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Extension to Sufficient Optimality Conditions in Complex Programming

E.A. Youness and M.E. Elbrolosy
Department of Mathematics, Faculty of Science, Tanta University, Tanta, Egypt

Abstract: In this study we extend the sufficient optimality conditions of complex programming problems to a large class of functions that include the two parts (real and imaginary) of the objective function. The sufficient conditions were established under generalized forms of convexity assumptions. The previous results that considered the real part only can be deduced as special cases.

Key words: Complex Programming Problems, Optimality Conditions, Polyhedral Cone, Pointed Cone and Generalized Convexity

INTRODUCTION

Mathematical programming in complex space was initiated by Levinson [1] who extended the duality theorems for complex linear programming. These results were extended to polyhedral cones in complex space by Ben-Israel [2] and Craven and Mond [3]. Abrams and Ben-Israel [4,5] surveyed contemporary works in complex programming and outlined some applications. Optimality conditions of nonlinear programming in complex space were discussed in many publications. More details, necessary conditions were established by Abrams and Ben-Israel [6]. They gave a complex version of the well-known Kuhn-Tucker conditions. Craven and Mond [7-9] gave a complex version of Fritz John conditions. Youness and Elbrolosy [10], in extending the above versions, gave the conditions considering the two parts (real and imaginary) of the objective function.

On the other hand, sufficient conditions were established by Abrams [11], Mond and Craven [9] with the usual concepts of convexity assumptions; Bector *et al.* [12], Gulati [13,14] and Mond and Craven [5] with the generalized convexity concepts. Other works in the field include Bhatia and Kaul [16], Rani and Kaul [17], Datta [18], Mond and Murray [19], Mond and Parida and Weir and Mond [21]. For more generalized and recent discussion, one may refer to Smart and Mond [22] where the concept of invexity is involved.

As well-known, the optimality conditions for the existence of an optimal solution in complex space induce their correspondents in real space [23] as special cases. Previously, the objective function in such theorems in complex programming problems was considered as the real part of a complex function, while the imaginary part is ignored. Here we extend the sufficient conditions with considering the two parts under usual and generalized forms of convexity notions. It is worthing that these results contain all the above formulations as special cases.

Notations and Preliminaries: Denote by C^n the n-dimensional complex space, and A^T, $\overline{A}$, A^H denote the transpose, the conjugate, the conjugate transpose of a matrix $A \in C^{m\times n}$.

The *dual* S^* of a cone $S \subset C^n$ is defined by:

$S^* = \{y \in C^n : x \in S \Rightarrow \text{Re} \quad y^H x \geq 0\}$.

The interiors of S and S^* are:

$int\, S = \{y \in S : 0 \neq x \in S^* \Rightarrow \text{Re} \quad y^H x > 0\}$

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One can easily show that if S and T are convex cones in C^n, then

a) S^* is a closed convex cone,
b) $S + int\, S \subseteq int\, S$,
c) $(S \times T)^* = S^* \times T^*$,
d) $int\,(S \times T) = int\, S \times int\, T$,
e) $S \subseteq T \Rightarrow T^* \subseteq S^*$,
f) $S \subseteq (S^*)^*$ and $S = (S^*)^*$ iff S is closed.
g) $(S \cap T)^* = cl(S^* + T^*)$ if S and T are closed; "cl" denotes closure.

Define the manifold Q by:

$Q = \left\{(\omega^1, \omega^2) \in C^{2n} : \omega^2 = \overline{\omega^1}\right\}$.

Define the canonical mapping $\rho : C^n \to R^{2n}$ by $x_i + iy_i \to (x_i, y_i)$, $i = 1, ..., n$.

A convex cone S is pointed if $S \cap -S = \{0\}$ (i.e.; if it does not contain a line) and it is solid if $int\, S \neq \emptyset$.

A pointed closed convex cone $S \subset C^n$ induces a partial order in C^n via $x \leqq y$ iff $y - x \in S$.

For an analytic function $f : C^n \longrightarrow C$ and a point $z_0 \in C^n$, the gradient of f at z_0 is denoted by:

$\nabla_z f(z_0) \equiv \left(\frac{\partial f(z_0)}{\partial z_i}\right), \qquad i = 1,\ldots,n$.

