Journal of Mathematics and Statistics

Original Research Paper

Existence Results for a Class of Nonlinear Hadamard Fractional with *p*-Laplacian Operator Differential Equations

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Article history
Received: 29-03-2021
Revised: 01-05-2021
Accepted: 10-05-2021

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Abstract: In this study, we are concerned with the existence and uniqueness of solution for some nonlinear Hadamard fractional differential equations. Our results are based on different classical fixed point theorems. Some useful examples are presented in order to illustrate the validity of our main results.

Keywords: Hadamard Fractional Derivative, Integral Equations, *p*-Laplacian Operator, Fixed Point Theorems

Introduction

Recently, fractional differential equations have been acquired much attention due to its applications in a number of fields such as physics, mechanics, chemistry, biology, signal and image processing, see for example the books (Baleanu *et al*., 2012; Kilbas *et al*., 2006; Lakshmikantham *et al*., 2009; Yang *et al*., 2015).

Some recent works on fractional differential equations involving Riemann Liouville and Caputo-type fractional derivatives are studied using nonlinear analysis methods such as Krasnoselskii fixed-point Theorems (Agarwal and O'Regan, 1998; Ghanmi and Horrigue, 2018; Guo *et al*., 2007; Guo *et al*., 2008), Leray-Schauder alternative (Ghanmi and Horrigue, 2019; Qi *et al*., 2017), sub-solution and super-solution methods (Wang *et al*., 2019; Mâagli *et al*., 2015) and iterative techniques (Liu *et al*., 2013).

Hadamard (1892) introduced an important fractional derivative, which differs from the above-mentioned ones because its definition involves logarithmic function of arbitrary exponent and named as Hadamard derivative. In the last few decades many authors are paying more and more attention to fractional differential equation involving Hadamard derivative, the study of the topic is still in its primary stage. For details and recent developments on Hadamard fractional differential equations, see (Huang and Liu, 2018; Wang *et al*., 2018; Zhai *et al*., 2018) and references therein. Recently, some researches have extensively interested in the study of the fractional differential equations with *p*-Laplacian operators see for examples (Chamekh *et al*., 2018; Ding *et al*., 2015).

From the above review of the literature concerning fractional differential equations, most of the authors investigated only the existence of solutions or positive solutions for Hadamard fractional differential equations without considering the *pi*-Laplacian operator. A very few authors established results along with *p*-Laplacian operator, us example in (Wang and Wang, 2016), the authors considered the following nonlinear Hadamard fractional differential problem:

$$\begin{cases} D^{\sigma}\left(\phi_p\left(D^{\alpha}u(t)\right)\right) = f\left(t,u(t)\right), & t\in(1,T), \\ u(T)=\lambda I^{\gamma}u(\eta), \quad D^{\alpha}u(1)=0, \quad u(1)=0, \end{cases}$$

where fore an appropriate ξ, D^{ξ} is the Hadamard fractional derivative of order ξ, $1 < \alpha \leq 2$, $0 < \sigma \leq 1$, $\gamma > 0$, $\lambda \in \mathbb{R}$ and $f \in C([1,T]\times \mathbb{R}, \mathbb{R})$. By using the Schauder fixed point Theorem, the existence of solutions is obtained. In Li and Lin (2013), the authors used the Guo-Krasnosel'skii fixed point Theorem to prove the existence and uniqueness of positive solutions of the following Hadamard fractional boundary value with $p-$Laplacian operator:

$$\begin{cases} D^{\sigma}\left(\phi_p\left(D^{\alpha}u(t)\right)\right) = f\left(t,u(t)\right), & t\in(1,e), \\ u(1)=u'(1)=u'(e)=0, \quad D^{\alpha}u(1)=D^{\alpha}u(e)=0, \end{cases}$$

where, $2 < \alpha \leq 3$, $1 < \sigma \leq 2$, $f \in C([1,e] \times [0,\infty),[0,\infty))$ and the function $\varphi_p (p > 1)$, is called *p*-Laplacian and is defined in R by $\varphi_p(s) = |s|^{p-2}s$. The authors in Zhang *et al*. (2018) established some existence of positive solutions for the following nonlinear Hadamard fractional differential equations with *p*-Laplacian operator:

$$\begin{cases} D^{\sigma}\left(\phi_p\left(D^{\alpha}u(t)\right)\right) = f\left(t,u(t)\right), t\in(1,e), \\ u(1)=u'(1)=u'(e)=0, \ D^{\alpha}u(1)=0, \\ \phi_p\left(D^{\alpha}u(e)\right)=\mu\int_1^e \phi_p\left(D^{\alpha}u(t)\right)\dfrac{dt}{t}, \end{cases}$$

Science Publications

DOI: 10.3844/ajabssp.2021.61.72

where, $2 < \alpha \le 3$, $1 < \sigma \le 2$, $0 \le \mu < \sigma$ and $f \in C([1,e] \times \mathbb{R}, \mathbb{R})$.

Motivated by the above mentioned papers, in this study, we study the following fractional Hadamard problem:

$$\begin{cases} D^{\sigma}\left(\phi_p\left(D^{\alpha}x\right)\right)(t) = f(t,x), \\ x(1) = \phi_p\left(D^{\alpha}x\right)(1) = 0, \\ A_1 I^{\gamma_1} x(\eta_1) + B_1 x(e) = c_1, \qquad 0 < \gamma_1 \\ A_2 I^{\gamma_2} x\left(\phi_p\left(D^{\alpha}x\right)\right)(\eta_2) + B_2 \phi_p(e) = c_2, \quad 0 < \gamma_2 \end{cases} \tag{1.1}$$

where, α and σ are in (1,2], η_1 and η_2 are in $(1,e)$, A_1, A_2, B_1, B_2, c_1 and c_2 are fixed real numbers.

For the sake of computational convenience, we set:

$$\Delta_1 = B_2 + \frac{A_2\Gamma(\sigma)}{\Gamma(\gamma_2+\sigma)}\left(\log\eta_2\right)^{\gamma_2+\sigma-1}, \tag{1.2}$$

$$\Delta_2 = B_1 + \frac{A_1\Gamma(\sigma)}{\Gamma(\gamma_1+\alpha)}\left(\log\eta_1\right)^{\gamma_1+\alpha-1}. \tag{1.3}$$

And we assume the following conditions.

(H_1) There exist nonnegative functions $a(t)$, $b(t) \in C([1,e], \mathbb{R})$, such that:

$$\left|f(t,x)\right| \le a(t) + b(t)\left|x\right|^{p-1}, \text{ for each } t \in [1,e] \text{ and } x \in \mathbb{R}.$$

$(H_2)\ \Theta_1 + \Theta_2 < 1$, where

$$\Theta_1 = \left(\frac{\|b\|\infty}{\Gamma(\sigma+1)}\right)^{q-1} \frac{\left(2^{q-1}-1\right)\Gamma\left(\sigma(q-1)+1\right)}{\Gamma\left(\alpha+\sigma(q-1)+1\right)} \left(1+\frac{|A_1|}{|\Delta_2|}+\left(2^{q-1}-1\right)\left(\frac{|A_2|+|B_2|}{|\Delta_1|}\right)\right)^{q-1}$$

and:

$$\Theta_2 = \frac{\|b\|\infty}{\Gamma(\sigma+1)\Gamma(\alpha+1)|\Delta_2|}\left(|B_1| + \frac{\left(|A_2|+|B_2|\right)}{|\Delta_1|}\right).$$

(H_3) There exist a positive continuous nondecreasing function g on $[0,\infty)$ and a function $p \in C([1,e],\mathrm{R}^+)$ such that:

$$\left|f(t,x)\right| \le p(t)\, g\left(\|x\|\right) \text{ for each } (t,u) \in [1,e]\times\mathbb{R}$$

