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Common Fixed Points of Generalized Cyclic *C* Class ψ-ϕ-Λ Weak Nonexpansive Mappings

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Abstract: This paper shows that if S and T are two joint *generalized* cyclic F-ψ-ϕ-Λ weak nonexpansive type mappings, then they have only one common fixed point. In particular, every *generalized* cyclic C class ψ-ϕ-Λ weak nonexpansive mapping has a unique fixed point. Hence it extends the results of the attached references of this paper.

Keywords: Fixed Point Theorems, *abc* Generalized Contractions and Nonexpansive Mappings, Cyclic Weak ϕ and Weak ψ-ϕ Contraction and Nonexpansive Mappings

Introduction and Preliminaries

Since 1922 till now many generalizations of Banach contraction principle (Banach, 1922) have been achieved. For cyclic ψ-ϕ mappings, we refer to the references below.

In particulr; Sahar Mohamed Ali Abou Bakr (2013) proved the existence of only one fixed point for both $\{a, b, c\}$-ntype and $\{a, b, c\}$-ctype types of mappings defined on closed convex weakly Cauchy subset C of a normed space X.

Definition 1

Let C be a subset of a normed space X and T be a mapping from C into C satisfying:

$$\|T(x)-T(y)\| \le a\|x-y\| + b\|x-T(x)\| + c\max\{\|y-T(y)\|, \|y-T(x)\|\} \ \forall x,y \in C, a,b,c \in [0,1].$$

Then:

(1) T is said to be $\{a, b, c\}$-ntype mapping, if $0 < a < 1$, $0 < b$, $0 \le c < 1 = 2$ and $a + b + c = 1$
(2) T is said to be $\{a, b, c\}$-ctype mapping, if $0 \le c < 1/2$ and $a + b + c < 1$

Sahar Mohamed Ali Abou Bakr and Ansari (2017) introduced new $\mathfrak{U}-T$ cyclic weak contraction C-class concept. Namely; $\mathfrak{U}-T$ cyclic weak F-ψ-ϕ-contraction type and proved some related fixed point theorems.

Definition 2

Let S and T be self mappings on X. Then S is $\mathfrak{U}-T$ cyclic F-ψ-ϕ *weak* contraction mapping on X iff there are:

(1) A collection of non empty sets $\mathfrak{U} = \{A_i\}_{i=1}^{j}$ with $X = \bigcup_{i=1}^{j} A_i$
(2) Non-decreasing functions ψ, ϕ: $[0, \infty] \to \mathbb{R}^+$, $\psi(t) = 0$ iff $t = 0$ and $\phi(t) = 0$ iff $t = 0$ with ψ continuous, and
(3) A C class function F: That is; F: $[0, \infty] \times [0, \infty] \to \mathbb{R}$ is continuous and satisfying $F(u, v) \le u$ for all $u, v \in [0, \infty]$ and if $F(u, v) = u$, then either $u = 0$ or $v = 0$ such that:

(1) $\mathfrak{U}$ is a T-cyclic representation of X with respect to S: That is; $T(S(A_1)) \subset A_2$, $T(S((A_2))) \subset A_3, \ldots, T(S(A_{j-1})) \subset A_j$ and $T(S((A_j))) \subset A_1$
(2) The following contractivity condition is satisfied:

$$\psi\Big(d\big(T(S(x)),T(S(y))\big)\Big) \le F\Big(\psi\big(d(T(x),T(y))\big),\phi\big(d(T(x),T(y))\big)\Big)$$

for every $x \in A_i$, $y \in A_{i+1}$, $i = 1,2,\ldots,j$, where $A_{j+1} = A_1$.

In this study; we define the real valued function $\Lambda_{S,(abc)}$: $X \times X \to R^+$ as follows:

$$\Lambda_{S,(abc)}(x,y) = a\, d(x,y) + b\, d(x,S(x)) + c\max\{d(y,S(y)), d(y,S(x))\} \ \forall x,y \in X,$$

where, a, b, c are three real numbers.

