Original Research Paper

Double Weighted Integrals Identities of Montgomery for Differentiable Function of Higher Order

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Abstract: We provide double weighted integrals identities of Montgomery for differentiable function of higher order for two variables and by help of those identities we get generalization of Ostrowski and Grüss type inequalities for weighted integrals for differentiable functions of higher order for two variables.

Keywords: Montgomery's Identities, Ostrowski Type Inequalities, Grüss Type Inequalities

Introduction and Preliminaries

The topic of Montgomery's identities have many applications and cover other important known identities and inequalities in that involve Ostrowski and Gru¨ss type inequalities, also see in present paper. There are many applications of Ostrowski inequalities in field of numerical integrations and probability theory (Dragomir *et al*., 2002; 2008; Liu, 2012). We can also get especial means using such inequalities (Alomari and Darus, 2010a; 2010b). The especial case of Ostrowski type inequalities is Čebyšev inequality which is very popular (Pečarić, 1984; 1987). Also there are many applications of Grüss type inequalities in the numerical integrations and other different fields (Buse *et al*., 2006; Cerone and Dragomir, 2007; Dragomir, 2012).

J is an interval in $\mathbb{R}$, throughout the article. Also throughout the article we would use the following notations for PDF = probability density function, $\mathbb{R} = (-\infty,\infty)$ and $\mathbb{R}_* = [0,\infty)$. The following identity of Montgomery is extracted from (Mitrinović *et al*., 1991; Pečarić and Vukelić, 2007).

Proposition 1.1.

Let f be a function and provided that f' is continuous in the interval $[a,b] \subset \mathbb{R}$. Then:

$$(b-a)f(y) = \int_a^b f(p)\,dp + \int_a^b r(y,p)f'(p)\,dp, \qquad (1.1)$$

holds for Peano kernel r, stated as:

$$r(y,p) = \begin{cases} p-a, & a \le p \le y, \\ p-b, & y < p \le b. \end{cases}$$

We also present here the generalization of above Equation (1.1) which is collected from (Pečarić, 1980).

Proposition 1.2

Let f be a function and provided that f' is continuous in the interval $[a,b]$. Then:

$$f(y) = \int_a^b \upsilon(p)f(p)\,dp + \int_a^b r_\upsilon(y,p)f'(p)\,dp,$$

holds for weighted Peano kernel r_υ, defined as:

$$r_\upsilon(y,p) = \begin{cases} V(p), & a \le p \le y, \\ V(p)-1, & y < p \le b, \end{cases}$$

where, $\upsilon : [a,b] \to \mathbb{R}_*$ is some PDF, i.e., it is *a* function that satisfies $\int_a^b \upsilon(p)\,dp = 1$ and:

$$V(p) = \begin{cases} 0, & p < a, \\ \int_a^p \upsilon(v)\,dv, & p \in [a,b], \\ 1, & p > b. \end{cases}$$

The following generalized identities are obtained from (Barnett and Dragomir, 2001) and (Dragomir *et al*., 2003) for functions of two independent variables.

Proposition 1.3

Let f be a function and provided that $f_{(1,1)}$ is continuous in the interval $[a,b]\times[c,d]$. Then:

$$\begin{aligned}&(d-c)(b-a)f(y,z)=-\int_a^b\int_c^d f(p,s)\,ds\,dp+(d-c)\\&\int_a^b f(p,z)\,dp\\&+(b-a)\int_c^d f(y,s)\,ds+\int_a^b\int_c^d r(y,p)q(z,s)f_{(1,1)}(p,s)\,ds\,dp,\\ \text{and}\quad&(d-c)(b-a)f(y,z)=\int_a^b\int_c^d f(p,s)\,ds\,dp\\&+\int_a^b\int_c^d r(y,p)f_{(1,0)}(p,s)\,ds\,dp\\&+\int_a^b\int_c^d q(z,s)f_{(0,1)}(p,s)\,ds\,dp+\int_a^b\int_c^d r(y,p)q(z,s)\\&f_{(1,1)}(p,s)\,ds\,dp,\end{aligned}$$

hold, here r and q are Peano kernals as defined above.

Pečarić and Vukelić (2007) authors gave the identities of weighted Montgomery for two variables functions.

Proposition 1.4

Let r: $[a,b]\times[c,d]\to\mathbb{R}$ be integrable function and Q is stated as:

$$Q(y,z)=\int_y^b\int_z^d r(v,\varsigma)\,d\varsigma\,dv. \tag{1.2}$$

If f be a function and provided that $f_{(1,1)}$ is continuous in the interval $[a,b]\times[c,d]$. Then:

$$\begin{aligned}&Q(a,c)f(y,z)=\int_a^b\int_c^d r(p,s)f(p,s)\,ds\,dp\\&+\int_a^b\hat{Q}(y,p)f_{(1,0)}(p,z)\,dp\\&+\int_c^d\tilde{Q}(z,s)f_{(0,1)}(y,s)\,ds\\&-\int_a^b\int_c^d\bar{Q}(y,p,z,s)f_{(1,1)}(p,s)\,ds\,dp,\end{aligned}\tag{1.3}$$

holds:

$$\text{where }\hat{Q}(y,p)=\begin{cases}\int_a^p\int_c^d r(v,\zeta)\,d\zeta\,dv, & a\le p\le y,\\ -Q(p,c), & y<p\le b,\end{cases}$$

$$\tilde{Q}(z,s)=\begin{cases}\int_a^b\int_c^s r(v,\zeta)\,d\zeta\,dv, & c\le s\le z,\\ -Q(a,s), & z<s\le d,\end{cases}$$

$$\text{and }\bar{Q}(y,p,z,s)=\begin{cases}\int_a^p\int_c^s r(v,\zeta)\,d\zeta\,dv, & a\le p\le y,\ c\le s\le z,\\ -\int_p^b\int_c^s r(v,\zeta)\,d\zeta\,dv, & y<p\le b,\ c\le s\le z,\\ -\int_a^p\int_s^d r(v,\zeta)\,d\zeta\,dv, & a\le p\le y,\ z<s\le d,\\ Q(p,s), & y<p\le b,\ z<s\le d,\end{cases}$$

Proposition 1.5

Let the suppositions of Proposition 1.4 be true, then identity:

$$\begin{aligned}&Q(a,c)f(y,z)=-\int_a^b\int_c^d r(p,s)f(p,s)\,ds\,dp\\&+\int_a^b\int_c^d r(p,s)f(p,z)\,ds\,dp\\&+\int_a^b\int_c^d r(p,s)f(y,s)\,ds\,dp\\&+\int_a^b\int_c^d\bar{Q}(y,p,z,s)f_{(1,1)}(p,s)\,ds\,dp,\end{aligned}\tag{1.4}$$

holds, where $\bar{Q}$ is as defined in Proposition 1.4.

Proposition 1.6

Let the suppositions of Proposition 1.4 be true, then:

$$\begin{aligned}&[Q(a,c)]^2f(y,z)=Q(a,c)\int_a^b\int_c^d r(p,s)f(p,s)\,ds\,dp\\&+\int_a^b\left(\int_a^b\int_c^d r(v,s)\hat{Q}(y,p)f_{(1,0)}(p,s)\,ds\,dp\right)dv\\&+\int_c^d\left(\int_a^b\int_c^d r(p,\zeta)\tilde{Q}(z,s)f_{(0,1)}(p,s)\,ds\,dp\right)d\zeta\\&+\int_a^b\int_c^d\breve{Q}(y,p,z,s)f_{(1,1)}(p,s)\,ds\,dp,\end{aligned}$$

holds, where $\hat{Q}$, $\tilde{Q}$ and $\bar{Q}$ are defined in Proposition 1.4 and:

$$\breve{Q}(y,p,z,s)=2\hat{Q}(y,p)\tilde{Q}(z,s)-Q(a,c)\bar{Q}(y,p,z,s).$$

The pattern of present paper is divided into four parts. The first part consists on introduction and preliminaries. In the second part, we would give generalized identities of Montgomery for higher order differentiable function of two independent variables. In the 3rd and 4th parts respectively we also get the generalization of Ostrowski and Grüss type inequalities for differentiable functions of higher order for two independent variables by applying identities which proved in 2nd part. Those identities and inequalities give the generalization of many important results (Barnett and Dragomir, 2001; Dragomir *et al*., 2003; 2000; Guezane-Lakoud and Aissaoui, 2011; Ostrowski, 1938; Pečarić and Vukelić, 2007).