(H_4) there exists $M_2 > 0$, such that:

$$\frac{M_2}{\frac{w_1}{\Gamma^{q-1}(\sigma+1)}\|p\|_\infty^{q-1} g(M_2)^{q-1} + w_2\|p\|_\infty g(M_2) + w_3} > 1,$$

where w_1, w_2 and w_3 is given respectively by:

$$w_1 = \frac{\Gamma\left(\sigma(q-1)+1\right)}{\Gamma\left(\alpha+\sigma(q-1)+1\right)}\left[1+\frac{|A_1|}{|\Delta_2|}\right] + \frac{\left(2^{q-1}-1\right)\left(|A_2|+|B_2|\right)^{q-1}}{|\Delta_1|^{q-1}} \left[1+\frac{|A_1|\Gamma\left(\sigma(q-1)+1\right)}{|\Delta_2|\Gamma(\alpha+\sigma)(q-1)+1}\right], \tag{1.4}$$

$$w_2 = \frac{|B_1|\Gamma(\sigma+1)}{\Gamma(\alpha+\sigma+1)} + \left(\frac{|A_2|+|B_2|}{\Gamma(\sigma+1)\Gamma(\alpha+1)|\Delta_1|}\right) \tag{1.5}$$

and:

$$w_3 = \frac{|c_1|}{|\Delta_2|} + \frac{\left(2^{q-1}-1\right)^2|c_2|^{q-1}}{|\Delta_1|^{q-1}}\left(1+\frac{|A_1|}{|\Delta_2|}\right) + \frac{|c_2||B_1|}{|\Delta_1\Delta_2|\Gamma(\alpha+1)\Gamma(\sigma+1)}. \tag{1.6}$$

The main results of this study are summarized in the following theorems.

Theorem 1.1

Let $q \ge 2$. If $f \in C([1,e] \times \mathbb{R}, \mathbb{R})$ such that hypothesis (H_1) and (H_2) are satisfied, then the Hadamard fractional boundary value problem (1.1) has a unique solution.

Theorem 1.2

If $f \in C([1,e] \times \mathbb{R}, \mathbb{R})$ and if hypothesis (H_3) and (H_4) are fulfilled. Then the fractional boundary value problem (1.1) has at least one solution.

This study is organized as follows, in Section 2 we present some preliminaries and usefully results which will be used in the proofs of the main results. Section 3 is devoted to the proof of Theorem 1.1 and Theorem 1.2. In Section 4, we present some important examples in order to illustrate the main results of this article.

Preliminaries

In this section, we recall some results and we prove key lemmas which we will use later in section 3. Also, we give some definitions and properties related on Hadamard fractional calculus, we refer the reader to Kilbas *et al*. (2006) for more details.

Definitions 2.1

The Hadamard fractional integral of order $q > 1$ for a function g: $[1, \infty) \to \mathbb{R}$ is defined as:

$$I^q g(t) = \frac{1}{\Gamma(q)}\int_1^t\left(\log\frac{t}{s}\right)^{q-1}\frac{g(s)}{s}ds, \tag{2.1}$$

$$= \frac{1}{\Gamma(q)}\left\{\left(\log(t)\right)^q\int_0^1(1-s)^{q-1} g\left(e^{s\log(t)}\right)ds\right\}, \tag{2.2}$$

where, $\log(.) = \log_e(.)$:

- The Hadamard derivative of fractional order α for a function g: $[1,\infty) \to \mathbb{R}$ is given by:

$$D^{\alpha} g(t) = \left(t\frac{d}{dt}\right)^{n} I^{n-\alpha}, \tag{2.3}$$

where, $n-1 < \alpha < n$, $n = [\alpha]+1$, $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.1

[12] For any $t \in [1,e]$, $c \in \mathbb{R}$ and any constants α,σ in [1,2], we have:

$$I^{\alpha}(c)(t) = cI^{\alpha}1(t) = c\frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} = \frac{c(\log t)^{\alpha}}{\Gamma(\alpha+1)}. \tag{2.4}$$

$$I^{\alpha}\left[(\log(.))^{\sigma}\right](t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \frac{\left[\log(s)\right]^{\sigma} ds}{s} = \frac{(\log t)^{\alpha+\sigma}\,\Gamma(\sigma+1)}{\Gamma(\alpha+\sigma+1)}. \tag{2.5}$$

We use also the following property:
If $q > 2$ and $max(|x|,|y|) \le R$, then:

$$\left|\Phi_q(x) - \Phi_q(y)\right| \le (q-1)R^{q-2}\left|x-y\right|. \tag{2.6}$$

The following elementary relation is usefully:
If $n > 1$, then for all positive numbers a and b, we have:

$$(a+b)^{n} \le \left(2^{n}-1\right)\left(a^{n}+b^{n}\right). \tag{2.7}$$

Lemma 2.2

Kilbas *et al*. (2006) Let $q > 0$ and $x \in C[1,\infty) \cap L^1[1,\infty)$. Then the Hadamard fractional differential equation $D^q x(t) = 0$ has the solution:

$$x(t) = \sum_{i=1}^{n} c_i (\log t)^{q-i},$$

and the following formula holds:

$$I^{q} D^{q} x(t) = x(t) + \sum_{i=1}^{n} c_i (\log t)^{q-i},$$

where $c_i \in \mathbb{R}$, $i = 1,2,...,n$ and $n-1 < q < n$.

Theorem 2.3

Given $y \in C([1,e],\mathbb{R})$, the unique solution of the problem:

$$\begin{cases} D^{\sigma}\left(\phi^{p}\left(D^{\alpha}x(t)\right)\right) = f(t,x), \\ x(1) = \phi^{p}\left(D^{\alpha}x(1)\right) = 0, \\ A_1 I^{\gamma_1} x(\eta_1) + B_1 x(e) = c_1, \\ A_2 I^{\gamma_2} x\left(\phi^{p}\left(D^{\alpha}x\right)\right)(\eta_2) + B_2\phi_p\left(D_{\alpha}x\right)(e) = c_2, \end{cases} \tag{2.8}$$

is given by:

$$x(t) = I^{\alpha}\phi_q\left[I^{\sigma} f(t,x) + c_3\left(\log(t)\right)^{\sigma-1}\right] + c_4(\log t)^{\alpha-1}, \tag{2.9}$$

where, c_3 and c_4 are given by:

$$c_3 = \frac{c_2 - A_2 I^{\sigma+\gamma_2}\left(f(\eta_2,x)\right) - B_2 I^{\sigma}\left(f(e,x)\right)}{\Delta_1}, \tag{2.10}$$

and:

$$c_4 = \frac{c_1 - A_1 I^{\alpha+\gamma_1}\phi_q\left[I^{\sigma}\left(f(\eta_1,x)\right) + c_3\left(\log\eta_1\right)^{\sigma-1}\right](\eta_1) - B_1 I^{\alpha}\left(I^{\sigma}\left(f(e,x)\right) + c_3\right)}{\Delta_2}, \tag{2.11}$$

where, Δ_1 and Δ_2 are given respectively in Eq. (1.2) and (1.3)

Proof

As argued in Kilbas *et al*. (2006), the Hadamard differential Equation in (2.8) can be written as:

$$\phi_p\left(D^{\alpha}x(t)\right) = I^{\sigma}\left(f(t,x)\right) + a(\log t)^{\sigma-1} + b(\log t)^{\sigma-2}.$$

Since $\phi_p(D^{\alpha}x(1)) = 0$, then $b = 0$. So, we obtain:

$$\phi_p\left(D^{\alpha}x(t)\right) = I^{\sigma} f(t,x) + a(\log t)^{\sigma-1}. \tag{2.12}$$

By applying I^{γ_2} on both sides of (2.12) for $t = \eta_2$ and using the property (2.5), we obtain:

$$I^{\gamma_2}\left(\phi_p\left(D^{\alpha}x(\eta_2)\right)\right) = I^{\gamma_2+\sigma} f(\eta_2,x) + a\frac{\Gamma(\sigma)}{\Gamma(\eta_2+\sigma)}\left(\log\eta_2\right)^{\gamma_2+\sigma-1}. \tag{2.13}$$

On the other hand, put $t = e$ in Equation (2.12), we obtain:

$$\phi_p\left(D^{\alpha}x(e)\right) = I^{\sigma} f(e,x) + a. \tag{2.14}$$

By combining Equations (2.13), (2.14) and the second boundary condition in (2.8), we obtain:

$$c_2 = A_2 I^{\gamma_2+\sigma} f(\eta_2, x)) + a\left[A_2 \frac{\Gamma(\sigma)}{\Gamma(\gamma_2+\sigma)}(\log\eta_2)^{\eta_2+\sigma-1} + B_2\right] + B_2 I^{\sigma} f(e,x).$$