Definition 3

Let (X, d) be metric space with $X = A \bigcup B$ and S be a self mapping on X with:

(1) $S(A) \subset B$ and $S(B) \subset A$ and
(2) There are real constants a, b, $c \in [0,1]$ with:

$$d\left(S(x),S(y)\right) \le \Lambda_{S,(abc)}(x,y) \quad \forall x \in A, y \in B.$$

Then S is said to be (A,B) generalized cyclic:

(1) Λ contraction iff $a + c + b < 1$
(2) Λ nonexpansive iff $a + c + b = 1$

Definition 4

Let S: $X \to X$ fulfill the condition:

$$d\left(S(x),S(y)\right) \le \Lambda_{S,(abc)}(x,y) - \phi\left(\Lambda_{S,(abc)}(x,y)\right) \quad \forall x \in A, y \in B.$$

where, ϕ is lower semi-continuous non-decreasing functions ϕ: $[0, \infty] \to [0, \infty]$ with $\phi(t) > 0$ for $t \in [0, \infty]$ and $\phi(0) = 0$. Then S is said to be (A, B) generalized cyclic:

(1) ϕ -Λ weak contraction iff $a + c + b < 1$,
(2) ϕ -Λ weak nonexpansive iff $a + c + b = 1$.

Definition 5

Let S: $X \to X$ be a mapping fulfill the condition:

$$\psi\left(d\left(S(x),S(y)\right)\right) \le \psi\left(\Lambda_{S,(abc)}(x,y)\right) - \phi\left(\Lambda_{S,(abc)}(x,y)\right) \quad \forall x \in A, y \in B,$$

where, ψ and ϕ are lower semi-continuous non-decreasing functions ψ, ϕ: $[0,\infty] \to [0, \infty]$ with $\psi(t) > 0$ for $t \in [0, \infty]$ and $\psi(0) = 0$ with $\phi(t) > 0$ for $t \in [0,\infty]$ and $\phi(0) = 0$. Then S is said to be (A, B) generalized cyclic:

(1) ψ-ϕ-Λ weak contraction iff $a + c + b < 1$,
(2) ψ-ϕ-Λ weak nonexpansive iff $a + c + b = 1$.

Definition 6

Let S: $X \to X$ be a mapping fulfill the condition:

$$\psi\left(d\left(S(x),S(y)\right)\right) \le F\left(\psi\left(\Lambda_{S,(abc)}(x,y)\right), \phi\left(\Lambda_{S,(abc)}(x,y)\right)\right) \quad \forall x \in A, y \in B,$$

where, ψ and ϕ are lower semi-continuous non-decreasing functions ψ, ϕ: $[0, \infty] \to [0, \infty]$ with $\psi(t) > 0$ for $t \in [0, \infty]$, $\psi(0) = 0$, $\phi(t) > 0$ for $t \in [0,\infty]$, $\phi(0) = 0$ and F is a C class function. Then S is said to be (A, B) generalized cyclic:

(1) F-ψ-ϕ-Λ weak contraction iff $a + c + b < 1$,
(2) F-ψ-ϕ-Λ weak nonexpansive iff $a + c + b = 1$.

Example

Let $X = [-1, 1]$, $A = [-1, 0]$, and $B = [0, 1]$. Define S: $X \to X$ as:

$$S(z) = \begin{cases} -\dfrac{z}{3}, & if\ z \in A \\ -\dfrac{z}{2}, & if\ z \in B \end{cases}$$