Weighted Integrals Identities of Montgomery for Differentiable Function of Higher Order for Two Independent Variables

In starting of this part, we would like to state some notations for simplification of the lengthy expressions as:

$$Q^{(j,l)}_{\binom{a}{c}\Rightarrow\binom{b}{d}}(y,z)=\int_a^b\int_c^d r(v,\zeta)\frac{(y-v)^j}{j!}\frac{(z-\eta)^l}{l!}\,d\zeta\,dv, \tag{2.1}$$

$$Q^{(0,l)}_{\binom{a}{c}\Rightarrow\binom{b}{d}}(z) = \int_a^b \int_c^d r(v,\xi)\frac{(z-\zeta)^l}{l!}d\zeta dv, \tag{2.2}$$

$$Q^{(j,0)}_{\binom{a}{c}\Rightarrow\binom{b}{d}}(y) = \int_a^b \int_c^d r(v,\zeta)\frac{(y-v)^j}{j!}d\zeta dv, \tag{2.3}$$

$$\begin{aligned} R(f:y,z) = &-\sum_{j=1}^{M}\sum_{l=1}^{N}(-1)^{j+1} f_{(j,l)}(y,z)Q^{(j,l)}_{\binom{a}{c}\Rightarrow\binom{b}{d}}(y,z) \\ &-\sum_{l=1}^{N}(-1)^{l} f_{(0,l)}(y,z)Q^{(0,l)}_{\binom{a}{c}\Rightarrow\binom{b}{d}}(z) \\ &-\sum_{j=1}^{M}(-1)^{j} f_{(j,0)}(y,z)Q^{(j,0)}_{\binom{a}{c}\Rightarrow\binom{b}{d}}(y). \end{aligned} \tag{2.4}$$

For our next main theorem of this recent part, we use a lemma from (Adnan *et al*., 2017) by using new notations as follows.

Lemma 2.1

Let f has continuous partial derivatives $f_{(j,l)}$ and r,f: $[a,b]\times[c,d] \to \mathbb{R}$ be both integrable functions, where $j \in \{0,1,\ldots,M,M+1\}$, $l \in \{0,1,\ldots,N,N+1\}$, then:

$$\begin{aligned} &\int_a^b \int_c^d r(y,z)f(y,z)dzdy \\ &= \sum_{j=0}^{M}\sum_{l=0}^{N} Q^{(j,l)}_{\binom{a}{c}\Rightarrow\binom{b}{d}}(b,d)(-1)^{j+1} f_{(j,l)}(b,d) \\ &+\sum_{l=0}^{N}\int_a^b Q^{(M,l)}_{\binom{a}{c}\Rightarrow\binom{s}{d}}(s,d)(-1)^{M+1+l} f_{(M+1,l)}(s,d)ds \\ &+\sum_{j=0}^{M}\int_c^d Q^{(j,N)}_{\binom{a}{c}\Rightarrow\binom{b}{t}}(b,t)(-1)^{j+N+1} f_{(j,N+1)}(b,t)dt \\ &+\int_a^b\int_c^d Q^{(M,N)}_{\binom{a}{c}\Rightarrow\binom{s}{t}}(s,t)(-1)^{M+N} f_{(M+1,N+1)}(s,t)dtds. \end{aligned} \tag{2.5}$$

Now, at this stage we are ready to give our important new theorems as follows.

Theorem 2.2

Let f has continuous partial derivatives $f_{(j,l)}$ and r,f: $[a,b]\times[c,d] \to \mathbb{R}$ be both integrable functions, where $j \in \{0,1,\cdots,M,M+1\}$, $l \in \{0,1,\ldots,N,N+1\}$, then:

$$\begin{aligned} f(y,z)Q(b,d) = R(f:y,z) &+ \int_a^b\int_c^d r(s,t)f(s,t)dtds \\ &+\sum_{l=0}^{N}\int_a^b(-1)^{M+1+l}\hat{Q}^{(M,l)}(y,s,z)f_{(M+1,l)}(s,z)ds \\ &+\sum_{j=0}^{M}\int_c^d(-1)^{j+N+1}\tilde{Q}^{(j,N)}(y,z,t)f_{(j,N+1)}(y,t)dt \\ &-\int_a^b\int_c^d(-1)^{M+N}\bar{Q}^{(M,N)}(y,s,z,t)f_{(M+1,N+1)}(s,t)dtds, \end{aligned} \tag{2.6}$$

where:

$$\hat{Q}^{(M,l)}(y,s,z) = \begin{cases} -Q^{(M,l)}_{\binom{a}{c}\Rightarrow\binom{s}{d}}(s,z), & a \le s \le y, \\ Q^{(M,l)}_{\binom{s}{c}\Rightarrow\binom{b}{d}}(s,z), & y < s \le b, \end{cases}$$

$$\tilde{Q}^{(j,N)}(y,z,t) = \begin{cases} -Q^{(j,N)}_{\binom{a}{c}\Rightarrow\binom{b}{t}}(y,t), & c \le t \le z, \\ Q^{(j,N)}_{\binom{a}{t}\Rightarrow\binom{b}{d}}(y,t), & z < t \le d, \end{cases}$$

and:

$$\bar{Q}^{(M,N)}(y,s,z,t) = \begin{cases} Q^{(M,N)}_{\binom{a}{c}\Rightarrow\binom{s}{t}}(s,t), & a \le s \le y,\ c \le t \le z, \\ Q^{(M,N)}_{\binom{b}{c}\Rightarrow\binom{s}{t}}(s,t), & y < s \le b,\ c \le t \le z, \\ Q^{(M,N)}_{\binom{a}{d}\Rightarrow\binom{s}{t}}(s,t), & a \le s \le y,\ z < t \le d, \\ Q^{(M,N)}_{\binom{b}{d}\Rightarrow\binom{s}{t}}(s,t), & y < s \le b,\ z < t \le d, \end{cases}$$

where, $Q^{(M,N)}_{\binom{\cdot}{\cdot}\Rightarrow\binom{\cdot}{\cdot}}(s,t)$, $Q^{(M,l)}_{\binom{\cdot}{\cdot}\Rightarrow\binom{\cdot}{\cdot}}(s,z)$, $Q^{(j,N)}_{\binom{\cdot}{\cdot}\Rightarrow\binom{\cdot}{\cdot}}(y,t)$, R (f: y, z) and R(f: y,z) are defined in (2.1), (2.2), (2.3) and (2.4) respectively.

Proof

Using Lemma 2.1 for [a, y]×[c, z] we get:

$$\begin{aligned} &\int_a^y\int_c^z r(s,t)f(s,t)dtds \\ &= \sum_{j=0}^{M}\sum_{l=0}^{N} Q^{(j,l)}_{\binom{a}{c}\Rightarrow\binom{y}{z}}(y,z)(-1)^{j+l} f_{(j,l)}(y,z) \\ &+\sum_{l=0}^{N}\int_a^y Q^{(M,l)}_{\binom{a}{c}\Rightarrow\binom{s}{z}}(s,z)(-1)^{M+1+l} f_{(M+1,l)}(s,z)ds \\ &+\sum_{j=0}^{M}\int_c^z Q^{(j,N)}_{\binom{a}{c}\Rightarrow\binom{y}{t}}(y,t)(-1)^{j+N+l} f_{(j,N+l)}(y,t)dt \\ &+\int_a^y\int_c^z Q^{(M,N)}_{\binom{a}{c}\Rightarrow\binom{s}{t}}(s,t)(-1)^{N+M} f_{(M+1,N+1)}(s,t)dtds \\ &= \sum_{j=0}^{M}\sum_{l=0}^{N}(-1)^{j+l} f_{(j,l)}(y,z)\Big[Q^{(j,l)}_{\binom{b}{d}\Rightarrow\binom{a}{c}}(y,z) - Q^{(j,l)}_{\binom{b}{d}\Rightarrow\binom{a}{z}}(y,z) \\ &-Q^{(j,l)}_{\binom{b}{d}\Rightarrow\binom{y}{c}}(y,z)+Q^{(j,l)}_{\binom{b}{d}\Rightarrow\binom{y}{z}}(y,z)\Big] + \sum_{l=0}^{N}\int_a^y(-1)^{M+1+l} f_{(M+1,l)}(s,z) \\ &\Big[Q^{(M,l)}_{\binom{b}{d}\Rightarrow\binom{a}{c}}(s,z) - Q^{(M,l)}_{\binom{b}{d}\Rightarrow\binom{a}{z}}(s,z) - Q^{(M,l)}_{\binom{b}{d}\Rightarrow\binom{s}{c}}(s,z) + Q^{(M,l)}_{\binom{b}{d}\Rightarrow\binom{s}{z}}(s,z)\Big]ds \\ &+\sum_{j=0}^{M}\int_c^z(-1)^{j+N+1} f_{(j,N+1)}(y,t)\times\Big[Q^{(j,N)}_{\binom{b}{d}\Rightarrow\binom{a}{c}}(y,t) - Q^{(j,N)}_{\binom{b}{d}\Rightarrow\binom{a}{t}}(y,t) \\ &-Q^{(j,N)}_{\binom{b}{d}\Rightarrow\binom{y}{c}}(y,t) + Q^{(j,N)}_{\binom{b}{d}\Rightarrow\binom{y}{t}}(y,t)\Big]dt + \int_a^y\int_c^z(-1)^{N+M} f_{(M+1,N+1)}(s,t) \\ &\Big[Q^{(M,N)}_{\binom{b}{d}\Rightarrow\binom{a}{c}}(s,t) - Q^{(M,N)}_{\binom{b}{d}\Rightarrow\binom{a}{t}}(s,t) - Q^{(M,N)}_{\binom{b}{d}\Rightarrow\binom{s}{c}}(s,t) + Q^{(M,N)}_{\binom{b}{d}\Rightarrow\binom{s}{t}}(s,t)\Big]dtds. \end{aligned}$$