For a complex-valued function $f(\omega^1,\omega^2)$ analytic in the $2n$ variables (ω^1,ω^2) at the point $(z_0,\overline{z}_0) \in C^n \times C^n$, denote the gradients by:

$$\nabla_z f(z_0,\overline{z}_0) \equiv \left(\frac{\partial f(z_0,\overline{z}_0)}{\partial \omega_i^1}\right), \quad i = 1,\ldots,n$$

and

$$\nabla_{\overline{z}} f(z_0,\overline{z}_0) \equiv \left(\frac{\partial f(z_0,\overline{z}_0)}{\partial \omega_i^2}\right), \quad i = 1,\ldots,n \cdot$$

For an analytic function $g: C^n \longrightarrow C^m$,

$$D_z g(z_0) \equiv \left(\frac{\partial g_i(z_0)}{\partial z_j}\right), \; i = 1,\ldots,m; \; j = 1,\ldots,n \cdot$$

($g: C^n \longrightarrow C^m$ is analytic if each of its components $g_i : C^n \longrightarrow C, \; i = 1,\ldots,m$ is analytic)

Similarly for a complex function $g: C^n \times C^n \to C^m$ analytic in the $2n$ variables (ω^1,ω^2) at the $(z_0,\overline{z}_0) \in C^n \times C^n$

$$D_z g(z_0,\overline{z}_0) \equiv \left(\frac{\partial g_i(z_0,\overline{z}_0)}{\partial \omega_j^1}\right), i = 1,\ldots,m; j = 1,\ldots,n$$

and

$$D_{\overline{z}} g(z_0,\overline{z}_0) \equiv \left(\frac{\partial g_i(z_0,\overline{z}_0)}{\partial \omega_j^2}\right), i = 1,\ldots,m; j = 1,\ldots,n \cdot$$

Definition 2.1: A set S is a polyhedral cone in C^n if it is the intersection of finitely many closed half spaces, each containing the origin in its boundary, i.e.;

$$S = \bigcap_{k=1}^{p} H_{u_k},$$

where, $\quad H_{u_k} = \{ z \in C^n \, ; \, \mathrm{Re}\, z^H u_k \geq 0 \}$,

for some vectors $u_1, \ldots, u_p \in C^n$ and $p \in N$.

This is equivalent to S is a polyhedral cone if there is an integer r and $A \in C^{r \times n}$ such that

$S = \{ z \in C^n \; : \; \mathrm{Re}\; Az \geq 0 \} \cdot$

Definition 2.2: Let S be a closed convex cone in C^n, and let $z_0 \in S$. Define $S(z_0)$, the cone S at z_0, as:

$$S(z_0) = \{ x \in C^n : \mathrm{Re}\, y^H z_0 = 0, \quad y \in S^* \Rightarrow \mathrm{Re}\, y^H x \geq 0 \}.$$

In the case that S is polyhedral, $S(z_0)$ is the intersection of those closed half spaces H_{u_k} which contain z_0 in their boundaries, i.e.,

$$B(z_0) = \{ k : \mathrm{Re}\; z_0^H \, u_k = 0 \} \Rightarrow S(z_0) = \bigcap_{k \in B(z_0)} H_{u_k},$$

with the convention that $S(z_0) = C^n$ when $B(z_0) = \varnothing$. Clearly $z_0 \in S(z_0)$, $-z_0 \in S(z_0)$ and $S \subseteq S(z_0)$.

Definition 2.3: Let S be a closed convex cone in C^m. At a point $(z_0,\overline{z}_0) \in Q$, the analytic function $f \,:\, C^{2n} \to C^m$ is said to be:

i) convex on Q with respect to S if

$$f(z,\overline{z}) - f(z_0,\overline{z}_0) - D_z f(z_0,\overline{z}_0)(z - z_0) - D_{\overline{z}} f(z_0,\overline{z}_0)(\overline{z} - \overline{z}_0) \in S,$$

for any $(z,\overline{z}) \in Q$;

ii) strictly convex with respect to S if

$$f(z,\overline{z}) - f(z_0,\overline{z}_0) - D_z f(z_0,\overline{z}_0)(z - z_0) - D_{\overline{z}} f(z_0,\overline{z}_0)(\overline{z} - \overline{z}_0) \in int\; S,$$

for any $(z,\overline{z}) \in Q$, $(z,\overline{z}) \neq (z_0,\overline{z}_0)$;

iii) quasi-convex with respect to S if

$$f(z_0,\overline{z}_0) - f(z,\overline{z}) \in S \Rightarrow D_z f(z_0,\overline{z}_0)(z_0 - z) + D_{\overline{z}} f(z_0,\overline{z}_0)(\overline{z}_0 - \overline{z}) \in S$$

for any $(z,\overline{z}) \in Q$;

iv) strictly quasi-convex with respect to S if

$$f(z_0,\overline{z}_0) - f(z,\overline{z}) \in S \Rightarrow D_z f(z_0,\overline{z}_0)(z_0 - z) + D_{\overline{z}} f(z_0,\overline{z}_0)(\overline{z}_0 - \overline{z}) \in int\;\; S,$$

for any $(z,\overline{z}) \in Q$, $(z,\overline{z}) \neq (z_0,\overline{z}_0)$;

v) pseudo-convex with respect to S if

$$D_z f(z_0,\overline{z}_0)(z_0 - z) + D_{\overline{z}} f(z_0,\overline{z}_0)(\overline{z}_0 - \overline{z}) \in S \Rightarrow f(z,\overline{z}) - f(z_0,\overline{z}_0) \in S,$$

for any for any $(z,\overline{z}) \in Q$;