It is clear that S is cyclic with respect to the representation $A \cup B$ of X. Endow X with the metric $d(x, y) = |x-y|$, cosider $\phi(t) = t$, $\psi(t) = t$, and $F(t, s) = t - \frac{1}{2}s$, then the operator S is generalized cyclic F-ψ-ϕ-$\Lambda\left(\frac{1}{9}\right)\left(\frac{1}{12}\right)\left(\frac{2}{3}\right)$ weak contraction w.r.t (A, B). In fact, let $x \in A$ and $y \in B$. Then we have:

$$\begin{aligned}
\Lambda_{S\left(\frac{1}{9}\right)\left(\frac{1}{12}\right)\left(\frac{2}{3}\right)}(x,y) &= \frac{1}{9}d(x,y) + \frac{1}{12}d\left(x,S(x)\right) + \frac{2}{3}\max\left\{d\left(y,S(y)\right), d\left(y,S(x)\right)\right\} \\
&= \frac{1}{9}(y-x) + \frac{1}{12}\left|x - \left(-\frac{x}{3}\right)\right| + \frac{2}{3}\max\left\{\left|y - \left(-\frac{y}{2}\right)\right|, \left|y - \left(-\frac{x}{3}\right)\right|\right\} \\
&= \frac{1}{9}(y-x) + \frac{1}{12}\left|x - \left(-\frac{x}{3}\right)\right| + \frac{2}{3}\left|y - \left(-\frac{y}{2}\right)\right| \\
&= \frac{1}{9}(y-x) + \frac{1}{12}\left|x + \frac{x}{3}\right| + \frac{2}{3}\left|y + \frac{y}{2}\right| = \frac{1}{9}(y-x) \\
&- \frac{x}{9} + y = \frac{2}{9}(5y - x).
\end{aligned}$$

$$\begin{aligned}
d\left(S(x),S(y)\right) &= \left|-\frac{x}{3} - \left(-\frac{y}{2}\right)\right| = \left|\frac{y}{2} - \frac{x}{3}\right| = \frac{y}{2} - \frac{x}{3} \\
&= \frac{1}{6}(3y - 2x) = \frac{2}{9}\left(\frac{9}{4}y - \frac{6}{4}x\right) \\
&= \frac{2}{9}\left[(5y - x) - \frac{1}{2}\left(\frac{11}{2}y - x\right)\right] \le \frac{2}{9}\left[(5y - x) - \frac{1}{2}(5y - x)\right] \\
&= F\left(\psi\left(\Lambda_{S,(abc)}(x,y)\right), \phi\left(\Lambda_{S,(abc)}(x,y)\right)\right) \quad \forall x \in A, y \in B.
\end{aligned}$$

Remark

If $\Lambda_{S,(abc)}(x,y) = ad(x,y) \quad \forall x, y \in X$, *that is if b=c=0, then we have the usual contraction or nonexpansive mapping acoording to the value of a, a<1 or not. One can see some related fixed point theorems proved in the attached references below.*

In the light of the particular cases; $F(u, v) = u-v$ and $\psi = Id$; the identity mapping, *we noticed the following:*

(1) *The class of all* (A,B) *generalized cyclic F-ψ-ϕ* Λ weak *non-expansive is wider than the class of all* (A, B) *generalized cyclic F-ψ-ϕ-*Λ weak *contraction.*
(2) *The class of all* (A, B) *generalized cyclic F-ψ-ϕ-*Λ weak *nonexpansive is wider than the class of all* (A, B) *generalized cyclic ψ-ϕ-* Λ weak *nonexpansive.*
(3) *The class of all* (A, B) *generalized cyclic F-ψ–ϕ-*Λ weak *contraction is wider than the class of all* (A, B) *generalized cyclic ψ-ϕ-* Λ weak *contraction.*
(4) *The class of all* (A,B) *generalized cyclic ψ-ϕ-*Λ weak *nonexpansive is wider than the class of all* (A,B) *generalized cyclic ψ-ϕ-* Λ weak *contraction.*
(5) *The class of all* (A,B) *generalized cyclic ψ-ϕ-*Λ weak *nonexpansive is wider than the class of all* (A,B) *generalized cyclic ϕ-*Λ weak *nonexpansive.*
(6) *The class of all* (A,B) *generalized cyclic ϕ-*Λ weak *nonexpansive is wider than the class of all* (A,B) *generalized cyclic ϕ-*Λ *contraction.*
(7) *The class of all* (A,B) *generalized cyclic* Λ *nonexpansive is wider than the class of all* (A,B) *generalized cyclic ϕ-*Λ *weak nonexpansive.*
(8) *The class of all* (A,B) *generalized cyclic* Λ *nonexpansive is wider than the class of all* (A,B) *generalized cyclic* Λ *contraction.*
(9) *The class of all* (A,B) *generalized cyclic* Λ *nonexpansive is wider than the class of all {a, b, c}-ntype mappings.*
(10) *The class of all {a, b, c}-ntype mappings is wider than the class of all {a, b, c}-ctype mappings.*

