right) + (\mu/\beta)\left(1 - e^{1\beta t}\right), t \geq 0 \tag{17}$$

and:

$$Y(t) = e^{\beta t}\left[y + Z(t) \right], t \geq 0$$

and from Eq. (17) above the asset process has the same distribution as:

$$Y(t) = \left(\frac{\sigma^2}{2\beta} \right)^{\frac{1}{2}} W\left(e^{-2\beta t} - 1\right) + y e^{\beta t} + (\mu/\beta)\left(1 - e^{1\beta t}\right), t \geq 0$$

with mean of $\mu(t) = \mu + \beta y$ and variance of $\delta^2(y) = \delta^2 y \in \mathbb{R}$.

Conclusion

We consider the financial situation of an insurance company at discrete times, say $t = 0,1,2,\ldots$ and we assume that there is an initial surplus $X_0 = x$, which is a non-negative integer.

The surplus X_{t+1} at time $t + 1$ results from X_t as follows ($t = 0,1,2,\ldots$):

$$Z_{t+1} = X_t - Z_t + Y_{i+1} \tag{18}$$

where, Z_t is the dividends paid to the shareholders at time t; Y_{t+1} represents the income (possible negative) of the company during the time interval from t to $t + 1$.

Both Z_t and Y_t are integer-valued and $Y'_t s$ are independent identically distributed random variables. By P_j, we denote the probability that the income resulting from a certain time interval (t, $t + 1$) will be equal to $j(i = 0, \pm 1, \pm 2,\ldots)$.

Assume the expectation of t_0 exists and is positive then:

$$\mu = \sum_{j=-\infty}^{j=+\infty} jP_j > 0 \tag{19}$$

Z_i*'os* are decision variables.
w = dividend strategy which determines how dividends will be paid out to the shareholders. In any event, Z_t will assume one of the values $0,1,2,\ldots, X_t$ (X_t be surplus at time t). Ruin occurs as soon as the surplus gets negative epoch τ such that $X_\tau<0$.

We are only interested in $t \geq \tau$. Accordingly, we put $Z_t = Z_{t+1} = \cdots = 0$.

We defined for each w and initial surplus x:

$$V(x,w) = E\left[\sum_{t=0}^{t=\tau-1} d^t Z_t \right. \tag{20}$$

$$V(x,w) \sup V(x,w), (xw, 1, 2\cdots) w \tag{21}$$

If $V(x, w) = V(x)$ for all $x = 0,1,2,\ldots,w$ is called dividend strategy optimal.

Basic Difference Equation Factorization

Let a be a non-negative integer and $V(x, a)$ be the expected sum of the discounted payments (initial surplus x) concerning the barrier strategy belonging to a.

Assuming $Y_i = j$ and applying a rule of total probability, we obtain the equations:

$$V(x,a) = d\sum_{j=-x}^{j=1} P_j V(x+j, a), (x = 0,1,\ldots,a-1) \tag{22}$$

with related boundary conditions:

$$V(a,a) = d\left\{ \sum_{j=-a}^{j=1} P_j V(a+j,a) + P_j V(a,a) + P_1 \right\} \tag{23}$$

Let us consider the following system of the equation:

$$W(s) = d\sum_{j=-x}^{j=0} P_j \left| N(x+j), (x = 0,1,2,\ldots) \right. \tag{24}$$

The system (24) has a unique solution $W(x)$ and can be seen by expressing $W(x+1)$ in terms of $W(0), W(1),\ldots, W(x)$.

Also, from (22) we have that $V(x, a)$ must be of the form:

$$V(x,a) = C(a)W(x)$$

From Eq. (23) which leads to:

$$C(a) = \left[W(a+1) - W(a)\right]^{-1}$$

And then we have:

$$V(x,a) = W(x) / \left[W(a+1) - W(a)\right], (x = 0,1,...,a) \quad (25)$$

The maximization of $V(x, a)$ amounts to minimizing the difference $w(a+1) - w(a)$.

We calculate $W(x)$ for $x = 1,2,3,\ldots$ successively [putting $W(0) = 1$] and examine where $W(a+1) - W(a)$ achieves its minimum and its finite.

Relation with the Probability of Ruin

We want to show a close connection between the fundamental function $W(x)$ defined in (24) and the probability of ruin of a certain conjugate random walk (Beekman, 1974).

For $s > 0$, we define $Ws(x) = S^{-x}W(x)$, $x = 0,1,2,\ldots$

Thus, Eq. (24) is equivalent to:

$$W_s(x) = d\sum_{j=-1}^{j=1} S^j P_j W_s(x+y), x = 0,1,2,..., \quad (26)$$

Close $s = S$, where S is the (unique) solution greater than one of:

$$d\sum_{j=-\infty}^{j=1} s^j P_j = 1 \quad (27)$$

The random walk belonging to $\left\{P_j^s = ds^j P_j\right\}$ is called a conjugate random walk of $\{P_j\}$ from $s > 1$ and (19) we have that the expectation of $\left\{P_j^s\right\}$ positive.

From Eq. (26), we have that $W_s(x)$ may be known as the probability of survival of the conjugate random walk (given initial surplus x and absorbing barrier at -1).

Thus, $\lim W_s(x) = 1(x \to \infty)$

Applying this and Eq. (25) we find that:

$$\lim_{a\to\infty} V(a,a) = \lim W_s(a) / \left[SW_s(a+1) - W_s(a)\right] = 1 | (s-1) \quad (28)$$

Ethics

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and no ethical issues involved.

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