vi) strictly pseudo-convex with respect to S if

$$D_z f(z_0,\overline{z}_0)(z_0 - z) + D_{\overline{z}} f(z_0,\overline{z}_0)(\overline{z}_0 - \overline{z}) \in S \Rightarrow f(z,\overline{z}) - f(z_0,\overline{z}_0) \in int\; S,$$

for any for any $(z,\overline{z}) \in Q$, $(z,\overline{z}) \neq (z_0,\overline{z}_0)$;

vii) concave, strictly concave, quasi-concave, strictly quasi-concave, pseudo-concave or strictly pseudo-concave with respect to S if f is convex, strictly convex, quasi-convex, strictly quasi-convex, pseudo-convex or strictly pseudo- convex with respect to $-S=\{z:-z\in S\}$, respectively;

viii) quasi-monotonic with respect to S if it is both quasi-convex and quasi-concave with respect to S.

Definitions of convexity of a function $f:C^n \to C^m$ are obtained, likewise, by replacing $f(z,\bar{z})$ with $f(z)$ and noting that $D_{\bar{z}} f(z)=0$.

For more illustration, let us give the following simple examples:

1- The function $f(z)=az+b$, where a and b are constant complex numbers, is convex with respect to any cone.

2- The function $f(z,\bar{z})=z\bar{z}+az$, where a is a constant complex number, is convex with respect to the cone $S=\{z\in C : \text{Re}[1 \quad -i \quad i]^T z\geq 0\}$

(note that $\rho(S)=\hat{S}=\{(x,y)\in R^2 : x\geq 0, y=0\}$).

3- Let $X=\{z\in C:\text{Re}\,[1-i \quad 1+i]^T \; z\geq 0\}$ and $S=\{z\in C:\text{Re}\,[1 \quad -i]^T z\geq 0\}$.

Then the function $f(z)=z^2$ is convex on X at the point $z=0$, with respect to S.

(note $\hat{X}=\{(x,y)\in R^2 : y+x\geq 0, y-x\leq 0\}$ and $\hat{S}=\{(x,y)\in R^2 : x\geq 0, y\geq 0\}$).

Let us, in the following, list a number of previously established results obtained in [15,24] that will be needed in proving our theorems.

Lemma 2.1: Let S be a closed convex cone in C^n, then S is pointed iff S^* is solid.

Lemma 2.2: Let S be a closed convex cone in C^n and $a\in S$, then $z\in[S(a)]^*$ iff $z\in S^*$ and $\text{Re}\, z^H a=0$.

Theorem 2.1: Let $0\neq A_1\in C^{m\times n_1}$, $A_2\in C^{m\times n_2}$, let T, S_1, S_2 be polyhedral cones in C^m, C^{n_1}, C^{n_2}, respectively and let S_1 be pointed.

Then exactly one of the following systems is consistent:

I- $A_1x_1+A_2x_2\in T,\ 0\neq x_1\in S_1,\ x_2\in S_2$,

II- $y\in -T^*$, $A_1^H\, y\in int\, S_1^*$, $A_2^H\, y\in S_2^*$.

Theorem 2.2: Let $A\in C^{m\times n}$, let K_1 and K_2 be pointed closed convex cones in C^n and C^m, respectively. Then exactly one of the following systems is consistent:

I- $Ax\in K_2$, $0\neq x\in K_1$,

II- $A^H y\in int\, K_1^*$, $-y\in int\, K_2^*$.

Theorems 2.1 and 2.2 are complex versions of the well-known transposition theorems of Motzkin and Gordan, respectively.

Complex Programming Problem Formulation: We will consider two classes of nonlinear programming problems. The first class consists of problems of the form:

$(T-P_1)$ $\quad \min f(z,\bar{z})$

subject to

$(z,\bar{z})\in M=\{(z,\bar{z})\in C^{\cdot\cdot} : g(z,\bar{z})\in S\}$, or

$(T-P_1')$ $\quad \min f(z,\bar{z})$

subject to

$(z,\bar{z})\in M'=\{\ (z,\bar{z})\in C^{\cdot\cdot} : g(z,\bar{z})\in S, h(z,\bar{z})=0\ \}$,

where, $f:C^{2n}\to C$, $g:C^{2n}\to C^m$ and $h:C^{2n}\to C^r$ are analytic functions, and S is a closed convex cone in C^m. A vector $(z_0,\bar{z}_0)\in C^{2n}$ is an optimal solution of $(T-P_1)$ or $(T-P_1')$ with respect to a pointed closed convex cone $T\subset C$, if $(z_0,\bar{z}_0)$ is feasible, and $f(z,\bar{z})-f(z_0,\bar{z}_0)\in T$, for every $(z,\bar{z})\in M$ or $(z,\bar{z})\in M'$, respectively.