In this study, the real valued function $\Lambda_{S,T,(abc)}$: $X \times X \to R^+$ is defined as:

$$\Lambda_{S,T(abc)}(x,y) = a\, d(x,y) + b\, d(x,S(x)) + c\max\{d(y,T(y)), d(y,S(x))\},$$

where, S, T: $X\to X$ are two self mappings and a, b, c are three real numbers.

We introduced the following fascinating definition for joint-cyclic mapping:

Definition 7

Let (X, d) be a metric space with $A \cup B$, S, T: $X\to X$ be two self mappings and a, b, $c \in [0, 1]$ be three real numbers satisfying:

(1) The cyclic condition: $S(A)\subset B$ and $T(B)\subset A$
(2) The contractivity condition:

$$d(S(x),T(y)) \le \Lambda_{S,T,(abc)}(x, y) - \phi(\Lambda_{S,T,(abc)}(x, y)) \quad \forall x \in A,\ y \in B,$$

where, ϕ is lower semi-continuous non-decreasing function ϕ: $[0, \infty]\to [0, \infty]$ with $\phi(t) > 0$ for $t\in[0, \infty]$ and $\phi(0) = 0$.

Then S and T are said to be joint (A, B) *generalized* cyclic:

(1) ϕ-Λ weak contraction types iff $a + c + b < 1$
(2) ϕ-Λ weak nonexpansive types iff $a + c + b = 1$

Definition 8

Let (X, d) be a metric space with $X = A\cup B$, S, T: $X\to X$ be two self mappings and a, b, $c\in[0, 1]$, $b \neq 0$ be three real numbers satisfying:

(1) The cyclic condition: $S(A)\subset B$ and $T(B) \subset A$
(2) The contractivity condition:

$$\psi\left(d(S(x),T(y))\right) \le \psi\left(\Lambda_{S,T,(abc)}(x,y)\right) - \phi\left(\Lambda_{S,T,(abc)}(x,y)\right) \ \forall x \in A, y \in B,$$

where, ψ and ϕ are non-decreasing functions ψ, ϕ: $[0, \infty]\to[0, \infty]$ with $\psi(t) > 0$, $\phi(t) > 0$ for $t \in[0, 1]$ and $\phi(0) = 0$, $\phi(0) = 0$, ψ is continuous and ϕ is lower semi-continuous.

Then S and T are said to be joint (A,B) cyclic generalized:

(1) ψ-ϕ-Λ weak contraction types iff $a + c + b < 1$
(2) ψ-ϕ-Λ weak nonexpansive types iff $a + c + b = 1$

Definition 9

Let (X, d) be a metric space with $X = A\cup B$, S,T: $X\to X$ be two self mappings and a, b, $c\in[0,1]$ $b \neq 0$ be three real numbers satisfying:

(1) The cyclic condition: $S(A)\subset B$ and $T(B) \subset A$
(2) The following contractivity condition:

$$\psi\left(d(S(x),T(y))\right) \le F\left(\psi\left(\Lambda_{S,T,(abc)}(x,y)\right), \phi\left(\Lambda_{S,T,(abc)}(x,y)\right)\right) \ \forall x \in A, y \in B,$$

where, ψ and ϕ are non-decreasing functions ψ, ϕ: $[0, \infty]\to [0, \infty]$ with $\psi(t) > 0$, $\phi(t) > 0$ for $t\in[0, \infty]$ and $\psi(0) = 0$, $\phi(0) = 0$, ψ is continuous, ϕ is lower semi-continuous and F is some C class function.Then S and T are said to be joint (A,B) generalized cyclic:

(1) *F*-ψ-ϕ-Λ weak contraction types iff $a + c + b < 1$
(2) *F*-ψ-ϕ-Λ weak nonexpansive types iff $a + c + b = 1$

We have the following:

Remarks

(1) *The class of all joint (A,B) generalized cyclic F-ψ-ϕ-Λ weak nonexpansive types is wider than that of joint (A,B) generalized cyclic ψ-ϕ-Λ weak nonexpansive types.*
(2) *The class of all joint* (*A*,*B*) *generalized cyclic F-ψ-ϕ-* Λ weak *nonexpansive types is wider than that of joint* (*A*,*B*) *generalized cyclic F-ψ-ϕ-Λ weak contraction types.*
(3) *The class of all joint* (*A*,*B*) *generalized cyclic ψ-ϕ-* Λ weak *nonexpansive types is wider than that of joint* (*A*,*B*) *generalized cyclic ψ-ϕ-Λ weak nonexpansive types.*
(4) *The class of all joint* (*A*,*B*) *generalized cyclic ψ-ϕ-* Λ weak *nonexpansive types is wider than that of joint* (*A*,*B*) *generalized cyclic ψ-ϕ-Λ weak contraction types.*
(5) *If S, T are continuous self mappings on* (*X*, *d*), *then restriction of the mapping* $\Lambda_{S,T,(abc)}$: $A\times B\rightarrow\mathbb{R}^+$ *for every $x\in A$, $y\in B$:*

$$\Lambda_{S,T,(abc)}(x,y) = a\,d(x,y) + b\,d(x,S(x)) + c\max\{d(y,T(y)),d(y,S(x))\}$$

is continuous.
(6) *If A,B are two compact subsets of the metric space* (*X*, *d*) *and S, T are continuous self mappings on X, then the restriction of the mapping* $\Lambda_{S,T,(abc)}$: $A\times B\rightarrow\mathbb{R}^+$ *attains its infimum as well as its supremum at some points in* $A\times B$

This paper shows that if *S* and *T* are two joint *generalized* cyclic *F*-ψ-ϕ-Λ weak nonexpansive types mappings, then they have only one common fixed point. In particular, every cyclic *C* class *generalized* ψ-ϕ-Λ weak nonexpansive mapping has a unique fixed point. The existing functions *F*, ψ and ϕ give extensions of many results of the references attached in this study.

Main Results

We have:

Theorem 1

Let (*X*, *d*) *be metric space and A,B be two compact subsets of which* $X = A\cup B$. *If S, T*: *X* $\rightarrow$ *X are continuous joint* (*A*, *B*) *generalized cyclic F-ψ-ϕ-Λ* weak *nonexpansive mappings on X, then there is only one point z* $\in$ *X such that* $S(z) = z = T(z)\in A\cap B$.

Proof

Let v_0 be arbitrarily chosen element in *X*. Then v_0 is either in *A* or in *B*, if v_0 is in *B*, then $v_1 = T(v_0) \in A$, $v_2 = S(v_1) \in B$, $v_3 = T(v_2)\in A$ and then define by induction:

$$v_{2n+2} = S(v_{2n+1})\in B \text{ and } v_{2n+1} = T(v_{2n})\in A \ \forall n\geq 0. \tag{2.1}$$