Therefore:

$$a = \frac{c_2 - A_2 I^{\gamma_2+\sigma} f(\eta_2,x) - B_2 I^{\sigma} f(e,x)}{\Delta_1} =: c_3 \tag{2.15}$$

where:

$$\Delta_1 = B_2 + \frac{A_2\Gamma(\sigma)}{\Gamma(\gamma_2+\sigma)}(\log\eta_2)^{\gamma 2+\sigma-1}.$$

Then, the solution can be written us follows:

$$x(t) = I^{\alpha}\left\{\varphi_q\left[I^{\sigma} f(t,x) + c_3(\log(t))^{\sigma-1}\right]\right\} + a^{0}(\log t)^{\alpha-1} + b^{0}(\log t)^{\alpha-2}.$$

Since $x(1) = 0$, then $b' = 0$ and we get:

$$x(t) = I^{\alpha}\left\{\phi_q\left[I^{\sigma} f(t,x) + c_3(\log t)^{\sigma-1}\right]\right\} + a'(\log t)^{\alpha-1}. \tag{2.16}$$

Now, if we apply I^{γ_1} to (2.16) and we replace t by η_1, then, using the property (2.5), we obtain:

$$I^{\gamma_1}\left(x(\eta_1)\right) = I^{\gamma_1+\alpha}\left[\phi_q\left\{I^{\sigma} f(\eta_1,x) + c_3(\log\eta_1)^{\sigma-1}\right\}\right] + a'\frac{\Gamma(\alpha)}{\Gamma(\eta_1+\alpha)}(\log\eta_1)^{\gamma_1+\alpha-1}. \tag{2.17}$$

On the other hand, Equation (2.16) with $t = e$, yields to:

$$x(e) = I^{\alpha}\left\{\phi_q\left[I^{\sigma} f(e,x) + c_3\right]\right\} + a'. \tag{2.18}$$

Finally, by combining Equations (2.17), (2.18) with the first boundary condition c_1, we obtain:

$$c_1 = A_1 I^{\gamma_1+\alpha}\left[\phi_q\left(I^{\sigma} f(\eta_1,x) + c_3(\log\eta_1)^{\sigma-1}\right)\right] + a'\left[A_1\frac{\Gamma(\alpha)}{\Gamma(\gamma_1+\alpha)}(\log\eta_1)^{\eta_1+\sigma-1} + B_1\right] + B_1 I^{\alpha}\left[\phi_q\left(I^{\sigma} f(e,x) + c_3\right)\right].$$

It follows that:

$$a' = \frac{c_1 - A_1 I^{\gamma_1+\alpha}\left[\phi_q\left(I^{\sigma} f(\eta_1,x) + c_3(\log\eta_1)^{\sigma-1}\right)\right] - B_1 I^{\alpha}\left[\phi_q\left(I^{\sigma} f(e,x) + c_3\right)\right]}{\Delta_2} =: c_4 \tag{2.19}$$

where:

$$\Delta_2 = B_1 + \frac{A_1\Gamma(\alpha)}{\Gamma(\gamma_1+\alpha)}(\log\eta_1)^{\gamma_1+\alpha-1}.$$

Substituting the values of c_3 and c_4 in (2.16), we obtain (2.9). This completes the proof.

To prove the main results of this study, we recall the following theorems.

Theorem 2.4

Smart (1974) Let X be a Banach space. If the operator $T: X \to X$ is completely continuous and if the set:

$$V = \{u \in X \mid u = \mu T u, 0 < \mu < 1\},$$

is bounded. Then T has a fixed point in X.

Theorem 2.5

Granas and Dugundji (2003) Let X be a Banach space, C be a closed, convex subset of X, U be an open subset of C and $0 \in C$. Suppose that the operator $F: U \to C$ is continuous and compact. Then either

(i). F has a fixed point in U, or
(ii). There is $u \in \partial U$ and $\lambda \in (0,1)$, with $u = \lambda F(u)$.

Proof of the Main Results

This section is devoted to prove existence results for the nonlinear boundary value problem (1.1). Also, we shall prove existence and uniqueness results by using different methods. Let us de ne the operator $Q: C([1,e], \mathbb{R}) \to C([1,e], \mathbb{R})$ by:

$$Qx(t) = I^{\alpha}\phi_q\left[I^{\sigma} f(t,x) + c_3(\log t)^{\sigma-1}\right] + c_4(\log t)^{\alpha-1}, \tag{3.1}$$

where c_3 and c_4 are given respectively by Equation (2.10) and (2.11). Notice that the existence of fixed points of the operator Q is equivalent to the existence of solutions for problem (1.1).

Proof of Theorem 1.1

In this subsection, by using the well known Banach's fixed point Theorem, we present the existence and uniqueness result for Problem (1.1).

Let $B_R = \{x \in C([1,e], \mathbb{R}): \|x\|_\infty \le R\}$. The proof is divided into tow steps.

Step 1: In this step, we will prove that the operator Q is completely continuous. Let O be an open bounded subset of $C([1,e], \mathbb{R})$. Since f is continuous, then, the operator Q is contunuous and so, $Q(O)$ is bounded. Next, we will show that Q is equicontinuous.

Let $1 \le t_1 < t_2 \le e$, then, from Inequality (2.6) we obtain:

$$\begin{aligned}&\left|Q(x)(t_2)-Q(x)(t_1)\right|\\ &\le\Big|I^\alpha\Big[\Phi_q\Big\{I^\sigma f(x,t_2)+c_3(\log t_2)^{\sigma-1}\Big\}-\Phi_q\\ &\Big\{I^\sigma f(x,t_1)+c_3(\log t_1)^{\sigma-1}\Big\}(t_1)\Big]\Big|\\ &+|c_4\Big[(\log t_2)^{\alpha-1}-(\log t_1)^{\alpha-1}\Big]\\ &\le(q-1)R^{q-2}\Big[I^\alpha I^\alpha(f)(x,t_2)-I^\alpha I^\alpha(f)(x,t_1)\Big]\\ &+(q-1)R^{q-2}|c_3|\Big|\Big[(\log t_2)^{\sigma-1}-(\log t_1)^{\sigma-1}\Big]\\ &+|c_4|\Big|\Big[(\log t_2)^{\alpha-1}-(\log t_1)^{\alpha-1}\Big].\end{aligned}\tag{3.2}$$

So, by using (2.1), (2.4) and the fact that f is bounded, we get:

$$\begin{aligned}\left|I^\sigma(f)(x,t)\right|&\le\frac{1}{\Gamma(\sigma)}\int_1^t\left(\log\frac{t}{s}\right)^{\sigma-1}\left|\frac{f(s,x(s))}{s}\right|ds\\ &\le\frac{1}{\Gamma(\sigma)}\int_1^t\left(\log\frac{t}{s}\right)^{\sigma-1}\frac{M}{s}ds\le\frac{M}{\Gamma(\sigma+1)}(\log t)^\sigma,\end{aligned}\tag{3.3}$$

for some $M > 0$. Hence, (2.5), implies that:

$$\left|I^\alpha I^\sigma f(x,t)\right|\le\frac{M(\log t)^{\alpha+\sigma}}{\Gamma(\alpha+\sigma+1)}.\tag{3.4}$$

By combining (3.2) and (3.4), we obtain:

$$\begin{aligned}&\left|Q(x)(t_2)-Q(x)(t_1)\right|\\ &\le\frac{(q-1)R^{q-2}M}{\Gamma(\alpha+\sigma+1)}\Big[(\log t_2)^{\alpha+\sigma}-(\log t_1)^{\alpha+\sigma}\Big]\end{aligned}\tag{3.5}$$