The second class consists of problems of the form:

$(T-P_2)$ $\quad \min f(z)$

subject to

$z\in M=\{z\in C^n : g(z)\in S\}$, or

$(T-P_2')$ $\quad \min f(z)$

subject to

$z\in M'=\{\ z\in C^n : g(z)\in S,\ \ h(z)=0\ \}$,

where $f:C^n\to C$, $g:C^n\to C^m$ and $h:C^n\to C^r$ are analytic; and optimality is defined as above.

RESULTS

Complex Programming Problem Without Equality Constraints

Theorem 1: Let f, g, S, T be as in problem $(T-P_1)$. If at some feasible point $(z_0, \overline{z}_0)$, f is convex with respect to T on the manifold Q and g is concave with respect to S on Q, then a sufficient condition for $(z_0, \overline{z}_0)$ to be an optimal solution of problem $(T-P_1)$ with respect to T is the existence of a $v_0 \in S^*$ such that

$$\operatorname{Re}\left\{\left[\tau^H \nabla_z f(z_0,\overline{z}_0) + \tau^T \overline{\nabla_{\overline{z}} f}(z_0,\overline{z}_0) - v_0^H D_z g(z_0,\overline{z}_0) - v_0^T \overline{D_{\overline{z}} g}(z_0,\overline{z}_0)\right](z-z_0)\right\} \geq 0,$$

for all $(z,\overline{z}) \in M$ and all $\tau \in T^*$, (1)

$$\operatorname{Re} v_0^H g(z_0, \overline{z}_0) = 0 \,. \qquad (2)$$

Proof: Since f is convex with respect to T, then for all $(z,\overline{z}) \in M$ and all $\tau \in T^*$,

$$\begin{aligned}
\operatorname{Re}\tau^H \left[f(z,\overline{z}) - f(z_0,\overline{z}_0)\right] &\geq \operatorname{Re}\left[\tau^H \nabla_z f(z_0,\overline{z}_0)(z-z_0) + \tau^H \nabla_{\overline{z}} f(z_0,\overline{z}_0)(\overline{z}-\overline{z}_0)\right] \\
&= \operatorname{Re}\left[\tau^H \nabla_z f(z_0,\overline{z}_0) + \tau^T \overline{\nabla_{\overline{z}} f}(z_0,\overline{z}_0)\right](z-z_0) \\
&\geq \operatorname{Re}\{[v_0^H D_z g(z_0,\overline{z}_0) + v_0^T \overline{D_{\overline{z}} g}(z_0,\overline{z}_0)](z-z_0)\} \quad \text{(by (1))} \\
&= \operatorname{Re}[v_0^H D_z g(z_0,\overline{z}_0)(z-z_0) + v_0^H D_{\overline{z}} g(z_0,\overline{z}_0)(\overline{z}-\overline{z}_0)] \\
&\geq \operatorname{Re} v_0^H [\, g(z,\overline{z}) - g(z_0,\overline{z}_0)] \quad \text{(by concavity of } g\text{)} \\
&= \operatorname{Re} v_0^H \; g(z,\overline{z}) \quad \text{(by (2))} \\
&\geq 0 \,. \quad \text{(since } v_0 \in S^* \text{ and } g(z,\overline{z}) \in S\text{)}
\end{aligned}$$

Thus $f(z,\overline{z}) - f(z_0,\overline{z}_0) \in T$, for any $(z,\overline{z}) \in M$ which means that $(z_0, \overline{z}_0)$ is an optimal solution for $(T-P_1)$ with respect to T.

Corollary 1: Let f, g, S, T be as in problem $(T-P_2)$. If at some feasible point z_0, f is convex with respect to T and g is concave with respect to S, then a sufficient condition for z_0 to be an optimal solution of problem $(T-P_2)$ with respect to T is the existence of a $v_0 \in S^*$ such that

$$\operatorname{Re}\left[\ {}^{H}\nabla_z f(z_0) - v_0^H D_z g(z_0)\right](z-z_0) \geq 0 \ \text{ for all } z \in M \text{ and all } \tau \in T^*, \qquad (3)$$

$$\operatorname{Re} v_0^H \; g(z_0) = 0 \qquad (4)$$

Theorem 2: Let f, g, S and T be as in problem $(T-P_1)$, with S to be solid. If at some feasible point $(z_0, \overline{z}_0)$, f is convex with respect to T and g is strictly concave with respect to $S(g(z_0,\overline{z}_0))$ on Q, then a sufficient condition for $(z_0, \overline{z}_0)$ to be an optimal solution of $(T-P_1)$, with respect to T, is the existence of a $\tau_0 \in T^*$ and $v_0 \in S^*$; $(\tau_0, v_0) \neq 0$ such that

$$\tau_0^H \nabla_z f(z_0,\overline{z}_0) + \tau_0^T \overline{\nabla_{\overline{z}} f}(z_0,\overline{z}_0) - v_0^H D_z g(z_0,\overline{z}_0) - v_0^T \overline{D_{\overline{z}} g}(z_0,\overline{z}_0) = 0, \qquad (5)$$