First, suppose *n* is an odd natural number. Then:

$$\begin{aligned}&\psi(d(v_{n+1},v_n)) = \psi(d(S(v_n),T(v_{n-1})))\\ &\leq F(\psi(\Lambda_{S,T,(abc)}(v_n,v_{n-1})),\phi(\Lambda_{S,T,(abc)}(v_n,v_{n-1})))\\ &\leq \psi(\Lambda_{S,T,(abc)}(v_n,v_{n-1})).\end{aligned} \tag{2.2}$$

Since ψ is non-decreasing, we see that:

$$\begin{aligned}&d(v_{n+1}v_n)\leq\Lambda_{S,T,(abc)}(v_n,v_{n-1})\\ &= a\ d(v_n,v_{n-1}) + b\,d(S(v_{n-1}),v_{n-1})\\ &+c\max\{d(T(v_n),v_n),d(S(v_{n-1}),v_n)\}\\ &= a\ d(v_n,v_{n-1}) + b\,d(S(v_{n-1}),v_{n-1})\\ &+c\max\{d(v_{n+1},v_n),d(v_n,v_n)\}\\ &= a\ d(v_n,v_{n-1}) + b\ d(v_n,v_{n-1}) + cd(v_{n+1},v_n)\\ &= (a+b)d(v_n,v_{n-1}) + cd(v_{n+1},v_n).\end{aligned}$$

Thus:

$$d(v_{n+1},v_n)\leq\left[\frac{a+c}{1-b}\right]d(v_n,v_{n-1}) = d(v_n,v_{n-1}).$$

Therefore:

$$\begin{aligned}&\Lambda_{S,T,(abc)}(v_n,v_{n-1})\leq(a+b)d(v_n,v_{n-1}) + c\,d(v_{n+1},v_n)\\ &\leq(a+b)d(v_n,v_{n-1}) + c\ d(v_n,v_{n-1})\\ &=(a+b+c)d(v_n,v_{n-1}) = d(v_n,v_{n-1}).\end{aligned}$$

hence:

$$d(v_{n+1},v_n)\leq\Lambda_{S,T,(abc)}(v_n,v_{n-1})\leq d(v_n,v_{n-1}). \tag{2.3}$$

Continuing gives:

$$\begin{aligned}&d(v_{n+1},v_n)\leq\Lambda_{S,T,(abc)}(v_n,v_{n-1})\\ &\leq d(v_n,v_{n-1})\leq\Lambda_{S,T,(abc)}(v_{n-1},v_{n-2})\end{aligned}. \tag{2.4}$$

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Second; by a similar method when n is an even natural number, we obtain the same conclusion as inequalities (2.4). Hence the sequences $\{d(v_{n+1}, v_n)\}_{n\in N}$ and $\{\Lambda_{S,T,(abc)}(v_{n+1}, v_n)\}_{n\in N}$ are monotonic non-increasing and bounded below by 0, thus their limit exist, each equals its infimum.

On the other side; they have the same infimum because of the inequalities (2.4), therefore if their infimum is r, then:

$$\lim_{n\to\infty} d\left(v_{n+1},v_n\right)=\lim_{n\to\infty}\Lambda_{S,T,(abc)}\left(v_n,v_{n-1}\right)=r\ .$$

Using the properties of ϕ:

$$\phi(r)\le\liminf_{n\to\infty}\phi\left(\Lambda_{S,T,(abc)}\left(v_n,v_{n-1}\right)\right).$$

Taking least upper limits on two sides of the inequality (2.2) as $n\to\infty$ gives:

$$\psi(r)\le F\left(\psi(r),\liminf_{n\to\infty}\phi\left(\Lambda_{S,T,(abc)}\left(v_n,v_{n-1}\right)\right)\le\psi(r).\quad (2.5)$$

Thus:

$$F\left(\psi(r),\liminf_{n\to\infty}\phi\left(\Lambda_{S,T,(abc)}\left(v_n,v_{n-1}\right)\right)\right)=\psi(r)\ .$$