Similarly for $[y,b]\times[c,z]$ we get:

$$\begin{aligned} &\int_y^b\int_c^z r(s,t)f(s,t)dtds = -\int_b^y\int_c^z r(s,t)f(s,t)dtds \\ &= -\sum_{j=0}^{M}\sum_{l=0}^{N}(-1)^{j+l} f_{(j,l)}(y,z)\Big[-Q^{(j,l)}_{\binom{b}{d}\Rightarrow\binom{y}{c}}(y,z) + Q^{(j,l)}_{\binom{b}{d}\Rightarrow\binom{y}{z}}(y,z)\Big] \end{aligned}$$

$$+\sum_{l=0}^{N}\int_y^b(-1)^{M+1+l}f_{(M+1,l)}(s,z)\left[-Q^{(M,l)}_{\binom{b}{d}\Rightarrow\binom{s}{c}}(s,z)+Q^{(M,l)}_{\binom{b}{d}\Rightarrow\binom{s}{z}}(s,z)\right]ds$$
$$-\sum_{j=0}^{M}\int_c^z(-1)^{j+N+1}f_{(j,N+1)}(y,t)\left[-Q^{(j,N)}_{\binom{b}{d}\Rightarrow\binom{y}{c}}(y,t)+Q^{(j,N)}_{\binom{b}{d}\Rightarrow\binom{y}{t}}(y,t)\right]dt$$
$$+\int_y^b\int_c^z(-1)^{N+M}f_{(M+1,N+1)}(s,t)\left[-Q^{(M,N)}_{\binom{b}{d}\Rightarrow\binom{s}{c}}(s,t)+Q^{(M,N)}_{\binom{b}{d}\Rightarrow\binom{s}{t}}(s,t)\right]dtds.$$

For $[a,y]\times[z,d]$:

$$\int_a^y\int_z^d r(s,t)f(s,t)dtds=-\int_a^y\int_d^z r(s,t)f(s,t)dtds$$
$$=-\sum_{j=0}^{M}\sum_{l=0}^{N}(-1)^{j+l}f_{(j,l)}(y,z)\left[-Q^{(j,l)}_{\binom{b}{d}\Rightarrow\binom{a}{z}}(y,z)+Q^{(j,l)}_{\binom{b}{d}\Rightarrow\binom{y}{z}}(y,z)\right]$$
$$-\sum_{l=0}^{N}\int_a^y(-1)^{M+1+l}f_{(M+1,l)}(s,z)\left[-Q^{(M,l)}_{\binom{b}{d}\Rightarrow\binom{a}{z}}(s,z)+Q^{(M,l)}_{\binom{b}{d}\Rightarrow\binom{s}{z}}(s,z)\right]ds$$
$$+\sum_{j=0}^{M}\int_z^d(-1)^{j+N+1}f_{(j,N+1)}(y,t)\left[-Q^{(j,N)}_{\binom{b}{d}\Rightarrow\binom{a}{t}}(\mathrm{y},t)+Q^{(j,N)}_{\binom{b}{d}\Rightarrow\binom{y}{t}}(\mathrm{y},\mathrm{t})\right]dt$$
$$+\int_a^y\int_z^d(-1)^{M+N}f_{(M+1,N+1)}(s,t)\left[-Q^{(M,N)}_{\binom{b}{d}\Rightarrow\binom{a}{t}}(s,t)+Q^{(M,N)}_{\binom{b}{d}\Rightarrow\binom{s}{t}}(s,t)\right]dtds.$$

At last for $[y,b]\times[z,d]$:

$$\int_y^b\int_z^d r(s,t)f(s,t)dtds=\int_b^y\int_d^z r(s,t)f(s,t)dtds$$
$$=\sum_{j=0}^{M}\sum_{l=0}^{N}(-1)^{j+l}f_{(j,l)}(y,z)Q^{(j,l)}_{\binom{b}{d}\Rightarrow\binom{y}{z}}(y,z)$$
$$-\sum_{l=0}^{N}\int_y^b(-1)^{M+1+l}f_{(M+1,l)}(s,z)+Q^{(M,l)}_{\binom{b}{d}\Rightarrow\binom{s}{z}}(s,z)ds$$
$$-\sum_{j=0}^{M}\int_z^d(-1)^{j+N+1}f_{(j,N+1)}(y,t)Q^{(j,N)}_{\binom{b}{d}\Rightarrow\binom{y}{t}}(\mathrm{y},t)dt$$
$$+\int_y^b\int_c^z(-1)^{M+N}f_{(M+1,N+1)}(s,t)Q^{(M,N)}_{\binom{b}{d}\Rightarrow\binom{s}{t}}(s,t)dtds.$$

Adding the four expressions we get desired result.

Theorem 2.3

Let f has continuous partial derivatives $f_{(M+1,N+1)}$ and f, r : $[a,b]\times[c,d]\rightarrow\mathbb{R}$ be both integrable functions. Then following identity:

$$f(y,z)Q(b,d)=R(f:y,z)-\int_a^b\int_c^d r(s,t)f(s,t)dtds$$
$$+\int_a^b\int_c^d r(s,t)f(s,z)dtds+\int_a^b\int_c^d r(s,t)f(y,t)dtds$$
$$+\sum_{l=1}^{N}\int_a^b\int_c^d(-1)^l r(s,\zeta)\frac{(\zeta-z)^l}{l!}f_{(0,l)}(s,z)d\varsigma ds \tag{2.7}$$
$$+\sum_{j=1}^{M}\int_a^b\int_c^d(-1)^j r(v,t)\frac{(v-y)^j}{j!}f_{(j,0)}(y,t)dtdv$$
$$+\int_a^b\int_c^d(-1)^{N+M}\bar{Q}^{(M,N)}(y,s,z,t)f_{(M+1,N+1)}(s,t)dtds,$$

holds, where $\bar{Q}^{(M,N)}(y,s,z,t)$ as in Theorem 2.2 and $Q(b,d)$ and $R(f{:}y,z)$ are as defined in (1.2) and (2.3) respectively.