$$\operatorname{Re} v_0^H g(z_0, \overline{z}_0) = 0 \,. \qquad (6)$$

Proof: From Lemma 2.2; $v_0 \in S^*$ and (6) imply $v_0 \in [S(g(z_0,\overline{z}_0))]^*$. Since there exists a non-zero vector $(\tau_0, v_0); \tau_0 \in T^*$, $v_0 \in [S(g(z_0,\overline{z}_0))]^*$, satisfying (3), then there exists a solution to the system

$$\begin{bmatrix} \nabla_z f(z_0,\overline{z}_0) \\ \overline{\nabla_{\overline{z}} f(z_0,\overline{z}_0)} \\ -D_z g(z_0,\overline{z}_0) \\ \overline{-D_{\overline{z}} g(z_0,\overline{z}_0)} \end{bmatrix}^H \begin{bmatrix} \tau \\ \overline{\tau} \\ v \\ \overline{v} \end{bmatrix} = 0,\ 0 \neq \begin{bmatrix} \tau \\ \overline{\tau} \\ v \\ \overline{v} \end{bmatrix} \in T^* \times \overline{T^*} \times [S(g(z_0,\overline{z}_0))]^* \times \overline{[S(g(z_0,\overline{z}_0))]^*}. \tag{7}$$

By Theorem 2.1 or 2.2, there exists no $p \in C^n$ such that

$$\begin{bmatrix} -\nabla_z f(z_0,\overline{z}_0) \\ -\overline{\nabla_{\overline{z}} f}(z_0,\overline{z}_0) \\ D_z g(z_0,\overline{z}_0) \\ \overline{D_{\overline{z}}} g(z_0,\overline{z}_0) \end{bmatrix} p \in int\ \ [T \times \overline{T} \times S(g(z_0,\overline{z}_0)) \times \overline{S(g(z_0,\overline{z}_0))}\ \ ],$$

i.e.; there exists no solution p to the system

$$\left.\begin{array}{l} -\nabla_z f(z_0,\overline{z}_0) \quad p \in \operatorname{int} T \\ -\overline{\nabla_{\overline{z}} f}(z_0,\overline{z}_0) \quad p \in \operatorname{int} \overline{T} \\ D_z g(z_0,\overline{z}_0) \quad p \in int \quad S(g(z_0,\overline{z}_0)) \\ \overline{D_{\overline{z}}} g(z_0,\overline{z}_0) \quad p \in int \quad \overline{S(g(z_0,\overline{z}_0))} \end{array}\right\}.$$

By conjugating the second and fourth, then adding to the first and third, respectively, there is no p satisfying

$$\left.\begin{array}{l} -[\nabla_z f(z_0,\overline{z}_0)p + \nabla_{\overline{z}} f(z_0,\overline{z}_0)\overline{p}] \in int\ T \\ D_z g(z_0,\overline{z}_0)p + D_{\overline{z}} g(z_0,\overline{z}_0)\overline{p} \in int\ \ S(g(z_0,\overline{z}_0)) \end{array}\right\}.$$

Since T is pointed, there is no p to the system

$$\left.\begin{array}{l} \nabla_z f(z_0,\overline{z}_0)p + \nabla_{\overline{z}} f(z_0,\overline{z}_0)\overline{p} \notin \operatorname{int} T \\ D_z g(z_0,\overline{z}_0)p + D_{\overline{z}} g(z_0,\overline{z}_0)\overline{p}\ \in int\ \ S(g(z_0,\overline{z}_0)) \end{array}\right\}. \tag{8}$$

Consequently, there exists no solution to the system

$$s.t.\quad \left.\begin{array}{l} f(z,\overline{z}) - f(z_0,\overline{z}_0) \ \notin T \\ \\ (z,\overline{z}) \in M \end{array}\right\}, \tag{9}$$

for if it did have a solution

$$\nabla_z f(z_0,\overline{z}_0)(z - z_0) + \nabla_{\overline{z}} f(z_0,\overline{z}_0)(\overline{z} - \overline{z}_0) +$$
$$[f(z,\overline{z}) - f(z_0,\overline{z}_0) - \nabla_z f(z_0,\overline{z}_0)(z - z_0) - \nabla_{\overline{z}} f(z_0,\overline{z}_0)(\overline{z} - \overline{z}_0)] \notin T.$$

It follows, from convexity of f with respect to T, that

$$\nabla_z f(z_0,\overline{z}_0)(z-z_0)+\nabla_{\overline{z}} f(z_0,\overline{z}_0)(\overline{z}-\overline{z}_0)\notin T, \quad (10)$$

otherwise $f(z,\overline{z})-f(z_0,\overline{z}_0)-\nabla_z f(z_0,\overline{z}_0)(z-z_0)-\nabla_{\overline{z}} f(z_0,\overline{z}_0)(\overline{z}-\overline{z}_0)\notin T$.