This insures that either $\psi(r) = 0$ or $\lim\inf_{n\to\infty} \phi(\Lambda_{S,T,(abc)}(v_n, v_{n-1}) = 0$. If $\psi(r) = 0$, then $r = 0$ and if $\lim\inf_{n\to\infty} \phi(\Lambda_{S,T,(abc)}(v_n, v_{n-1}) = 0$ while $r > 0$, then we have the following contradiction:

$$0<\phi(r)\le\liminf_{n\to\infty}\phi\left(\Lambda_{S,T,(abc)}\left(v_n,v_{n-1}\right)\right)=0.$$

Hence:

$$\lim_{n\to\infty} d\left(v_{n+1},v_n\right)=\lim_{n\to\infty}\Lambda_{S,T,(abc)}\left(v_n,v_{n-1}\right)=0.\qquad (2.6)$$

This insures that:

$$\inf\left\{\Lambda_{S,T,(abc)}(x,y):x\in A,y\in B\right\}=0.$$

Since $\Lambda_{S,T,(abc)}$ attains its infimum on $A\times B$, there is $x_0\in A$ and $y_0\in B$ such that:

$$\Lambda_{S,T,(abc)}\left(x_0,y_0\right)=0.$$

This gives:

$$ad\left(x_0,y_0\right)+b\ d\left(x_0,S\left(x_0\right)\right)$$
$$+c\max\left\{d\left(y_0,T\left(y_0\right)\right),d\left(y_0,S\left(x_0\right)\right)\right\}=0.$$

Since all are nonnegative real numbers, clearly:

$$d\left(x_0,y_0\right)=d\left(x_0,S\left(x_0\right)\right)=d\left(y_0,T\left(y_0\right)\right)=d\left(y_0,S\left(x_0\right)\right)=0,$$

and we have:

$$x_0=y_0=S\left(x_0\right)=S\left(y_0\right).$$

Notice that the converse is also true, if $x_0 = y_0 = S(x_0) = T(x_0) = S(y_0) = T(y_0)$, then $\Lambda_{S,T,(abc)}(x_0, y_0) = 0$ is clear. On the other side, this showed that $x_0 = y_0\in A\cap B$. If there exists another point $v\in A\cap B$ such that $S(v) = v = T(v)$ with $v \ne y_0$, then we get:

$$\begin{aligned}\psi\left(d\left(v,y_0\right)\right)&=\psi\left(d\left(S(v),T\left(y_0\right)\right)\right)\\&\le F\left(\psi\left(\Lambda_{S,T,(abc)}\left(v,y_0\right)\right),\phi\left(\Lambda_{S,T,(abc)}\left(v,y_0\right)\right)\right).\\&\le\psi\left(\Lambda_{S,T,(abc)}\left(v,y_0\right)\right).\end{aligned}$$

Hence; the following is a contradiction:

$$\begin{aligned}&d\left(v,y_0\right)\le\Lambda_{S,T,(abc)}\left(v,y_0\right)\\&\le a\ d\left(v,y_0\right)+b\ d\left(S(v),v\right)+c\max\left\{d\left(y_0,T\left(y_0\right)\right),d\left(y_0,S(v)\right)\right\}\\&=a\ d\left(v,y_0\right)+cd\left(v,y_0\right)=(a+c)d\left(v,y_0\right)<d\left(v,y_0\right)\end{aligned}$$

This shows that $d(v, y_0) = 0$, that is $v = y_0$.
We have:

Proposition 1

The sequence defined iteratively by the induction (2.1) is convergent to the unique common fixed point of S and T:

$$\lim_{n\to\infty} v_n=v.$$

Proof

Let v be the unique common fixed point of S and T, in addition suppose that $\lim_{n\to\infty} v_n = u$ with $v \ne u$. Then there is $n\in N$ such that:

$$\begin{aligned}\psi\left(d\left(v_n,v\right)\right)&=\psi\left(d\left(S\left(v_{n-1}\right),T(v)\right)\right)\\&\le F\left(\psi\left(\Lambda_{S,T,(abc)}\left(v_{n-1},v\right)\right),\phi\left(\Lambda_{S,T,(abc)}\left(v_{n-1},v\right)\right)\right)\\&\le\psi\left(\Lambda_{S,T,(abc)}\left(v_{n-1},v\right)\right).\end{aligned}$$