Proof

Firstly, finding expression for the following:

$$\int_a^b(-1)^{M+1+l}\hat{Q}^{(M,l)}(y,s,z)f_{(M+1,l)}(s,z)ds$$

using integration by parts as:

$$\int_c^d(-1)^{M+1+l}\hat{Q}^{(M,l)}(y,s,z)f_{(M+1,l)}(s,z)ds$$
$$=-\int_a^y(-1)^{M+1+l}Q^{(M,l)}_{\binom{a}{c}\Rightarrow\binom{s}{d}}(s,z)f_{(M+1,l)}(s,z)ds$$
$$+\int_y^b(-1)^{M+1+l}Q^{(M,l)}_{\binom{s}{c}\Rightarrow\binom{b}{d}}(s,z)f_{(M+1,l)}(s,z)ds$$
$$=-\Bigg(\int_a^y(-1)^{M+1+l}Q^{(M,l)}_{\binom{a}{c}\Rightarrow\binom{s}{d}}(s,z)f_{(M+1,l)}(s,z)ds$$
$$+\int_y^b(-1)^{M+1+l}Q^{(M,l)}_{\binom{b}{c}\Rightarrow\binom{s}{d}}(s,z)f_{(M+1,l)}(s,z)ds\Bigg)$$
$$=Q^{(M,l)}_{\binom{a}{c}\Rightarrow\binom{y}{d}}(y,z)(-1)^{M+l}f_{(M,l)}(y,z)$$
$$+\int_a^y Q^{(M-1,l)}_{\binom{a}{c}\Rightarrow\binom{s}{d}}(s,z)(-1)^{M+l}f_{(M,l)}(s,z)ds$$
$$+Q^{(M,l)}_{\binom{y}{c}\Rightarrow\binom{b}{d}}(y,z)(-1)^{M+l}f_{(M,l)}(y,z)$$
$$+\int_y^b Q^{(M-1,l)}_{\binom{b}{c}\Rightarrow\binom{s}{d}}(s,z)(-1)^{M+l}f_{(M,l)}(s,z)ds$$
$$=Q^{(M,l)}_{\binom{a}{c}\Rightarrow\binom{b}{d}}(y,z)(-1)^{M+l}f_{(M,l)}(y,z)$$
$$+\int_a^y Q^{(M-1,l)}_{\binom{a}{c}\Rightarrow\binom{s}{d}}(s,z)(-1)^{M+l}f_{(M,l)}(s,z)ds$$
$$+\int_y^b Q^{(M-1,l)}_{\binom{b}{c}\Rightarrow\binom{s}{d}}(s,z)(-1)^{M+l}f_{(M,l)}(s,z)ds$$
$$=Q^{(M,l)}_{\binom{a}{c}\Rightarrow\binom{b}{d}}(y,z)(-1)^{M+l}f_{(M,l)}(y,z)$$
$$+\int_a^b(-1)^{M+1}\hat{Q}^{(M-1,l)}(y,s,z)f_{(M,l)}(s,z)ds,$$

continuing in this way we finally get:

$$\int_a^b(-1)^{M+1+l}\hat{Q}^{(M,l)}(y,s,z)f_{(M+1,l)}(s,z)ds$$
$$=\int_a^b\int_c^d r(v,\zeta)\frac{(\zeta-z)^l}{l!}\left[\sum_{i=0}^{M}\frac{(v-y)^i}{i!}(-1)^{i+1}f_{(i,l)}(y,z)\right]d\zeta dv \tag{2.8}$$
$$-\int_a^b\int_c^d r(s,\zeta)\frac{(\zeta-z)^l}{l!}(-1)^l f_{(0,l)}(s,z)d\zeta ds.$$

Similarly:

$$\int_c^d(-1)^{j+N+1}\tilde{Q}^{(j,N)}(y,z,t)f_{(j+N+1)}(y,t)dt$$
$$=\int_a^b\int_c^d r(v,\zeta)\frac{(v-y)^j}{j!}\left[\sum_{k=0}^{N}\frac{(\zeta-z)^k}{k!}(-1)^{j+k}f_{(j,k)}(y,z)\right]d\zeta dv \tag{2.9}$$
$$-\int_a^b\int_c^d r(v,t)\frac{(v-y)^j}{j!}(-1)^j f_{(j,0)}(y,t)dtdv.$$

We get desired identity by putting all above values in (2.6) then some cancelation and rearrangements.

Theorem 2.4

Let $f{:}[a,b]\times[c,d]\rightarrow\mathbb{R}$ be function and $f\in C^{(2N+1,2M+1)}$ and $\forall\ (y,z)\in[a,b]\times[c,d]$ we have:

$$f(y,z)Q(b,d)^2 = Q(b,d)R(f:y,z)+Q(b,d)$$
$$\int_a^b\int_c^d r(s,t)f(s,t)dtds+\sum_{l=0}^{N}\int_a^b(-1)^{M+1+l}\hat{Q}^{(M,l)}(y,s,z)$$
$$R\left(f_{(M+1,l)}:s,z\right)ds+\sum_{j=0}^{M}\int_c^d(-1)^{j+N+1}\tilde{Q}^{(j,N)}(y,z,t)$$
$$R\left(f_{(j+N+1)}:y,t\right)dt$$
$$+\sum_{j=0}^{M}\sum_{l=0}^{N}\int_a^b\int_a^b\int_c^d(-1)^{M+1+j+l}\hat{Q}^{(M,l)}(y,s,z)$$
$$r(v,t)\frac{(v-y)^j}{j!}f_{(M+1+j,l)}(s,t)dtdsdv$$
$$+\sum_{j=0}^{M}\sum_{l=0}^{N}\int_c^d\int_a^b\int_c^d(-1)^{j+l+N+1}\tilde{Q}^{(j,N)}(y,z,t)$$
$$r(s,\zeta)\frac{(\zeta-z)^l}{l!}f_{(j,N+1+l)}(s,t)dtdsd\zeta$$
$$+2\int_a^b\int_c^d\sum_{j=0}^{M}\sum_{l=0}^{N}(-1)^{j+l+M+N}\hat{Q}^{(M,l)}(y,s,z)$$
$$\tilde{Q}^{(j,N)}(y,z,t)f_{(M+1+j,N+1+l)}(s,t)dtds,$$
$$-Q(b,d)\int_a^b\int_c^d(-1)^{M+N}\bar{Q}^{(M,N)}(y,s,z,t)f_{(M+1,N+1)}(s,t)dtds, \quad (2.10)$$

where, $\hat{Q}^{(M,l)}(y,s,z)\,\tilde{Q}^{(N,j)}(y,z,t)\,\bar{Q}^{(M,N)}(y,s,z,t)$, Q and r are as in Theorem 2.2.

Proof

By Sums (2.8) for $l = 0,\ldots,N$ and (2.9) for $j = 0,\ldots,M$ we obtain respectively:

$$f(y,z)Q(b,d) = R(f:y,z)$$
$$+\sum_{l=0}^{N}\int_a^b\int_c^d(-1)^l r(s,\zeta)\frac{(\zeta-z)^l}{l!}f_{(0,l)}(s,z)d\zeta ds \quad (2.11)$$
$$+\sum_{l=0}^{M}\int_a^b(-1)^{M+1+l}\hat{Q}^{(M,l)}(y,s,z)f_{(M+1,l)}(s,z)ds,$$

and:

$$f(y,z)Q(b,d) = R(f:y,z)$$
$$+\sum_{j=0}^{M}\int_a^b\int_c^d(-1)^j r(v,t)\frac{(v-y)^j}{j!}f_{(j,0)}(y,t)dtdv \quad (2.12)$$
$$+\sum_{j=0}^{M}\int_c^d(-1)^{j+N+1}\tilde{Q}^{(j,N)}(y,z,t)f_{(j,N+1)}(y,t)dt,$$

$\forall$ $(y,z) \in [a,b]\times[c,d]$.

Formula (2.11) used for partial derivatives $f_{(j,N+1)}$ for $j = 0,1,\ldots,M$, then:

$$f_{(j,N+1)}(y,t)Q(b,d) = R\left(f_{(j,N+1)}:y,t\right)$$
$$+\sum_{l=0}^{N}\int_a^b\int_c^d(-1)^l r(s,\zeta)\frac{(\zeta-t)^l}{l!}f_{(j,N+1+l)}(s,t)d\varsigma ds \quad (2.13)$$
$$+\sum_{l=0}^{N}\int_a^b(-1)^{M+1+l}\hat{Q}^{(M,l)}(y,s,t)f_{(M+1+j,N+1+l)}(s,t)ds.$$

Formula (2.12) used for partial derivatives $f_{(M+1,l)}$ for $l = 0,1,\ldots,N$, then:

$$f_{(M+1,l)}(s,z)Q(b,d) = R\left(f_{(M+1,l)}:s,z\right)$$
$$+\sum_{j=0}^{M}\int_a^b\int_c^d(-1)^j r(v,t)\frac{(v-s)^j}{j!}f_{(M+1+j,l)}(s,t)dtdv \quad (2.14)$$
$$+\sum_{j=0}^{M}\int_c^d(-1)^{j+N+1}\tilde{Q}^{(j,N)}(s,z,t)f_{(M+1+j,N+1+l)}(s,t)dt.$$