Now for any $u\in\left[S(g(z_0,\overline{z}_0))\right]^*$ we see, using Lemma 2.2, that

$$\operatorname{Re}u^H g(z,\overline{z})\geq 0=\operatorname{Re}u^H g(z_0,\overline{z}_0),$$

i.e., $\operatorname{Re} u^H\left[g(z,\overline{z})-g(z_0,\overline{z}_0)\right]\geq 0$,

hence $g(z,\overline{z})-g(z_0,\overline{z}_0)\in S(g(z_0,\overline{z}_0))$.

It follows, from the strictly concavity of g with respect to $S(g(z_0,\overline{z}_0))$, that

$$g(z,\overline{z})-g(z_0,\overline{z}_0)+\left[D_z g(z_0,\overline{z}_0)(z-z_0)+D_{\overline{z}} g(z_0,\overline{z}_0)(\overline{z}-\overline{z}_0)-g(z,\overline{z})+g(z_0,\overline{z}_0)\right]$$

$$\in S(g(z_0,\overline{z}_0))+int\quad S(g(z_0,\overline{z}_0))\subseteq int\quad S(g(z_0,\overline{z}_0)),$$

i.e.,

$$D_z g(z_0,\overline{z}_0)(z-z_0)+D_{\overline{z}} g(z_0,\overline{z}_0)(\overline{z}-\overline{z}_0)\in int\quad S(g(z_0,\overline{z}_0)). \quad (11)$$

Thus, if (9) has a solution, setting $p=z-z_0$ in (10) and (11), we have a solution to (8), contradicting the fact that the system (7) has a solution. Hence (9) has no solution, which implies that $(z_0,\overline{z}_0)$ is an optimal solution of problem $(T-P_1)$ with respect to T.

Corollary 2: Under conditions similar to Theorem 2, a sufficient condition for z_0 to be an optimal solution of problem $(T-P_2)$ with respect to T is the existence of a $\tau_0\in T^*$ and $v_0\in S^*$; $(\tau_0,v_0)\neq 0$ such that

$$\tau_0^H\nabla_z f(z_0)-v_0^H D_z\, g(z_0)=0, \quad (12)$$

$$\operatorname{Re}\quad v_0^H\quad g(z_0)=0.$$

Theorem 3: Let f, g, S and T be as in problem $(T-P_1)$. If at some feasible point $(z_0,\overline{z}_0)$, f is pseudo-convex with respect to T and g is quasi-concave with respect to $S(g(z_0,\overline{z}_0))$, then a sufficient condition for $(z_0,\overline{z}_0)$ to be an optimal solution of $(T-P_1)$, with respect to T, is the existence of $v_0\in S^*$ such that (1) and (2) above are satisfied.

Proof: Let $v\in\left[S(g(z_0,\overline{z}_0))\right]^*$, using Lemma 2.2; we have for any $(z,\overline{z})\in M$

$$\operatorname{Re}v^H g(z,\overline{z})\geq 0=\operatorname{Re}v^H g(z_0,\overline{z}_0),$$

so, $\operatorname{Re}v^H\left[g(z,\overline{z})-g(z_0,\overline{z}_0)\right]\geq 0$,

thus $g(z,\overline{z})-g(z_0,\overline{z}_0)\in S(g(z_0,\overline{z}_0))$.

By the quasi-concavity of g with respect to $S(g(z_0,\overline{z}_0))$,

$$D_z\, g(z_0,\overline{z}_0)(z-z_0)+D_{\overline{z}}\, g(z_0,\overline{z}_0)(\overline{z}-\overline{z}_0)\in S(g(z_0,\overline{z}_0)).$$

From (2) with $v_0\in S^*$, Lemma 2.2 implies $v_0\in\left[S(g(z_0,\overline{z}_0))\right]^*$.

Therefore $\operatorname{Re}v_0^H\left[D_z\, g(z_0,\overline{z}_0)(z-z_0)+D_{\overline{z}}\, g(z_0,\overline{z}_0)(\overline{z}-\overline{z}_0)\right]\geq 0$,

i.e.; $\operatorname{Re}\left\{\left[v_0^H D_z\, g(z_0,\overline{z}_0)+v_0^T\overline{D_{\overline{z}}\, g}(z_0,\overline{z}_0)\right](z-z_0)\right\}\geq 0$.

It follows, from (1), that for all $\tau\in T^*$ and all $(z,\overline{z})\in M$

$$\operatorname{Re}\left\{\left[\tau^H\nabla_z f(z_0,\overline{z}_0)+\tau^T\overline{\nabla_{\overline{z}} f}(z_0,\overline{z}_0)\right](z-z_0)\right\}\geq 0\quad,$$

i.e.; $\operatorname{Re}\,\tau^H\left[\nabla_z f(z_0,\overline{z}_0)(z-z_0)+\nabla_{\overline{z}} f(z_0,\overline{z}_0)(\overline{z}-\overline{z}_0)\right]\geq 0$,

then $\nabla_z f(z_0,\overline{z}_0)(z-z_0)+\nabla_{\overline{z}} f(z_0,\overline{z}_0)(\overline{z}-\overline{z}_0)\in T$.