On the other hand, from (2.10) and (3.3), we obtain:

$$\begin{aligned}|c_3|&\le\frac{|c_2|+\left|A_2I^{\sigma+\gamma_2}f(x,\eta_2)\right|+\left|B_2I^\sigma f(x,e)\right|}{|\Delta_1|}\\ &\le\frac{|c_2|+|A_2|\frac{M}{\Gamma(\sigma+\gamma_2+1)}(\log\eta_2)^{\sigma+\gamma_2}+|B_2|\frac{M}{\Gamma(\sigma+1)}}{|\Delta_1|}\\ &\le\frac{|c_2|+M\frac{|A_2|}{\Gamma(\sigma+\gamma_2+1)}(\log\eta_2)^{\sigma+\gamma_2}+\frac{|B_2|}{\Gamma(\sigma+1)}}{|\Delta_1|},\end{aligned}\tag{3.6}$$

Therefore using (2.6), (2.11) and (3.4), we get:

$$\begin{aligned}|c_4|&\le\frac{|c_1|+|A_1|\left|(q-1)R^{q-2}I^{\alpha+\gamma_1}\left[I^\sigma f(\eta_1,x)+c_3(\log\eta_1)^{\sigma-1}\right]\right|+|B_1|\left|I^\alpha(I^\sigma f(e,x)+c_3)\right|}{|\Delta_2|}\\ &\le\frac{|c_1|+|A_1|\left|(q-1)R^{q-2}\left[\frac{(\log\eta_1)^{\alpha+\sigma+\gamma_1}M}{\Gamma(\alpha+\sigma+\gamma_1+1)}+\frac{|c_3|(\log\eta_1)^{\alpha+\gamma_1+\sigma-1}\Gamma(\sigma-2)}{\Gamma(\alpha+\gamma_1+\sigma)}\right]\right|}{|\Delta_2|}+\frac{|B_1|\left[\frac{M}{\Gamma(\alpha+\sigma+1)}+\frac{c_3}{\Gamma(\alpha+1)}\right]}{|\Delta_2|}\\ &\le\frac{|c_1|+M\left[(q-1)R^{q-2}|A_1|\left[\frac{(\log\eta_1)^{\alpha+\sigma+\gamma_1}}{\Gamma(\alpha+\sigma+\gamma_1+1)}\right]+\frac{|B_1|}{\Gamma(\alpha+\sigma+1)}\right]}{|\Delta_2|}\\ &+\frac{\left(|c_2|+M\left(\frac{A_2}{\Gamma(\sigma+\gamma_2+1)}(\log\eta_2)^{\sigma+\gamma 2}+\frac{|B_2|}{\Gamma(\sigma+1)}\right)\right)\left(\frac{(\log\eta_1)^{\alpha+\gamma_1+\sigma-1}\Gamma(\sigma-2)}{\Gamma(\alpha+\gamma_1+\sigma)}+\frac{1}{\Gamma(\alpha+1)}\right)}{|\Delta_1\Delta_2|}.\end{aligned}\tag{3.7}$$

By combining (3.5), (3.6) and (3.7), one has:

$$\lim_{t_2 \to t_1} \left| Q(x)(t_2) - Q(x)(t_1) \right| = 0.$$

Finally, the Arzelá-Ascoli Theorem implies that $Q: C([1, e], \mathbb{R}) \to C([1, e], \mathbb{R})$ is completely continuous.

Step 2: In this step, we will pove that the set $D = \{x \in C([1, e], \mathbb{R}): x = \mu Q(x), \mu \in (0,1)\}$ is bounded. From hypothesis (H_1) and using (2.7), we have:

$$\begin{aligned}
&|x(t)| = |\mu(Qx)(t)| \\
&\leq \max_{t\in[1,e]} \left| I^{\alpha}\Phi_q\left(I^{\sigma}(f)(x,t) + c_3(\log t)^{\sigma-1}\right) + c_4(\log t)^{\alpha-1} \right| \\
&\leq \max_{t\in[1,e]} \left| I^{\alpha}\Phi_q\left(I^{\sigma}(A)(x,t) + c_3(\log t)^{\sigma-1}\right) + |c_4|(\log t)^{\alpha-1} \right| \\
&\leq \max_{t\in[1,e]} \left[I^{\alpha}\Phi_q\left(A\left(\frac{(\log t)^{\sigma}}{\Gamma(\sigma+1)}\right) + |c_3|(\log t)^{\sigma-1} \right) + |c_4|(\log t)^{\alpha-1} \right] \\
&\leq \max_{t\in[1,e]} \left[I^{\alpha}\left\{ \left(\frac{A}{\Gamma(\sigma+1)} + |c_3|\right)^{q-1} (\log t)^{\sigma(q-1)} \right\} + |c_4|(\log t)^{\alpha-1} \right] \\
&\leq \max_{t\in[1,e]} \left[\left(\frac{A}{\Gamma(\sigma+1)} + |c_3|\right)^{q-1} \frac{(\log t)^{\alpha+\sigma(q-1)}\Gamma(\sigma(q-1)+1)}{\Gamma(\alpha+\sigma(q-1)+1)} + |c_4|(\log t)^{\alpha-1} \right] \\
&\leq \left(\frac{A}{\Gamma(\sigma+1)} + |c_3|\right)^{q-1} \frac{\Gamma(\sigma(q-1)+1)}{\Gamma(\alpha+\sigma(q-1)+1)} + |c_4| \\
&\leq \left(2^{q-1}-1\right)\left[\left(\frac{A}{\Gamma(\sigma+1)}\right)^{q-1} + |c_3|^{q-1} \right] \frac{\Gamma(\sigma(q-1)+1)}{\Gamma(\alpha+\sigma(q-1)+1)} + |c_4|,
\end{aligned}$$

where, $A = \|a\|_\infty + \|b\|_\infty \|x\|_\infty^{p-1}$.

On the other hand, we have:

$$\begin{aligned}
|c_3| &\leq \frac{|c_2| + \left|A_2 I^{\sigma+\gamma_2} f(x,\eta_2)\right| + \left|B_2 I^{\sigma} f(x,e)\right|}{|\Delta_1|} \\
&\leq \frac{|c_2| + |A_2| \frac{A}{\Gamma(\sigma+\gamma_2+1)}(\log\eta_2)^{\sigma+\gamma_2} + |B_2|\frac{A}{\Gamma(\sigma+1)}}{|\Delta_1|} \\
&\leq \frac{|c_2| + A\left(\frac{|A_2|}{\Gamma(\sigma+\gamma_2+1)}(\log\eta_2)^{\sigma+\gamma_2} + \frac{|B_2|}{\Gamma(\sigma+1)}\right)}{|\Delta_1|} \qquad (3.8) \\
&\leq \frac{|c_2| + A\left(\frac{|A_2|}{\Gamma(\sigma+\gamma_2+1)} + \frac{|B_2|}{\Gamma(\sigma+1)}\right)}{|\Delta_1|}, \\
&\leq \frac{|c_2|}{|\Delta_1|} + \frac{A}{|\Delta_1|\Gamma(\sigma+1)}\left(|A_2| + |B_2|\right).
\end{aligned}$$

Then, using Inequality (2.7), we obtain

$$|c_3|^{q-1} \leq \left(2^{q-1}-1\right)\left[\left(\frac{|c_2|}{|\Delta_1|}\right)^{q-1} + \left(\frac{A\left(|A_2|+|B_2|\right)}{\Gamma(\sigma+1)|\Delta_1|}\right)^{q-1} \right], \qquad (3.9)$$

and:

$$\begin{aligned}
|c_4| &\leq \frac{|c_1| + |A_1| \left| I^{\alpha+\gamma_1}\left[\phi_q\left(I^{\sigma} f(\eta_1,x) + c_3(\log\eta_1)^{\sigma-1}\right)\right]\right| + |B_1| \left| I^{\alpha}\left(I^{\sigma} f(e,x) + c_3\right)\right|}{|\Delta_2|} \\
&\leq \frac{|c_1| + |A_1| \left| I^{\alpha+\gamma_1}\left[\phi_q\left(I^{\sigma}(A)(\eta_1) + |c_3|(\log\eta_1)^{\sigma-1}\right)\right]\right| + |B_1| \left| I^{\alpha}\left(I^{\sigma}(A)(e) + c_3\right)\right|}{|\Delta_2|} \\
&\leq \frac{|c_1| + |A_1| \left| I^{\alpha+\gamma_1}\left[A\frac{(\log\eta_1)^{\sigma}}{\Gamma(\sigma+1)} + |c_3|(\log\eta_1)^{\sigma-1} \right]^{q-1} \right| + |B_1| \left| I^{\alpha}\left(\frac{A}{\Gamma(\sigma+1)} + |c_3|(e)\right)\right|}{|\Delta_2|} \\
&\leq \frac{|c_1| + |A_1| \left[\frac{A}{\Gamma(\sigma+1)} + |c_3|\right]^{q-1} \frac{(\log\eta_1)^{\alpha+\gamma_1+\sigma(q-1)}\Gamma(\sigma(q-1)+)}{\Gamma(\alpha+\gamma_1+\sigma(q-1)+1)} + \left(\frac{|B_1|A}{\Gamma(\sigma+1)} + |c_3|\right)/\Gamma(\alpha+1)}{|\Delta_2|} \\
&\leq \frac{|c_1| + |A_1| \left[\frac{A}{\Gamma(\sigma+1)} + |c_3|\right]^{q-1} \frac{\Gamma(\sigma(q-1)+1)}{\Gamma(\alpha+\gamma_1+\sigma(q-1)+1)} + \left(\frac{|B_1|A}{\Gamma(\sigma+1)} + |c_3|\right)/\Gamma(\alpha+1)}{|\Delta_2|} \\
&\leq \frac{|c_1| + |A_1| \left(2^{q-1}-1\right)\left[\left(\frac{A}{\Gamma(\sigma+1)}\right)^{q-1} + |c_3|^{q-1}\right] \frac{\Gamma(\sigma(q-1)+1)}{\Gamma(\alpha+\gamma_1+\sigma(q-1)+1)} + \left(\frac{|B_1|A}{\Gamma(\sigma+1)} + |c_3|\right)/\Gamma(\alpha+1)}{|\Delta_2|}. \qquad (3.10)
\end{aligned}$$

Therefore, it follows from (2.7), (3.8), (3.9) and (3.10), that:

$$\begin{aligned}
&|x(t)| = |\mu(Qx)(t)| \\
&\le \left(2^{q-1}-1\right)\left[\left(\frac{A}{\Gamma(\sigma+1)}\right)^{q-1} + \left(2^{q-1}-1\right)\left[\left(\frac{|c_2|}{|\Delta_1|}\right)^{q-1} + \left(\frac{A\left(|A_2|+|B_2|\right)}{\Gamma(\sigma+1)|\Delta_1|}\right)^{q-1}\right]\right]\frac{\Gamma(\sigma(q-1)+1)}{\Gamma(\alpha+\sigma(q-1)+1)} \\
&+\frac{|c_1| + \left(2^{q-1}-1\right)|A_1|\left[\left(\frac{A}{\Gamma(\sigma+1)}\right)^{q-1} + \left(|c_3|\right)^{q-1}\right]\frac{\Gamma(\sigma(q-1)+1)}{\Gamma(\alpha+\gamma_1+\sigma(q-1)+1)}}{|\Delta_2|} \\
&+\frac{\left(\frac{\left(|B_1|A\right)}{\Gamma(\sigma+1)\Gamma(\alpha+1)} + \frac{|c_2|}{|\Delta_1|\Gamma(\alpha+1)} + \frac{A}{|\Delta_1|\Gamma(\sigma+1)\Gamma(\alpha+1)}\left(|A_2|+|B_2|\right)\right)}{|\Delta_2|} \\
&\le \left(\frac{A}{\Gamma(\sigma+1)}\right)^{q-1}\frac{\left(2^{q-1}-1\right)\Gamma(\sigma(q-1)+1)}{\Gamma(\alpha+\sigma(q-1)+1)}\left(1+\frac{|A_1|}{|\Delta_2|} + \left(2^{q-1}-1\right)\left(\frac{\left(|A_2|+|B_2|\right)}{|\Delta_1|}\right)\right)^{q-1} \\
&+\frac{|c_1|}{|\Delta_2|} + \frac{|c_2|}{\Gamma(\alpha+1)|\Delta_1\Delta_2|} + \left(2^{q-1}-1\right)^2\left(\frac{|c_2|}{|\Delta_1|}\right)^{q-1}\left(1+\frac{|A_1|}{|\Delta_2|}\right)\frac{\Gamma(\sigma(q-1)+1)}{\Gamma(\alpha+\sigma(q-1)+1)} \\
&+\frac{A}{\Gamma(\sigma+1)\Gamma(\alpha+1)|\Delta_2|}\left(|B_1| + \frac{|A_2+|+|B_2|}{|\Delta_1|}\right).
\end{aligned}$$

Using the inequality (2.7), we have:

$$\begin{aligned}
A^{q-1} &= \left(\|a\|_\infty + \|b\|_\infty\|x\|_\infty^{p-1}\right)^{q-1} \\
&\le \left(\|a\|_\infty^{q-1} + \|b\|_\infty^{q-1}\|x\|_\infty\right).
\end{aligned}$$

Since $q \ge 2$, then, $p \le 2$ and $\|x\|_\infty^{p-1} \le \|x\|\infty$. Thus, we have:

$$\|x\|_\infty < \left(\Theta_1 + \Theta_2\right)\|x\|_\infty + \Theta_3,$$

where:

$$\begin{aligned}
\Theta_1 &= \left(\frac{\|b\|_\infty}{\Gamma(\sigma+1)}\right)^{q-1}\frac{\left(2^{q-1}-1\right)\Gamma(\sigma(q-1)+1)}{\Gamma(\alpha+\sigma(q-1)+1)} \\
&\left(1+\frac{|A_1|}{|\Delta_2|} + \left(2^{q-1}-1\right)^2\left(\frac{|A_2|+|B_2|}{|\Delta_1|}\right)^{q-1}\right), \qquad ,\\
\Theta_2 &= \frac{\|b\|\infty}{\Gamma(\sigma+1)\Gamma(\alpha+1)|\Delta_2|}\left(|B_1| + \frac{\left(|A_2|+|B_2|\right)}{|\Delta_1|}\right),
\end{aligned}$$

and:

$$\begin{aligned}
\Theta_3 &= \frac{|c_1|}{|\Delta_2|} + \frac{|c_2|}{\Gamma(\alpha+1)|\Delta_1\Delta_2|} + \frac{\left(2^{q-1}-1\right)\Gamma(\sigma(q-1)+1)}{\Gamma(\alpha+\sigma(q-1)+1)} \\
&\left(\frac{|c_2|}{|\Delta_1|}\right)^{q-1}\left(\left(2^{q-1}-1\right)+\frac{|A_1|}{|\Delta_2|}\right) + \left(\frac{\|a\|\infty}{\Gamma(\sigma+1)}\right)^{q-1}\frac{\left(2^{q-1}-1\right)\Gamma(\sigma(q-1)+)}{\Gamma(\alpha+\sigma(q-1)+1)} \\
&\left(1\frac{|A_1|}{|\Delta_2|} + \left(2^{q-1}-1\right)\left(\frac{|A_2|+|B_2|}{|\Delta_1|}\right)^{q-1}\right) + \frac{\|a\|\infty}{\Gamma(\sigma+1)\Gamma(\alpha+1)}\left(|B_1| + \frac{|A_2|+|B_2|}{|\Delta_1|}\right).
\end{aligned}$$

This together with condition (H_2), gives $||u||_\infty < M_1$. That is D is bounded, so the operator Q has at lest one fixed point. Which implies that the problem (1.1) has at least one solution.

Proof of Theorem 1.2

In this subsection, by using Leray-Schauder's nonlinear alternative Theorem, we give the proof of Theorem 1.2. The proof is divided into several steps.