Substituting (2.13) and (2.14) into (2.6) we get:

$$f(y,z)Q(b,d) = R(f:y,z)+\int_a^b\int_c^d r(s,t)f(s,t)dtds$$
$$+\frac{1}{Q(b,d)}\sum_{l=0}^{N}\int_a^b(-1)^{M+1+l}\hat{Q}^{(M,l)}(y,s,z)\Big[R\left(f_{(M+1,l)}:s,z\right)$$
$$+\sum_{j=0}^{M}\int_a^b\int_c^d(-1)^j r(v,t)\frac{(v-s)^j}{j!}f_{(M+1+j,l)}(s,t)dtdv$$
$$+\sum_{j=0}^{M}\int_c^d(-1)^{j+N+1}\tilde{Q}^{(j,N)}(s,z,t)f_{(M+1+j,N+1+l)}(s,t)dt\Big]ds$$
$$+\frac{1}{Q(b,d)}\sum_{j-0}^{M}\int_c^d(-1)^{j+N+1}\tilde{Q}^{(j,N)}(y,z,t)\Big[R\left(f_{(j,N+1)}:y,t\right)$$
$$+\sum_{l=0}^{N}\int_a^b\int_c^d(-1)^l r(s,\zeta)\frac{(\zeta-t)^l}{l!}f_{(j,N+1+l)}(s,t)d\zeta ds$$
$$+\sum_{l=0}^{N}\int_a^b(-1)^{M+1+l}\hat{Q}^{(M,l)}(y,s,t)f_{(M+1+j,N+1+l)}(s,t)ds\Big]dt$$
$$-\int_a^b\int_c^d(-1)^{M+N}\bar{Q}^{(M,N)}(y,s,z,t)f_{(M+1,N+1)}(s,t)dtds.$$

We get desired result, after some rearrangements and using Theorem of Fubini.

Remark 2.5

For $M = N = 0$, we can get especial cases of Theorem 2.2, 2.3 and 2.4 as similar as Proposition 1.4, 1.5 and 1.6 respectively (also see similar case in (Pečarić and Vukelić, 2007).

2.6. Especial Cases

If $r(s,t) = q(s)p(t)$ in identities (2.6), (2.7) and (2.10) then we obtain the following especial cases respectively:

$$f(y,z)Q_{a\to b}(q)Q_{c\to d}(p) = S(f:y,z)$$
$$+\int_a^b\int_c^d p(t)q(s)f(s,t)dtds$$
$$+\sum_{l=0}^{N}\int_a^b(-1)^{M+1+l}\hat{S}^{(M,l)}(y,s,z)f_{(M+1,l)}(s,z)ds$$
$$+\sum_{j=0}^{M}\int_c^d(-1)^{j+N+1}\tilde{S}^{(j,N)}(y,z,t)f_{(j,N+1)}(y,t)dt$$
$$-\int_a^b\int_c^d\bar{S}^{(M,N)}(y,s,z,t)(-1)^{M+N}f_{(M+1,N+1)}(s,t)dtds,$$

$$
\begin{aligned}
&f(y,z)Q_{a\to b}(q)Q_{c\to d}(p)=S(f:y,z)\\
&-\int_a^b\int_c^d q(s)p(t)f(s,t)dtds\\
&+\int_a^b\int_c^d p(t)q(s)f(s,z)dtds\\
&+\int_a^b\int_c^d p(t)q(s)f(y,t)dtds\\
&+\sum_{l=1}^{N}\int_a^b q(s)(-1)^l f_{(0,l)}(s,z)ds S_{c\to d}^{(l)}(p,z)\\
&+\sum_{j=1}^{M}S_{c\to b}^{(j)}(q,y)\int_c^d p(t)(-1)^j f_{(j,0)}(y,t)dt\\
&+\int_a^b\int_c^d(-1)^{M+N}\,\overline{S}^{(M,N)}(y,s,z,t)f_{(M+1,N+1)}(s,t)dtds,
\end{aligned}
$$

$$
\begin{aligned}
&f(y,z)\left[Q_{a\to b}(q)Q_{c\to d}(p)\right]^2=Q_{a\to b}(q)Q_{c\to d}(p)S(f:y,z)\\
&+Q_{a\to b}(q)Q_{c\to d}(p)\int_a^b\int_c^d q(s)p(t)f(s,t)dtds\\
&+\sum_{l=0}^{N}\int_a^b(-1)^{M+1+l}\,\hat{S}^{(M,l)}(y,s,z)S\left(f_{(M+1,l)}:s,z\right)ds\\
&+\sum_{j=0}^{M}\int_c^d(-1)^{j+N+1}\,\tilde{S}^{(j,N)}(y,z,t)S\left(f_{(j,M+1)}:y,t\right)dt\\
&+\sum_{j=0}^{M}\sum_{l=0}^{N}S_{a\to b}^{(j)}(q,y)\int_a^b\int_c^d(-1)^{M+1+j+l}\,\hat{S}^{(M,l)}(y,s,z)p(t)f_{(M+1+j,l)}(s,t)dtds\\
&+\sum_{j=0}^{M}\sum_{l=0}^{N}S_{c\to d}^{(l)}(p,z)\int_a^b\int_c^d(-1)^{j+N+1+l}\,\tilde{S}^{(j,N)}(y,z,t)q(s)f_{(j,N+1+l)}(s,t)dtds\\
&+2\int_a^b\int_c^d\sum_{j=0}^{M}\sum_{l=0}^{N}(-1)^{j+M+l+N}\,\hat{S}^{(M,l)}(y,s,z)\tilde{S}^{(j,N)}(y,z,t)f_{(M+1+j,N+1+l)}(s,t)dtds\\
&-Q_{a\to b}(q)Q_{c\to d}(p)\int_a^b\int_c^d(-1)^{N+M}\,\overline{S}^{(M,N)}(y,s,z,t)f_{(M+1,N+1)}(s,t)dtds,
\end{aligned}
$$

where:

$$
Q_{a\to b}(q)=\int_a^b q(s)ds,\quad S_{a\to b}^{(j)}(q,y)=\int_a^b q(v)\frac{(y-v)^j}{j!}dv,
$$

$$
S_{\binom{a}{c}\Rightarrow\binom{b}{d}}^{(j,l)}(y,z)=S_{a\to b}^{(j)}(q,y)S_{c\to d}^{(l)}(p,z),
$$

$$
S_{\binom{a}{c}\Rightarrow\binom{b}{d}}^{(0,l)}(z)=Q_{a\to b}(q)S_{c\to d}^{(l)}(p,z),
$$

$$
S_{\binom{a}{c}\Rightarrow\binom{b}{d}}^{(j,0)}(y)=S_{a\to b}^{(j)}(q,y)Q_{c\to d}(p),
$$

$$
\begin{aligned}
S(f:y,z)=&-\sum_{j=1}^{M}\sum_{l=1}^{N}f_{(j,l)}(y,z)S_{\binom{a}{c}\Rightarrow\binom{b}{d}}^{(j,l)}(y,z)\\
&-\sum_{l=1}^{N}f_{(0,l)}(y,z)S_{\binom{a}{c}\Rightarrow\binom{b}{d}}^{(0,l)}(z)-\sum_{j=1}^{M}f_{(j,0)}(y,z)S_{\binom{a}{c}\Rightarrow\binom{b}{d}}^{(j,0)}(y),
\end{aligned}
$$

$$
\hat{S}^{(M,l)}(y,s,z)=\begin{cases}-S_{\binom{a}{c}\Rightarrow\binom{s}{d}}^{(M,l)}(s,z), & a\le s\le y,\\ +S_{\binom{s}{c}\Rightarrow\binom{b}{d}}^{(M,l)}(s,z), & y<s\le b,\end{cases}
$$

$$
\tilde{S}^{(j,N)}(y,z,t)=\begin{cases}-S_{\binom{a}{c}\Rightarrow\binom{b}{t}}^{(j,N)}(y,t), & c\le t\le z,\\ +S_{\binom{a}{t}\Rightarrow\binom{b}{d}}^{(j,0)}(y,t), & z<t\le d,\end{cases}
$$

and:

$$
\overline{S}^{(M,N)}(y,s,z,t)=\begin{cases}S_{\binom{a}{c}\Rightarrow\binom{s}{t}}^{(M,N)}(s,t), & a\le s\le y,\ c\le t\le z,\\ S_{\binom{b}{c}\Rightarrow\binom{s}{t}}^{(M,N)}(s,t), & y<s\le b,\ c\le t\le z,\\ S_{\binom{a}{d}\Rightarrow\binom{s}{t}}^{(M,N)}(s,t), & a\le s\le y,\ z<t\le d,\\ S_{\binom{b}{d}\Rightarrow\binom{s}{t}}^{(M,N)}(s,t), & y<s\le b,\ z<t\le d,\end{cases}
$$

Ostrowski Type Inequalities for Double Weighted Integrals for Higher Order Differentiable Function of Two Variables

Under this heading we would recall well known Ostrowski inequality which is extracted from (Ostrowski, 1938):

$$
\begin{aligned}
&\left|f(y)-\frac{1}{b-a}\int_a^b f(t)dt\right|\\
&\le M\left[\frac{\left(y-\frac{a+b}{2}\right)^2}{(b-a)^2}+\frac{1}{1}\right](b-a);\quad y\in[b,a]
\end{aligned}
\tag{3.1}
$$

where, f is a real valued function in the interval [a, b] and provided that f' is continuous and satisfying $f'(y)| \le M$, $\forall y\in[a, b]$. This inequality has helped to provide many generalizations and in Pečarić and Vukelić (2007) have also provided the generalizations of this inequality using identities (1.3) and (1.4). Now by using identities (2.6) and (2.7), we would get results for generalization of Ostrowski type for differentiable function of higher order for two variables as below.