Hence:

$$\begin{aligned} d(v_n,v) &= d(S(v_{n-1}),T(v)) \le \Lambda_{S,T,(abc)}(v_{n-1},v) \\ &\le a\, d(v_{n-1},v) + b\, d(S(v_{n-1}),v_{n-1}) \\ &+ c\max\{d(T(v),v), d(S(v_{n-1}),v)\} \\ &\le a\, d(v_{n-1},v) + b\, d(v_n,v_{n-1}) + c\max\{d(v,v), d(v_n,v)\} \\ &\le a\, d(v_{n-1},v) + b\, d(v_n,v_{n-1}) + c\, d(v_n,v). \end{aligned}$$

That is:

$$d(v_n,v) \le \frac{1}{1-c}\left[a\, d(v_{n-1},v) + b\, d(v_n,v_{n-1})\right].$$

Using Equation (2.6) with the limiting approach as $n \to \infty$ prove that:

$$d(u,v) \le \frac{a}{1-c} d(u,v)$$

hence; $\left(1-\frac{a}{1-c}\right)d(u,v) \le 0$, since $\neq 1-c$, we get $d(u,v)=0$, that is; $v = u$.

Corollary 1

Let (X, d) *be metric space and* A,B *two compact subsets of which* $X = A \cup B$. *If* $S: X \to X$ *is continuous* (A,B) *generalized cyclic* F-ψ-ϕ-Λ weak *nonexpansive mapping on* X, *then there is only one point* $v \in X$ *such that* $S(v) = v \in A \cap B$. *Moreover; for any* $v_0 \in X$, *we have* $\lim_{n\to\infty} S^n(v_0) = v$.

Proof

Using Theorem (1) with $S = T$ completes the prove.

Corollary 2

Let (X, d) *be metric space and* A, B *two compact subsets of which* $X = A \cup B$. *If* $S: X \to X$ *is continuous* (A, B) *generalized cyclic* ψ-ϕ-Λ weak *nonexpansive mapping on* X, *then there is only one point* $v \in X$ *such that* $S(v) = v \in A \cap B$. *Moreover; for any* $v_0 \in X$, *we have* $\lim_{n\to\infty} S^n(v_0) = v$.

Proof

Using Theorem (1) with $S = T$ and taking $F(t, s) = \psi(t)$-$\phi(s)$ complete the prove.

Corollary 3

Let (X, d) *be metric space and* A, B *two compact subsets of which* $X = A \cup B$. *If* $S: X \to X$ *is continuous* (A, B) *generalized cyclic* ϕ-Λ weak *nonexpansive mapping on* X, *then there is only one point* $v \in X$ *such that* $S(v) = v \in A \cap B$. *Moreover; for any* $v_0 \in X$, *we have* $\lim_{n\to\infty} S^n(v_0) = v$.

Proof

Using Theorem (1) with $S = T$, taking $F(t, s) = \psi(t)$-$\phi(s)$ and $\psi(t) = t \forall t \in [0,\infty]$ complete the prove.

Conclusion

This paper shows that if S and T are two joint *generalized* cyclic F-ψ-ϕ-Λ weak nonexpansive type mappings, then they have only one common fixed point. In particular, every *generalized* cyclic C class ψ-ϕ-Λ weak nonexpansive mapping has a unique fixed point. Hence continuing restrictions of F, ψ and ϕ to be taken special cases gives extensions of many fixed point in the filed of fixed point theory. In particular, it extends the results of attached references in this study.

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Competing Interests

The author has no conflict of interests.

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