Step 1: In this step, We prove that Q maps bounded sets into equicontinuous sets of $C([1,e], \mathbb{R})$. Let $t_1,t_2 \in [1, e]$ with $t_1 < t_2$ and $x \in B_r$. Then, we have:

$$\begin{aligned}
&I_\sigma f(x,t_1) - I_\sigma f(x,t_2) \\
&\le \frac{g(r)\|p\|_\infty}{\Gamma(\sigma)}\left[\int_1^{t_1}\left(\log\frac{t_1}{s}\right)^{\sigma-1}\frac{ds}{s} - \int_1^{t_2}\left(\log\frac{t_2}{s}\right)^{\sigma-1}\frac{ds}{s}\right] \\
&\le \frac{g(r)\|p\|_\infty}{\Gamma(\sigma)}\left[\int_1^{t_1}\left(\log\frac{t_1}{s}\right)^{\sigma-1} - \left(\log\frac{t_2}{s}\right)^{\sigma-1}\right]\frac{ds}{s} - \int_1^{t_2}\left(\log\frac{t_2}{s}\right)^{\sigma-1}\frac{ds}{s}\Bigg].
\end{aligned}$$

So, using Inequality (2.6), we obtain:

$$\begin{aligned}
&\left|(Qx)(t_1)-(Qx)(t_2)\right| \\
&\le I_\alpha \Phi_q\left[I_\sigma f(x,t_1)+c_3(\log t_1)^{\sigma-1}\right]-I_\alpha \Phi_q\left[I_\sigma f(x,t_2)+c_3(\log t_1)^{\sigma-1}\right]+c_4\left[(\log t_2)^{\alpha-1}-(\log t_2)^{\alpha-1}\right] \\
&\le I_\alpha\left\{\Phi_q\left[I_\sigma f(x,t_1)+c_3(\log t_1)^{\sigma-1}\right]-\Phi_q\left[I_\sigma f(x,t_2)+c_3(\log t_2)^{\sigma-1}\right]\right\}+c_4\left[(\log t_1)^{\alpha-1}-(\log t_2)^{\alpha-1}\right] \\
&\le I_\alpha\left\{(q-1)r^{q-2}\left[I_\sigma f(x,t_1)-I_\sigma f(x,t_2)\right]+c_3\left[(\log t_1)^{\sigma-1}-(\log t_2)^{\sigma-1}\right]\right\}+c_4\left[(\log t_1)^{\alpha-1}-(\log t_2)^{\alpha-1}\right] \\
&\le I_\alpha\left\{\frac{(q-1)r^{q-2}g(r\|p\|)}{\Gamma(\sigma)}\left[\int_1^{t_1}\left(\log\frac{t_1}{s}\right)^{\sigma-1}-\left(\log\frac{t_2}{s}\right)^{\sigma-1}\right]\frac{ds}{s}-\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\sigma-1}\frac{ds}{s}\right]+c_3\left[(\log t_1)^{\sigma-1}-(\log t_2)^{\sigma-1}\right]\right\}+c_4\left[(\log t_1)^{\alpha-1}-(\log t_2)^{\alpha-1}\right] \\
&\le I_\alpha\left\{\frac{(q-1)r^{q-2}g(r\|p\|)}{\Gamma(\sigma+1)}\left[(\log t_1)^{\sigma-1}-(\log t_2)^{\sigma-1}\right]-\left(\log\frac{t_2}{t_1}\right)^{\sigma}\right]+c_3\left[(\log t_1)^{\sigma-1}-(\log t_2)^{\sigma-1}\right]\right\}+c_4\left[(\log t_1)^{\alpha-1}-(\log t_2)^{\alpha-1}\right] \\
&\le\left\{\frac{(q-1)r^{q-2}g(r\|p\|)}{\Gamma(\sigma+\sigma+1)}\left[(\log t_1)^{\alpha+\sigma-1}-(\log t_2)^{\alpha+\sigma-1}\right]-\left(\log\frac{t_2}{t_1}\right)^{\alpha+\sigma}\right]+c_3\left[(\log t_1)^{\sigma-1}-(\log t_2)^{\sigma-1}\right]\right\}+c_4\left[(\log t_1)^{\alpha-1}-(\log t_2)^{\alpha-1}\right].
\end{aligned}$$

Obviously the right-hand side of the above inequality tends to zero independently of $x \in B_r$ as $t_1 - t_2 \to 0$.

Step 2: In this step, we will prove that Q maps bounded sets (balls) into bounded sets in $C([1, e], \mathbb{R})$.

Let $r > 0$, $t \in [1, e]$ and $x \in B_r$, then, by using the hypothesis (H_3) and the inequality (2.7), we obtain:

$$\begin{aligned}
&\left|Q(x)(t)\right| \le \max_{t\in[1,e]}\left|I^\alpha\phi_q\left[I^\sigma f(t,x)+c_3(\log t)^{\sigma-1}\right]+c_4(\log t)^{\alpha-1}\right| \\
&\le \max_{t\in[1,e]}\left|I^\alpha\phi_q\left[\frac{1}{\Gamma(\sigma)}\int_1^t\left(\log\frac{t}{s}\right)^{\sigma-1}\frac{f(s,x)}{s}ds+c_3(\log t)^{\sigma-1}\right]+c_4(\log t)^{\alpha-1}\right| \\
&\le \max_{t\in[1,e]}\left|I^\alpha\phi_q\left[\frac{\|p\|_\infty g(\|x\|_\infty)}{\Gamma(\sigma)}\int_1^t\left(\log\frac{t}{s}\right)^{\sigma-1}\frac{1}{s}ds+c_3(\log t)^{\sigma-1}\right]+c_4(\log t)^{\alpha-1}\right| \\
&\le \max_{t\in[1,e]}\left|I^\alpha\phi_q\left[\frac{\|p\|_\infty g(\|x\|_\infty)(\log t)^{\sigma}}{\Gamma(\sigma+1)}+c_3(\log t)^{\sigma-1}\right]+c_4(\log t)^{\alpha-1}\right| \\
&\le \max_{t\in[1,e]}\left|I^\alpha\left[\frac{\|p\|_\infty g(\|x\|_\infty)(\log t)^{\sigma}}{\Gamma(\sigma+1)}+c_3(\log t)^{\sigma-1}\right]^{q-1}+c_4(\log t)^{\alpha-1}\right| \\
&\le \max_{t\in[1,e]}\left|I^\alpha\left[\frac{\|p\|_\infty^{q-1} g(\|x\|_\infty)^{q-1}(\log t)^{\sigma(q-1)}}{\Gamma(\sigma+1)^{q-1}}+c_3(\log t)^{(\sigma-1)(q-1)}\right](2^{q-1}-1)+c_4(\log t)^{\alpha-1}\right| \\
&\le \max_{t\in[1,e]}\left|\left[\frac{\|p\|_\infty^{q-1} g(\|x\|_\infty)^{q-1}(\log t)^{\alpha+\sigma(q-1)}\Gamma(\sigma(q-1)+1)}{\Gamma(\alpha+\sigma(q-1)+1)\Gamma(\sigma+1)^{q-1}}+c_3(\log t)^{(\sigma-1)(q-1)}\right](2^{q-1}-1)+c_4\right| \\
&\le (2^{q-1}-1)\left[\frac{\|p\|_\infty^{q-1} g(\|x\|_\infty)^{q-1}\Gamma(\sigma(q-1)+1)}{\Gamma(\alpha+\sigma(q-1)+1)\Gamma(\sigma+1)^{q-1}}+c_3^{q-1}\right]+c_4. \qquad (3.11)
\end{aligned}$$