Theorem 3.1

Let f be a real valued function in the interval $[a,b]\times[c,d]$ and $f\in C^{(M+1,N+1)}$ in the same interval. Then:

$$
\begin{aligned}
&\left|f(y,z)-\frac{1}{Q(b,d)}\int_a^b\int_c^d r(s,t)f(s,t)dtds\right|\le O(y,z)\\
&+\sum_{l=0}^{N}\hat{O}^{(0,l)}(y,z)+\sum_{j=0}^{M}\tilde{O}^{(j,0)}(y,z)+\overline{O}(y,z)
\end{aligned}
$$

holds, $\forall$(y,z) $\in$ [a,b]$\times$ [c,d]
where:

$$
O(y,z)=\frac{1}{|Q(b,d)|}\left|R(f:y,z)\right|,
$$

$$
\hat{O}^{(0,l)}(y,z)=\frac{1}{|Q(b,d)|}\left(\sum_{l-0}^{N}\int_a^b\left|\hat{Q}^{(M,l)}(y,s,z)\right|^{\hat{q}l}ds\right)^{1/\hat{q}l}.\left\|(-1)^{M+1+l}f_{(M+1+l)}\right\|_{\hat{r}j},
$$

$$
provided\ that\ f_{(M+1,l)}\in L_{\hat{r}1}\left([a,b]\times[c,d]\right),\ 1/\tilde{r}_l+1/\tilde{q}_l=1,
$$

$$
\tilde{O}^{(j,0)}(y,z)=\frac{1}{|Q(b,d)|}\left(\sum_{j-0}^{M}\int_c^d\left|\tilde{Q}^{(j,N)}(y,z,t)\right|^{\overline{q}_j}dt\right)^{1/\tilde{q}j}.\left\|(-1)^{j+N+1}f_{(j+N+1)}\right\|\tilde{r}_j,
$$

$$
provided\ that\ f_{(j,N+1)}\in L_{\hat{r}j}\left([a,b]\times[c,d]\right),1/\tilde{r}_j+1/\tilde{q}_j=1,
$$

$$
\overline{O}(y,z)=\frac{1}{|Q(b,d)|}\left(\int_a^b\int_c^d\left|\overline{Q}^{(M,N)}(y,s,z,t)\right|^{\overline{q}}dtds\right)^{1/\overline{q}}.\left\|(-1)^{M+N}f_{(M+1,N+1)}\right\|_{\overline{r}},
$$

$$
provided\ that\ f_{(M+1,N+1)}\in L_{\overline{r}}\left([a,b]\times[c,d]\right),1/\overline{r}+1/\overline{q}=1,
$$

where, $\hat{Q}^{(M,l)}(y,s,z)$, $\tilde{Q}^{(j,N)}(y,z,t)$, $\bar{Q}^{(M,N)}(y,s,z,t)$ are as in Theorem 2.2 where as Q and $R(f: y, z)$ are stated in (1.2) and (2.3) respectively.

Proof

Identity (2.6) may be written as:

$$f(y,z)-\frac{1}{Q(b,d)}\int_a^b\int_c^d r(s,t)f(s,t)dtds$$
$$=\frac{1}{Q(b,d)}\Big[R(f:y,z)+\sum_{l=0}^{N}\int_a^b(-1)^{M+1+l}\hat{Q}^{(M,l)}(y,s,z)f_{(M+1,l)}(s,z)ds$$
$$+\sum_{j=0}^{M}\int_c^d(-1)^{j+N+1}\tilde{Q}^{(j,N)}(y,z,t)f_{(j,N+1)}(y,t)dt$$
$$-\int_a^b\int_c^d(-1)^{M+N}\bar{Q}^{(M,N)}(y,s,z,t)f_{(M+1,N+1)}(s,t)dtds\Big].$$

We can easily get desired result by taking absolute value and by using inequality of Hőlder for double integrals.

Remark 3.2

For $M = N = 0$, we can get especial case of Theorem 3.1 as similar as Theorem 4 of (Pečarić and Vukelić, 2007).

Theorem 3.3

Let f be a real valued continuous function in the interval $[a,b]\times[c,d]$ ∋, $f\in C^{(M+1,N+1)}$ in the same interval and $|f_{(M+1,N+1)}|^r$ be integrable function i.e.,:

$$\|(-1)^{N+M}f_{(M+1,N+1)}\|_r:=\left(\int_a^b\int_c^d\left|(-1)^{N+M}f_{(M+1,N+1)}(s,t)\right|^r dtds\right)^{1/r}<\infty,$$

where, $1/r+1/q = 1$. Then, it follows:

$$\Big|\int_a^b\int_c^d r(s,t)f(s,t)dtds-\Big[R(f:y,z)+\int_a^b\int_c^d r(s,t)f(s,z)dtds$$
$$+\int_a^b\int_c^d r(s,t)f(y,t)dtds$$
$$+\sum_{l=1}^{N}\int_a^b\int_c^d(-1)^l r(s,\zeta)\frac{(\zeta-z)^l}{l!}f_{(0,l)}(s,z)d\zeta ds$$
$$+\sum_{j=1}^{M}\int_a^b\int_c^d(-1)^j r(v,t)\frac{(v-y)^j}{j!}f_{(j,0)}(y,t)dtdv-f(y,z)Q(b,d)\Big]\Big|$$
$$\le\left(\int_a^b\int_c^d|\bar{Q}^{(M,N)}(y,s,z,t)|^r dtds\right)^{1/r}\left\|(-1)^{M+N}f_{(M+1,N+1)}\right\|_q.$$

for all $(y,z)\in[a,b]\times[c,d]$.

Proof

Identity (2.7) can be written like:

$$\int_a^b\int_c^d f(s,t)r(s,t)dtds-\Big[R(f:y,z)+\int_a^b\int_c^d r(s,t)f(s,z)dtds$$
$$+\int_a^b\int_c^d r(s,t)f(y,t)dtds+\sum_{l=1}^{N}\int_a^b\int_c^d(-1)^l r(s,\zeta)\frac{(\zeta-z)^l}{l!}f_{(0,l)}(s,z)d\zeta ds$$
$$+\sum_{j=1}^{M}\int_a^b\int_c^d(-1)^j r(v,t)\frac{(v-y)^j}{j!}f_{(j,0)}(y,t)dtdv-f(y,z)Q(b,d)\Big]$$
$$=\int_a^b\int_c^d(-1)^{M+N}\bar{Q}^{(M,N)}(y,s,z,t)f_{(M+1,N+1)}(s,t)dtds.$$

We can easily get desired result by taking absolute value and by using inequality of Hőlder for double integrals.

Remark 3.4

For $M = N = 0$, we can obtain especial case of Theorem 3.3 as similar as Theorem 5 of (Pečarić and Vukelić, 2007)

Double Weighted Integrals of Grüss type Inequalities for Differentiable Functions of Higher Order with two Variables

Grüss (1935) proved remarkable integral inequality in 1935, which is as follows (see also [Mitrinović *et al*., 1991: 296]):

$$\left|\frac{1}{b-a}\int_a^b f(y)g(y)dy-\left(\frac{1}{b-a}\int_a^b f(y)dy\right)\left(\frac{1}{b-a}\int_a^b g(y)dy\right)\right|$$
$$\le\frac{1}{4}(N-n)(M-m)$$

provided that g and f are both integrable functions in the interval $[a,b]$ and satisfying the following:

$$n\le g(y)\le N,\ m\le f(y)\le M,\ \forall y\in[a,b]$$

where, m,n,M,N real constants.