The pseudo-convexity of f with respect to T gives $f(z,\overline{z})-f(z_0,\overline{z}_0)\in T$, for any $(z,\overline{z})\in M$, which means that $(z_0,\overline{z}_0)$ is an optimal solution of $(T-P_1)$ with respect to T .

Complex Programming Problem with Equality Constraints

Theorem 4: Let f,g,h,S,T be as in problem $(T-P_1')$. If at some feasible point $(z_0,\overline{z}_0)$, f is pseudo-convex with respect to *T*, *g* is quasi-concave with respect to $S(g(z_0,\overline{z}_0))$ and *h* is quasi-monotonic with respect to *a* pointed closed convex cone *K* in C^r, then a sufficient condition for $(z_0,\overline{z}_0)$ to be an optimal solution of problem $(T-P_1')$ is the existence of a $v_0\in S^*$ and $u_0\in C^r$ such that

$$\operatorname{Re}\left\{\left[\tau^H\nabla_z f(z_0,\overline{z}_0)+\tau^T\overline{\nabla_{\overline{z}} f}(z_0,\overline{z}_0)-v_0^H D_z g(z_0,\overline{z}_0)-v_0^T\overline{D_{\overline{z}} g}(z_0,\overline{z}_0)+u_0^H D_z h(z_0,\overline{z}_0)\right.\right.$$

$$\left.\left.+u_0^T\overline{D_{\overline{z}} h}(z_0,\overline{z}_0)\right](z-z_0)\right\}\geq 0,\ \text{for all } (z,\overline{z})\in M' \text{ and all } \tau\in T^*, \tag{13}$$

$$\operatorname{Re}\, v_0^H g(z_0,\overline{z}_0)=0\ .$$

Proof: As in the above Theorem, for any $(z,\overline{z})\in M'$

$$\operatorname{Re} v_0^H\left[D_z\, g(z_0,\overline{z}_0)(z-z_0)+D_{\overline{z}}\, g(z_0,\overline{z}_0)(\overline{z}-\overline{z}_0)\right]\geq 0\ . \tag{14}$$

Now, by the quasi-convexity of h with respect to K

$h(z_0,\overline{z}_0)-h(z,\overline{z})=0\in K\Rightarrow D_z h(z_0,\overline{z}_0)(z_0-z)+D_{\overline{z}} h(z_0,\overline{z}_0)(\overline{z}_0-\overline{z})\in K$,

and by quasi-concavity of h with respect to *K*

$h(z,\overline{z})-h(z_0,\overline{z}_0)=0\in K\Rightarrow D_z h(z_0,\overline{z}_0)(z_0-z)+D_{\overline{z}} h(z_0,\overline{z}_0)(\overline{z}_0-\overline{z})\in -K$.

Since K is pointed, i.e.; $K\bigcap -K=\{0\}$, we have

$D_z h(z_0,\overline{z}_0)(z_0-z)+D_{\overline{z}} h(z_0,\overline{z}_0)(\overline{z}_0-\overline{z})=0$.

Thus

$$\operatorname{Re}\ u_0^H D_z h(z_0,\overline{z}_0)(z_0-z)+u_0^T\overline{D_{\overline{z}} h}(z_0,\overline{z}_0)(\overline{z}_0-\overline{z})=0\ . \tag{15}$$

It follows, from (13)-(15), that for all $(z,\overline{z})\in M'$ and all $\tau\in T^*$

$$\operatorname{Re}\ \left\{\left[\tau^H\nabla_z f(z_0,\overline{z}_0)+\tau^T\overline{\nabla_{\overline{z}} f}(z_0,\overline{z}_0)\right](z-z_0)\right\}\geq 0,$$

i.e.; $\operatorname{Re}\,\tau^H\left[\nabla_z f(z_0,\overline{z}_0)(z-z_0)+\nabla_{\overline{z}} f(z_0,\overline{z}_0)(\overline{z}-\overline{z}_0)\right]\geq 0$.

Hence $\nabla_z f(z_0,\overline{z}_0)(z-z_0)+\nabla_{\overline{z}} f(z_0,\overline{z}_0)(\overline{z}-\overline{z}_0)\in T$,

which by pseudo-convexity of f with respect to T , gives $f(z,\overline{z})-f(z_0,\overline{z}_0)\in T$, for any $(z,\overline{z})\in M$. Therefore $(z_0,\overline{z}_0)$ is an optimal solution of $(T-P_1')$ with respect to T .

Remark 1: The conclusions of the above Theorem remain valid under some modified hypotheses; if *K* is not pointed, then restrict u_0 to $cl\,[K^*+(-K^*)]=[K\bigcap(-K)]^*$; if h is quasi-convex only with respect to K, then restrict u_0 to K^*.