In the other hand, we have:

$$\left|c_3\right| \le \frac{\left|c_2\right| + \left|A_2 I^{\sigma+\gamma_2} f\left(x,\eta_2\right)\right| + \left|B_2 I^{\sigma} f\left(x,e\right)\right|}{\left|\Delta_1\right|} \le \frac{\left|c_2\right| + \left|A_2\right| \frac{\|p\|_\infty g\left(\|x\|_\infty\right)}{\Gamma\left(\sigma+\gamma_2+1\right)}\left(\log\eta_2\right)^{\sigma+\gamma_2} + \left|B_2\right| \frac{\|p\|_\infty g\left(\|x\|\infty\right)}{\Gamma\left(\sigma+1\right)}}{\left|\Delta_1\right|}$$

$$\le \frac{\left|c_2\right| + \|p\|_\infty g\left(\|x\|_\infty\right)\left(\frac{\left|A_2\right|}{\Gamma\left(\sigma+\gamma_2+1\right)}\left(\log\eta_2\right)^{\sigma+\gamma_2} + \frac{\left|B_2\right|}{\Gamma\left(\sigma+1\right)}\right)}{\left|\Delta_1\right|} \le \frac{\left|c_2\right| + \|p\|_\infty g\left(\|x\|_\infty\right)\left(\frac{\left|A_2\right|}{\Gamma\left(\sigma+\gamma_2+1\right)} + \frac{\left|B_2\right|}{\Gamma\left(\sigma+1\right)}\right)}{\left|\Delta_1\right|} \tag{3.12}$$

$$\le \frac{\left|c_2\right| + \|p\|_\infty g\left(\|x\|_\infty\right)\left(\left|A_2\right| + \left|B_2\right|\right)}{\Gamma\left(\sigma+1\right)\left|\Delta_1\right|},$$

$$\left|c_3\right|^{q-1} \le \left(2^{q-1}-1\right)\frac{\left|c_2\right|^{q-1} + \|p\|_\infty^{q-1} g\left(\|x\|_\infty\right)^{q-1}\left(\left|A_2\right| + \left|B_2\right|\right)^{q-1}}{\left(\Gamma\left(\sigma+1\right)\left|\Delta_1\right|\right)^{q-1}}, \tag{3.13}$$

and:

$$\left|c_4\right| = \left|\frac{c_1 - A_1 I^{\alpha+\gamma_1}\phi_q\left[I^{\sigma}F\left(\eta_1,x\right) + c_3\left(\log\eta_1\right)^{\sigma-1}\right] - B_1 I^{\alpha}\left(I^{\sigma} f\left(e,x\right) + c_3\right)}{\Delta_2}\right|$$

$$\le \frac{\left|c_1\right| + \left|A_1\right|\left|I^{\alpha+\gamma_1}\phi_q\left[\frac{\|p\|_\infty g\left(\|x\|\infty\right)\left(\log\eta_1\right)^{\sigma}}{\Gamma\left(\sigma+1\right)} + c_3\left(\log\eta_1\right)^{\sigma-1}\right]\right| + \left|B_1\right|\left|I^{\alpha}\left(\frac{\|p\|_\infty g\left(\|x\|_\infty\right)\left(\log(.)\right)^{\sigma}}{\Gamma\left(\sigma+1\right)} + c_3\right)(e)\right|}{\left|\Delta_2\right|}$$

$$\le \frac{\left|c_1\right| + \left|A_1\right|\left|I^{\alpha+\gamma_1}\left[\frac{\|p\|_\infty g\left(\|x\|\infty\right)\left(\log\eta_1\right)^{\sigma}}{\Gamma\left(\sigma+1\right)} + c_3\left(\log\eta_1\right)^{\sigma-1}\right]^{q-1}\right| + \left|B_1\right|\left(\frac{\|p\|_\infty g\left(\|x\|_\infty\right)\Gamma\left(\sigma+1\right)}{\Gamma\left(\alpha+\sigma+1\right)} + \frac{c_3}{\Gamma\left(\alpha+1\right)}\right)}{\left|\Delta_2\right|}$$

$$\le \frac{\left|c_1\right| + \left|A_1\right|\left|\left[\frac{\|p\|_\infty g\left(\|x\|_\infty\right)}{\Gamma\left(\sigma+1\right)} + c_3\right]^{q-1} I^{\alpha+\gamma_1}\left[\left(\log\eta_1\right)^{\sigma(q-1)}\right]\right| + \left|B_1\right|\left(\frac{\|p\|_\infty g\left(\|x\|_\infty\right)\Gamma\left(\sigma+1\right)}{\Gamma\left(\alpha+\sigma+1\right)} + \frac{c_3}{\Gamma\left(\alpha+1\right)}\right)}{\left|\Delta_2\right|}$$

$$\le \frac{\left|c_1\right| + \left|A_1\right|\left[\frac{\|p\|_\infty g\left(\|x\|_\infty\right)}{\Gamma\left(\sigma+1\right)} + c_3\right]^{q-1}\frac{\Gamma\left(\sigma\left(q-1\right)+1\right)\left(\log\eta_1\right)^{\alpha+\eta_1+\sigma(q-1)}}{\Gamma\left(\alpha+\gamma 1+\sigma\left(q-1\right)+1\right)} + \left|B_1\right|\left(\frac{\|p\|_\infty g\left(\|x\|_\infty\right)\Gamma\left(\sigma+1\right)}{\Gamma\left(\alpha+\sigma+1\right)} + \frac{c_3}{\Gamma\left(\alpha+1\right)}\right)}{\left|\Delta_2\right|}$$

$$\le \frac{\left|c_1\right| + \left|A_1\right|\left[\frac{\|p\|_\infty g\left(\|x\|_\infty\right)}{\Gamma\left(\sigma+1\right)} + c_3\right]^{q-1}\frac{\Gamma\left(\sigma\left(q-1\right)+1\right)}{\Gamma\left(\alpha+\gamma 1+\sigma\left(q-1\right)+1\right)} + \left|B_1\right|\left(\frac{\|p\|_\infty g\left(\|x\|_\infty\right)\Gamma\left(\sigma+1\right)}{\Gamma\left(\alpha+\sigma+1\right)} + \frac{c_3}{\Gamma\left(\alpha+1\right)}\right)}{\left|\Delta_2\right|}$$

$$\le \frac{\left|c_1\right| + \left(2^{q-1}-1\right)\left|A_1\right|\left[\frac{\|p\|_\infty^{q-1} g\left(\|x\|_\infty\right)^{q-1}}{\Gamma^{q-1}\left(\sigma+1\right)} + c_3^{q-1}\right]\frac{\Gamma\left(\sigma\left(q-1\right)+1\right)}{\Gamma\left(\alpha+\gamma 1+\sigma\left(q-1\right)+1\right)} + \left|B_1\right|\left(\frac{\|p\|_\infty g\left(\|x\|_\infty\right)\Gamma\left(\sigma+1\right)}{\Gamma\left(\alpha+\sigma+1\right)} + \frac{c_3}{\Gamma\left(\alpha+1\right)}\right)}{\left|\Delta_2\right|} \tag{3.14}$$

$$\le \frac{\left|c_1\right| + \left(2^{q-1}-1\right)\left|A_1\right|\left[\frac{\|p\|_\infty^{q-1} g\left(\|x\|_\infty\right)^{q-1}}{\Gamma^{q-1}\left(\sigma+1\right)} + c_3^{q-1}\right]\frac{\Gamma\left(\sigma\left(q-1\right)+1\right)}{\Gamma\left(\alpha+\sigma\left(q-1\right)+1\right)} + \left|B_1\right|\left(\frac{\|p\|_\infty g\left(\|x\|_\infty\right)\Gamma\left(\sigma+1\right)}{\Gamma\left(\alpha+\sigma+1\right)} + \frac{c_3}{\Gamma\left(\alpha+1\right)}\right)}{\left|\Delta_2\right|}.$$

It follows, from Inequalities (3.12), (3.13) and (3.14) that the inequality (3.11) becomes:

$$\|Q(x)\| \le \frac{\omega_1\left(2^{q-1}-1\right)}{\Gamma^{q-1}(\sigma+1)}\|p\|_\infty^{q-1} g\left(\|x\|_\infty\right)^{q-1} + \frac{\omega_2\|p\|_\infty g\left(\|x\|_\infty\right)}{|\Delta_2|} + \omega_3,$$

where:

$$\omega_1 = \frac{\Gamma(\sigma(q-1)+1)}{\Gamma(\alpha+\sigma(q-1)+1)}\left[1+\frac{|A_1|}{|\Delta_2|}\right] + \frac{\left(2^{q-1}-1\right)\left(|A_2|+|B_2|\right)^{q-1}}{|\Delta_1|^{q-1}}$$
$$\left[1+\frac{|A_2|\Gamma(\sigma(q-1)+1)}{|\Delta_2|\Gamma(\alpha+\sigma(q-1)+1)}\right]$$
$$\omega_2 = \frac{|B_1|\Gamma(\sigma+1)}{\Gamma(\alpha+\sigma+1)} + \left(\frac{|A_2|+|B_2|}{\Gamma(\sigma+1)\Gamma(\alpha+1)|\Delta_1|}\right),$$

and:

$$\omega_3 = \frac{|c_1|}{|\Delta_2|} + \frac{\left(2^{q-1}-1\right)^2|c_2|^{q-1}}{|\Delta_1|^{q-1}}\left(1+\frac{|A_1|}{|\Delta_2|}\right) + \frac{|c_1||B_1|}{|\Delta_1\Delta_2|\Gamma(\alpha+1)\Gamma(\sigma+1)}.$$