Pečarić and Vukelić, (2007), using identities (1.3) and (1.4) Peĉarić and Vukelić have given new Grüss type inequalites for double weighted integrals. Now by using differentiable functions of higher order for two variables, we get results for more generalization but for this purpose we use following notations for simplification instead of detailed presentations:

$$B^{(j,l)}(y,z)=r(y,z)\big[f_{(j,l)}(y,z)g(y,z)$$
$$+g_{(j,l)}(y,z)f(y,z)\big]Q^{(j,l)}_{\binom{a}{c}\Rightarrow\binom{b}{d}}(y,z),\tag{4.1}$$

$$B(y,z)=r(y,z)\int_a^b\int_c^d r(s,t)$$
$$\big[f(s,t)g(y,z)+g(s,t)f(y,z)\big]dtds,\tag{4.2}$$

$$\hat{B}^{(M,l)}(y,z)=r(y,z)\int_a^b\big[f_{(M+1,l)}(s,z)g(y,z)$$
$$+g_{(M+1,l)}(s,z)f(y,z)\big]\hat{Q}^{(M,l)}(y,s,z)ds,\tag{4.3}$$

$$\tilde{B}^{(j,N)}(y,z)=r(y,z)\int_c^d\big[g(y,z)f_{(j,N+1)}(y,t)$$
$$+g_{(j,N+1)}(y,t)f(y,z)\big]\tilde{Q}^{(j,N)}(y,z,t)dt,\tag{4.4}$$

$$\bar{B}^{(M,N)}(y,z)=r(y,z)\int_a^b\int_c^d\big[g(y,z)f_{(M+1,N+1)}(s,t)$$
$$+g_{(M+1,N+1)}(s,t)f(y,z)\big]\times\bar{Q}^{(M,N)}(y,s,z,t)dtds,\tag{4.5}$$

$$C^{(j,l)}(y,z)=|r(y,z)g(y,z)|\left\|f_{(j,l)}(y,z)\right\|_{\infty}+|r(y,z)f(y,z)|\left\|g_{(j,l)}(y,z)\right\|_{\infty}, \tag{4.6}$$

$$D^{(j,l)}(y,z)=\frac{\left(\max\{b-y,y-a\}\right)^{j+1}}{(j+1)!}\frac{\left(\max\{d-z,z-c\}\right)l+1}{(l+1)!}\int_a^b\int_c^d|r(v,\varsigma)|d\varsigma dv, \tag{4.7}$$

$$D^{(0,l)}(z)=(b-a)\frac{\left(\max\{d-z,z-c\}\right)^{l+1}}{(l+1)!}\int_a^b\int_c^d|r(v,\varsigma)|d\varsigma dv, \tag{4.8}$$

$$D^{(j,0)}(y)=(d-c)=\frac{\left(\max\{b-y,y-a\}\right)^{j+1}}{(j+1)!}\int_a^b\int_c^d|r(v,\varsigma)|d\varsigma dv, \tag{4.9}$$

$$\hat{D}^{(M,l)}(y,z)=\int_a^b|\hat{Q}^{(M,l)}(y,s,z)|\,ds, \tag{4.10}$$

$$\tilde{D}^{(j,N)}(y,z)=\int_c^d|\tilde{Q}^{(j,N)}(y,z,t)|\,dt, \tag{4.11}$$

$$\bar{D}^{(M,N)}(y,z)=\int_a^b\int_c^d\left|\bar{Q}^{(M,N)}(y,s,z,t)\right|dtds, \tag{4.12}$$

$$\begin{aligned}G_f(y,z)&=R(f:y,z)+\int_a^b\int_c^d r(s,t)f(s,z)dtds\\&+\int_a^b\int_c^d r(s,t)f(y,t)dtds\\&+\sum_{l=1}^{N}\int_a^b\int_c^d(-1)^l r(s,\zeta)\frac{(\zeta-z)^l}{l!}f_{(0,l)}(s,z)d\zeta ds\\&+\sum_{j=1}^{M}\int_a^b\int_c^d(-1)^j r(v,t)\frac{(v-y)^j}{j!}f_{(j,0)}(y,t)dtdv,\end{aligned} \tag{4.13}$$

$$\begin{aligned}G_g(y,z)&=R(g:y,z)+\int_a^b\int_c^d r(s,t)g(s,z)dtds\\&+\int_a^b\int_c^d r(s,t)g(y,t)dtds\\&+\sum_{l=1}^{N}\int_a^b\int_c^d(-1)^l r(s,\zeta)\frac{(\zeta-z)^l}{l!}g_{(0,l)}(s,z)d\zeta ds\\&+\sum_{j=1}^{M}\int_a^b\int_c^d(-1)^j r(v,t)\frac{(v-y)^j}{j!}g_{(j,0)}(y,t)dt\,dv,\end{aligned} \tag{4.14}$$

where, $\hat{Q}^{(M,l)}(y,s,z)$, $\tilde{Q}^{(j,N)}(y,z,t)$, $\bar{Q}^{(M,N)}(y,s,z,t)$ are as in Theorem 2.2 whereas Q and $R(f:y,z)$ are stated in (1.2) and (2.3) respectively.

Now, we are ready to get our important results of current section using notations defined above, which are as follows:

Theorem 4.1

Let $r,f,g:[a,b]\times[c,d]\to\mathbb{R}$ be three functions $\ni f, g\in C^{(M+1,N+1)}([a,b]\times[c,d])$ and r is an integrable. Then:

$$\begin{aligned}&\left|\frac{1}{Q(b,d)}\int_a^b\int_c^d r(y,z)f(y,z)g(y,z)dzdy\right.\\&-\left(\frac{1}{Q(b,d)}\int_a^b\int_c^d r(y,z)f(y,z)dzdy\right)\\&\left.\times\left(\frac{1}{Q(b,d)}\int_a^b\int_c^d r(y,z)g(y,z)dzdy\right)\right|\\&\le\frac{1}{2[Q(b,d)]^2}\int_a^b\int_c^d\left[\sum_{j=1}^{M}\sum_{l=1}^{N}\right.\\&\times C^{(j,l)}(y,z)D^{(j,l)}(y,z)\\&+\sum_{l=1}^{N}C^{(0,l)}(z)D^{(0,l)}(z)+\sum_{j=1}^{M}C^{(j,0)}(y)D^{(j,0)}(y)\\&+C^{(M+1,l)}(y,z)\hat{D}^{(M,l)}(y,z)+C^{(j,N+1)}(y,z)\tilde{D}^{(j,N)}(y,z)\\&\left.+C^{(M+1,N+1)}(y,z)\bar{D}^{(M,N)}(y,z)\right]dzdy.\end{aligned}$$

Proof

From (2.6) we have below identities:

$$\begin{aligned}f(y,z)Q(b,d)&=R(f:y,z)+\int_a^b\int_c^d r(s,t)f(s,t)dtds\\&+\sum_{l=0}^{N}\int_a^b(-1)^{M+1+l}\hat{Q}^{(M,l)}(y,s,z)f_{(M+1,l)}(s,z)ds\\&+\sum_{j=0}^{M}\int_c^d(-1)^{j+N+1}\tilde{Q}^{(j,N)}(y,z,t)f_{(j,N+1)}(y,t)dt\\&-\int_a^b\int_c^d(-1)^{M+N}\bar{Q}^{(M,N)}(y,s,z,t)f_{(M+1,N+1)}(s,t)dtds\end{aligned} \tag{4.15}$$

$$\begin{aligned}g(y,z)Q(b,d)&=R(g:y,z)+\int_a^b\int_c^d r(s,t)g(s,t)dtds\\&+\sum_{l=0}^{N}\int_a^b(-1)^{M+1+l}\hat{Q}^{(M,l)}(y,s,z)g_{(M+1,l)}(s,z)ds\\&+\sum_{j=0}^{M}\int_c^d(-1)^{j+N+1}\tilde{Q}^{(j,N)}(y,z,t)g_{(j,N+1)}(y,t)dt\\&-\int_a^b\int_c^d(-1)^{N+M}\bar{Q}^{(M,N)}(y,s,z,t)g_{(M+1,N+1)}(s,t)dtds\end{aligned} \tag{4.16}$$