Theorem 5: Let f, g, h, S and T be as in $(T-P_1')$; with *S* and *T* to be solid. If at some feasible point $(z_0,\overline{z}_0)$, f is pseudo-convex with respect to *T*, *g* is strictly quasi-concave with respect to $S(g(z_0,\overline{z}_0))$; and *h* is strictly quasi-convex with respect to *a* solid pointed closed convex cone *K* in C^r, then a sufficient condition for

$(z_0,\overline{z}_0)$ to be an optimal solution of problem $(T-P_1')$ is the existence of a $\tau_0 \in T^*$, $v_0 \in S^*$ and $u_0 \in T^*$, not all zero, such that

$$\tau_0^H \nabla_z f(z_0,\overline{z}_0) + \tau_0^T \overline{\nabla_{\overline{z}} f}(z_0,\overline{z}_0) - v_0^H D_z g(z_0,\overline{z}_0) - v_0^T \overline{D_{\overline{z}} g}(z_0,\overline{z}_0)$$

$$+u_0^H D_z h(z_0,\overline{z}_0) + u_0^T \overline{D_{\overline{z}} h}(z_0,\overline{z}_0) = 0 \ , \qquad (16)$$

$$\text{Re } v_0^H g(z_0,\overline{z}_0) = 0 .$$

Proof: A procedure similar to that in the proof of Theorem 2, yields the consistency of the system

$$\begin{bmatrix} \nabla_z f \\ \overline{\nabla_{\overline{z}} f} \\ -D_z g \\ -\overline{D_{\overline{z}} g} \\ D_z h \\ \overline{D_{\overline{z}} h} \end{bmatrix}^H \begin{bmatrix} \tau \\ \overline{\tau} \\ v \\ \overline{v} \\ u \\ \overline{u} \end{bmatrix} = 0, 0 \neq \begin{bmatrix} \tau \\ \overline{\tau} \\ v \\ \overline{v} \\ u \\ \overline{u} \end{bmatrix} \in T^* \times \overline{T^*} \times [S(g(z_0,\overline{z}_0))]^* \times \overline{[S(g(z_0,\overline{z}_0))]^*} \times K^* \times \overline{K^*} .$$

So, there exists no solution $p \in C^n$ to the system

$$\left.\begin{array}{l} \nabla_z f(z_0,\overline{z}_0)p + \nabla_{\overline{z}} f(z_0,\overline{z}_0)\overline{p} \notin int\ T \\ D_z g(z_0,\overline{z}_0)p + D_{\overline{z}} g(z_0,\overline{z}_0)\overline{p} \in int\ \ S(g(z_0,\overline{z}_0)) \\ -D_z h(z_0,\overline{z}_0)p - D_{\overline{z}} h(z_0,\overline{z}_0)\overline{p} \in int\ K \end{array}\right\} .$$

Consequently, there exists no solution to the system

$$s.t. \quad \left.\begin{array}{l} f(z,\overline{z}) - f(z_0,\overline{z}_0) \notin T \\ (z,\overline{z}) \in M' \end{array}\right\} ,$$

for if it did have a solution, the pseudo-convexity of f with respect to T gives

$$\nabla_z f(z_0,\overline{z}_0)(z-z_0) + \nabla_{\overline{z}} f(z_0,\overline{z}_0)(\overline{z}-\overline{z}_0) \notin T ,$$

and so

$$\nabla_z f(z_0,\overline{z}_0)(z-z_0) + \nabla_{\overline{z}} f(z_0,\overline{z}_0)(\overline{z}-\overline{z}_0) \notin \text{int} T .$$

Since, as in Theorem 2,

$$g(z,\overline{z}) - g(z_0,\overline{z}_0) \in S(g(z_0,\overline{z}_0)),$$

then the strict quasi-concavity of g with respect to $S(g(z_0))$ implies

$$D_z g(z_0,\overline{z}_0)(z-z_0) + D_{\overline{z}} g(z_0,\overline{z}_0)(\overline{z}-\overline{z}_0) \in int\ \ S(g(z_0,\overline{z}_0)) .$$

Finally, the strict quasi-convexity of h with respect to K implies

$$h(z_0,\overline{z}_0) - h(z,\overline{z}) = 0 \in K \Rightarrow -D_z h(z_0,\overline{z}_0)(z-z_0) - D_{\overline{z}} h(z_0,\overline{z}_0)(\overline{z}-\overline{z}_0) \in \text{int} K .$$

Hence, as in the termination of the proof of Theorem 2, we conclude that $(z_0,\overline{z}_0)$ is an optimal solution of $(T-P_1')$ with respect to T.

Remark 2: When f, g and h are functions of z, conditions (13) and (16), in the above theorems, can be formulated in similar forms, for problem $(T-P_2')$, to conditions (3) and (12) in Corollaries (1) and (2).

Special Cases: If the objective function in this study is considered as $\text{Re } f(z,\overline{z})$ instead of $f(z,\overline{z})$, then our theorems yield, as special cases, the results discussed before in complex space C^{2n} by taking $T = R_+$, the cone of non-negative half line of R.

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