Consequently, as $x \in B_r$, we have:

$$\|Q(x)\| \le \left|\frac{\omega_1\left(2^{q-1}-1\right)}{\Gamma^{q-1}(\sigma+1)}\|p\|_\infty^{q-1} g(r)^{q-1} + \frac{\omega_2}{\Delta_2}\|p\|_\infty g(r) + \omega_3.\right.$$

Therefore, the Arzelá-Ascoli Theorem implies that $Q: C([1, e], \mathbb{R}) \to C([1, e], \mathbb{R})$ is completely continuous. Let x be a solution. Then, for $t \in [1, e]$, as in the first step, we have:

$$\|x\|_\infty \le \frac{\omega_1\left(2^{q-1}-1\right)}{\Gamma^{q-1}(\sigma+1)}\|p\|_\infty^{q-1} g\left(\|x\|_\infty\right)^{q-1} + \frac{\omega_2\|p\|_\infty g\left(\|x\|_\infty\right)}{|\Delta_2|} + \omega_3,$$

which implies that:

$$\frac{\|x\|_\infty}{\frac{\omega_1\left(2^{q-1}-1\right)}{\Gamma^{q-1}(\sigma+1)}\|p\|_\infty^{q-1} g\left(\|x\|_\infty\right)^{q-1} + \frac{\omega_2\|p\|_\infty g\left(\|p\|_\infty\right)}{|\Delta_2|} + \omega_3} \le 1.$$

In view of (H_4), there exists M_2 such that $\|x\| 6 = M_2$. Let us set:

$$U = \left\{u \in C([1,e], \square): \text{k}u\text{k} \le M_2\right\}.$$

Note that the operator $Q: U \to C([1,e], \mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $u \in \partial \bar{U}$ such that $u = \lambda Qu$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray- Schauder type (Lemma 2.5), we deduce that Q has a fixed point $u \in U$ which is a solution of problem (1.1). This completes the proof.

Examples

Example 4.1 Consider the Problem

$$\begin{cases} D^{7/4}\left(\phi_{3/2}\left(D^{3/2}x(t)\right)\right) = f(t,x), \\ x(1) = \phi_{3/2}\left(D^{3/2}x(1)\right) = 0, \\ I^{1/2}x(2) + 2x(e) = 0, \\ I^{1/4}\left(\phi_{3/2}\left(D^{3/2}x\right)\right)(5/2) - \phi_{3/2}\left(D^{3/2}x\right)(e) = 0, \end{cases} \tag{4.1}$$

where $\sigma = \frac{7}{4}, \alpha = \frac{3}{2}, p = \frac{3}{2}, q = 3, A_1 = A_2 = 1, B_1 = 2, B_2 = -1, \gamma_1 = \frac{1}{2}, \gamma_2 = \frac{1}{4}, \eta_1 = 2, \eta_2 = \frac{5}{2}, c_1 = 2$ and $c_2 = 0$. Then, we have:

$$\Delta_1 = -0.157\, and\, \Delta_2 = 2.61.$$

Consider problem (4.1) with:

$$f(t,x) = \frac{1}{10}e^{-t^2}\sqrt{|x(t)|} + arctan(1+t).$$

Clearly:

$$|f(t,x)| \le a(t) + b(t)\|x\|_\infty,$$

where, $a(t)$ = arctan(1 + t) and $b(t) = \frac{1}{10}e^{-t^2}$. It follows that:

$$\|a\|_\infty \le \frac{\pi}{2}, \|b\|_\infty = \frac{1}{10e}.$$

With the given values, we find that:

$$\theta_1 \le 8.58 \times 10^{-3}\, and\, \theta_2 \le 0.121.$$

So, $\theta_1 + \theta_2 \le 1$. Thus, all hypothesis of Theorem 1.1 hold. Therefore, the Hadamard fractional integral boundary value problem (4.1) has at least one solution.

Consider Problem (4.1) with:

$$f(t,x) = \frac{\log t}{1+x^2}.$$

It follows that:

$$\left|f(t,x)\right| \le p(t)\, g\left(\|x\|_\infty\right),$$

for $p(t) = \log t$ and $g(\|x\|_\infty) = 1$. Then, we have:

$$\|p\|_\infty = 1, \omega_1 = 488.4, \omega_2 = 6.405 \, and \, \omega_3 = 0.766.$$

Further, the hypothesis (H_4), it is equivalent to show existence of M_2 such that:

$$M_2 > 496.$$

Thus, all hypothesis of Theorem 1.2 hold. Therefore, the Hadamard fractional integral boundary value problem (4.1) has at least one solution.

Example 4.2

Consider the problem, for $\lambda > 1$:

$$\begin{cases} D^\sigma\left(\phi^p\left(D^\alpha x(t)\right)\right) = \frac{\lambda}{10}\log t x^{p-1}, \\ x(1) = \phi_p\left(D^\alpha x(1)\right) = 0, \\ A_1 I^{\gamma_1} x(\eta_1) + B_1 x(e) = c_1, \\ A_2 I^{\gamma_2}\left(\phi^p\left(D^\alpha x\right)\right)(\eta_2) + B_2\phi^p\left(D^\alpha x\right)(e) = c_2,. \end{cases} \tag{4.2}$$

1. Let, $\alpha = \sigma = 7/4$, $p = 5/4$, $q = 5$, $A_1 = B_1 = -1/4$, $A_2 = B_2 = 1/4$, $c_1 = 1$, $c_2 = 0$, $\eta_1 = 2$, $\eta_2 = 5/2$, $\gamma_1 = 3/2$ and $\gamma_2 = 5/4$. Then, it follows that:

$$\|a\|_\infty = 0, \|b\|_\infty = \frac{\lambda}{10},$$
$$\Delta_1 = 0.34, \Delta_2 = -0.4,$$
$$\theta_1 = 1.89\left(\frac{\lambda}{10}\right)^4 and \, \theta^2 = 1.68\frac{\lambda}{10}.$$

The hypothesis (H_2) is equivalent to give a positive real M_1 such that:

$$1.89\left(\frac{\lambda}{10}\right)^4 + 1.68\frac{\lambda}{10} \le 1.$$

So, for $\lambda < 5$, there exist $M_1 > 0$ satisfying the above inequality. Thus, all hypothesis of Theorem 1.1 hold. Therefore, the Hadamard fractional integral boundary value problem (4.1) has at least one solution:

2. Let $\alpha = 3/2$, $\sigma = 7/4$, $p = 3$, $q = 3/2$, $A_1 = -1$, $B_1 = 2$, $A_2 = 1$, $B_2 = 0$, $c_1 = c_2 = 0$, $\eta_1 = 5/2$, $\eta_2 = 3/2$, $\gamma_1 = 1/2$ and $\gamma_2 = 3/2$

It follows that:

$$\left|f(t,x)\right| \le p(t)\, g\left(\|x\|_\infty\right),$$
$$for\; p(t) = \frac{\lambda}{10}\log t \, and \, g\left(\|x\|_\infty\right) = \|x\|_\infty^2 .Then, we\, have$$

$$\|p\|_\infty = \left|\frac{\lambda}{10}\right., \omega_1 = 5.5, \; \omega_2 = 7 and\, \omega_3 = 0.$$

Further, the hypothesis (H_4), it is equivalent to show existence of M_2 such that:

$$0.7\lambda M_2^2 + \left(1.5\sqrt{\lambda} - 1\right)M_2 \ge 0.$$

Then, for every $\lambda > 1$, there exist $M_2 > \frac{1.5\sqrt{\lambda}-1}{0.7\lambda}$. Thus, all hypothesis of Theorem 1.2 hold. Therefore, the Hadamard fractional integral boundary value problem (4.1) has at least one solution.

Ethics

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and no ethical issues involved.

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