$\forall$ $(y,z)\in[a,b]\times[c,d]$. Multiply (4.15) by $r(y,z)g(y,z)$ and (4.16) by $r(y,z)f(y,z)$ and then summing these identities, we get:

$$\begin{aligned}&2Q(b,d)r(y,z)f(y,z)g(y,z)\\&=-\sum_{j=1}^{M}\sum_{l=1}^{N}(-1)^{j+l}B^{(j,l)}(y,z)-\sum_{l=1}^{N}(-1)^l B^{(0,l)}(z)\\&-\sum_{j=1}^{M}(-1)^j B^{(j,0)}(y)+B(y,z)+(-1)^{M+1+l}\hat{B}^{(M,l)}(y,z)\\&+(-1)^{j+N+1}\tilde{B}^{(j,N)}(y,z)-(-1)^{M+N}\bar{B}^{(M,N)}(y,z)\end{aligned} \tag{4.17}$$

Integrate the above equation over $[a,b]\times[c,d]$ and divided by $2Q(b,d)$ on both sides and obtain:

$$\int_a^b \int_c^d f(y,z)\, g(y,z) r(y,z) dzdy$$
$$= \frac{1}{2Q(b,d)} \int_a^b \int_c^d \Bigg[-\sum_{j=1}^{M} \sum_{l=1}^{N} (-1)^{j+l} B^{(j,l)}(y,z)$$
$$-\sum_{l=1}^{N} (-1)^{l} B^{(0,l)}(z) - \sum_{j=1}^{M} (-1)^{j} B^{(j,0)}(y) + B(y,z)$$
$$+(-1)^{M+1+l} \hat{B}^{(M,l)}(y,z) + (-1)^{j+N+1} \tilde{B}^{(j,N)}(y,z)$$
$$-(-1)^{N+M} \overline{B}^{(M,N)}(y,z) \Bigg] dzdy$$

It may be written as:

$$\frac{1}{Q(b,d)} \int_a^b \int_c^d r(y,z) f(y,z) g(y,z) dzdy$$
$$-\left(\frac{1}{Q(b,d)} \int_a^b \int_c^d f(y,z) r(y,z) dzdy \right)$$
$$\times \left(\frac{1}{Q(b,d)} \int_a^b \int_c^d r(y,z) g(y,z) dzdy \right)$$
$$= \frac{1}{2\left[Q(b,d)\right]^2} \int_a^b \int_c^d \Bigg[-\sum_{j=1}^{M} \sum_{l=1}^{N} (-1)^{j+l} B^{(j,l)}(y,z) \tag{4.18}$$
$$-\sum_{l=1}^{N} (-1)^{l} B^{(0,l)}(z)$$
$$-\sum_{j=1}^{M} (-1)^{j} B^{(j,0)}(y) + (-1)^{M+1+l} \hat{B}^{(M,l)}(y,z)$$
$$+(-1)^{j+N+1} \tilde{B}^{(j,N)}(y,z) - (-1)^{M+N} \overline{B}^{(M,N)}(y,z) \Bigg] dzdy.$$

Using (4.1),...,(4.12) we get below inequalities $\forall (y,z) \in [a,b]\times[c,d]$

$$\left|(-1)^{j+l} B^{(j,l)}(y,z)\right| \le C^{(j,l)}(y,z) D^{(j,l)}(y,z),$$
$$\left|(-1)^{l} B^{(0,l)}(z)\right| \le C^{(0,l)}(z) D^{(0,l)}(z),$$
$$\left|(-1)^{j} B^{(j,0)}(y)\right| \le C^{(j,0)}(y) D^{(j,0)}(y),$$
$$\left|(-1)^{M+1+l} \hat{B}^{(M,l)}(y,z)\right| \le C^{(M+1,l)}(y,z) \hat{D}^{(M,l)}(y,z),$$
$$\left|(-1)^{j+N+1} \tilde{B}^{(j,N)}(y,z)\right| \le C^{(j,N+1)}(y,z) \tilde{D}^{(j,N)}(y,z),$$
$$\left|(-1)^{M+N} \overline{B}^{(M,N)}(y,z)\right| \le C^{(M+1,N+1)}(y,z) \overline{D}^{(M,N)}(y,z).$$

We can easily get desired result by taking absolute value in (4.18) on both sides and by using above these inequalites in it.

Theorem 4.2

Let $r, f, g: [a,b] \times [c,d] \to \mathbb{R}$ be three functions $\ni$ $f, g \in C^{(M+1,N+1)}$ in the same interval and r is an integrable. Then:

$$\Bigg| \frac{1}{Q(b,d)} \int_a^b \int_c^d r(y,z) f(y,z) g(y,z) dzdy$$
$$+\left(\frac{1}{Q(b,d)} \int_a^b \int_c^d r(y,z) f(y,z) dzdy \right)\left(\frac{1}{Q(b,d)} \int_a^b \int_c^d r(y,z) g(y,z) dzdy \right)$$
$$-\frac{1}{2\left[Q(b,d)\right]^2} \int_a^b \int_c^d r(y,z)\left[g(y,z) G_f(y,z) + f(y,z) G_g(y,z) \right] dzdy \Bigg|$$
$$\le \frac{1}{2\left[Q(b,d)\right]^2} \int_a^b \int_c^d C^{(M+1,N+1)}(y,z) \overline{D}^{(M,N)}(y,z) dzdy$$

Proof

From (2.7) we have below identities:

$$f(y,z) Q(b,d) = G_f(y,z) - \int_a^b \int_c^d r(s,t) f(s,t) dtds$$
$$+\int_a^b \int_c^d (-1)^{N+M} \overline{Q}^{(M,N)}(y,s,z,t) f_{(M+1,N+1)}(s,t) dtds, \tag{4.19}$$

$$g(y,z) Q(b,d) = G_g(y,z) - \int_a^b \int_c^d r(s,t) g(s,t) dtds$$
$$+\int_a^b \int_c^d (-1)^{N+M} \overline{Q}^{(M,N)}(y,s,z,t) g_{(M+1,N+1)}(s,t) dtds, \tag{4.20}$$

for $(y,z) \in [a, b]\times[c, d]$. Multiply (4.19) by $r(y, z)g(y, z)$ and (4.20) by $r(y,z)f(y,z)$ and then summing these identities, we get:

$$2Q(b,d) r(y,z) f(y,z) g(y,z)$$
$$= r(y,z) g(y,z) G_f(y,z)$$
$$+r(y,z) f(y,z) G_g(y,z) \tag{4.21}$$
$$-B(y,z) + (-1)^{M+N} B^{(M+N)}(y,z)$$

Integrate the above equation over $[a, b]\times[c, d]$ and divided by $2Q(b, d)$ on both sides and obtain:

$$\int_a^b \int_c^d r(y,z) f(y,z) g(y,z) dzdy$$
$$= \frac{1}{2Q(b,d)} \int_a^b \int_c^d r(y,z)\left[g(y,z) G_f(y,z) + f(y,z) G_g(y,z) \right] dzdy$$
$$-\frac{1}{Q(b,d)} \left(\int_a^b \int_c^d f(y,z) r(y,z) dzdy \right)\left(\int_a^b \int_c^d g(y,z) r(y,z) dzdy \right) \tag{4.22}$$
$$+\frac{1}{2Q(b,d)} \int_a^b \int_c^d (-1)^{M+N} \overline{B}^{(M,N)}(y,z) dzdy.$$

since we have:

$$\left|(-1)^{M+N} \overline{B}^{(M,N)}(y,z)\right| \le C^{(M+1,N+1)}(y,z) \overline{D}^{(M,N)}(y,z). \tag{4.23}$$

From (4.22) and (4.23) we obtain our required inequality.

Remark 4.3

For $N = M = 0$, we can obtain especial cases of Theorems 4.1 and 4.2 as similar as Theorems 6 and 7 of (Pečarić and Vukelić, 2007) respectively , we may also obtain as similar results as in (Guezane-Lakoud and Aissaoui, 2011).

Conclusion

In the last section of this article, we would present our conclusion of this article is that we gave the more generalized identities of Montgomery for differentiable function of higher order for two independent variables as compare to Pečarić and Vukelić (2007) results which were given in (Pečarić and Vukelić, 2007) and we gave especial cases of identities of Montgomery by putting $r(s,t) = q(s)p(t)$. We also obtained more generalized Ostrowski and Grüss type inequalities for differentiable functions of higher order for two independent variables by the help of generalized identities of Montgomery as compare to the results of (Pečarić and Vukelić, 2007).

Author's Contributions

All authors equally contributed in this work.

Ethics

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and no ethical issues involved.

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