Original Research Paper

Anthropomorphic Solid Structures n-R Kinematics

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Abstract: This paper presents and treats (in an original way) the specific elements of the structures of robotic solid mobile anthropomorphic type. Are "placed on the wallpaper", the geometry and kinematics of the anthropomorphic robotic solid systems, in an original vision of the authors. One presents the inverse kinematics of anthropomorphic systems, with mechanical elements and points: Geometry, cinematic, positions, displacements, velocities and accelerations. They will be presented further two methods (as the most representatives): First one the method trigonometric and second one the geometric method.

Keywords: Anthropomorphic Robots, Direct Kinematic, Inverse Kinematic, 3R Systems, Velocities, Accelerations

Introduction

Today, anthropomorphic structures are used more and more in almost all the fields of industrial. Robotic structures have emerged from the need for automation and robotics of the industrial processes. The first industrial robots were called upon by the heavy industry and in particular by the automobiles industry. The automotive industry not only has requested the appearance of the industrial robots but even their subsequent development (Angeles, 1989; Atkenson *et al.*, 1986; Avallone and Baumeister, 1996; Baili, 2003; Baron and Angeles, 1998; Borrel and Liegeois, 1986; Burdick, 1988).

The most used were and have remained, the robots anthropomorphic, because they are more easily designed, built, maintained, are easily to handle, more dynamics, robust, economic and in general they have a broadly working area. The structures of the solid anthropomorphic robots are made up of elements and the couples of rotation, to which can add on an occasional basis and one or more couplers with translational moving. The couplers of rotation have been proven their effectiveness by moving them easier, more dynamic, step by step and especially being the most reliable. In general the couplers of rotation are moving more easily and more continuous, are actuated better and easier, control is less expensive and more reliable and programming the movements of rotation is also much simpler and more efficient (Ceccarelli, 1996; Choi *et al.*, 2004; Denavit, 1964; Di Gregorio and Parenti-Castelli, 2002; Goldsmith, 2002; Grotjahn *et al.*, 2004; Guegan and Khalil, 2002; Kim and Tsai, 2002; Lee and Sanderson, 2001; Liu and Kim, 2002).

All anthropomorphic structures are made up of a basic structure 3R (Fig. 1). Starting from the basic structure 3R may build then various robots 4R, 5R, 6R, 7R..., by the addition of moving elements and the couplers. Regardless of how many degrees of mobility has a solid structure final mobile, the basis for the thing is always represented by the solid structure 3R shown in Fig. 1. For this reason, the calculations presented in this study will be developed for the basic structure 3R. The structure in Fig. 1 consists of three moving elements linked together by the couplers of rotation. It is a spatial structure, with a large area of movement (working).
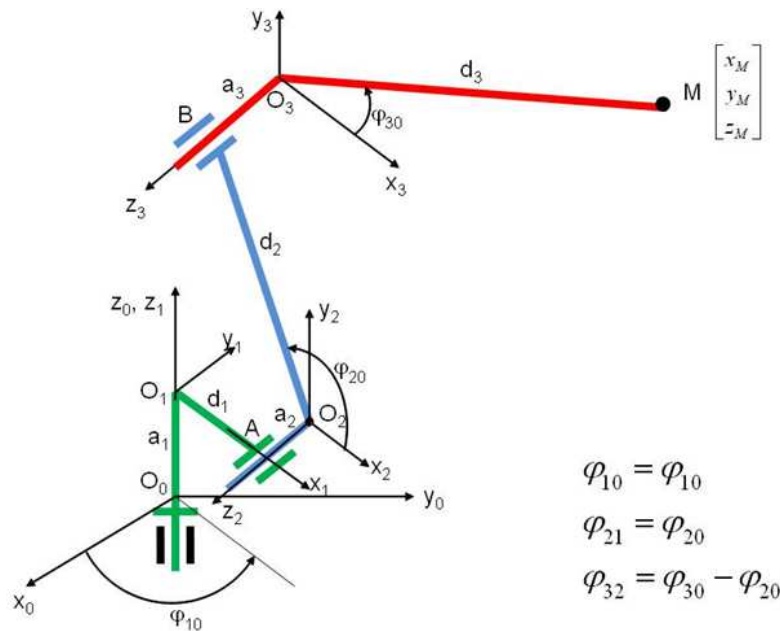
Fig. 1. Schema geometry-cinematic of a structure 3R modern (anthropomorphic)

The platform (system) as shown in Fig. 1 has three degree of mobility, carried out by three actuators (electric motors). The first electric motor drives the entire system in a rotating movement around a vertical shaft $O_0 z_0$. The engine (actuator) number 1 is mounted on the fixed element (frame 0) and causes the mobile element 1 in a rotation movement around a vertical shaft $O_0 O_1$. On the mobile element 1 are constructed then all the other elements (components) of the system.

It follows a kinematic chain plan (vertical), composed of two elements in movement and two couplers cinematic engines. It is about the kinematic elements in movement 2 and 3, the assembly 2-3 being moved by the actuator of the second fitted in the coupler A, fixed on the element 1. Therefore the second electric motor attached to the component 1 will drive the element 2 in rotational motion relative to the item 1 and at the same time it will move the entire kinematic chain 2-3. The last actuator (electric motor) fixed by item 2, in B, will turn up the item 3 (relative in relation to the 2).

The rotation of the φ_{10} carried out by the first actuator, is and relative (between items 1 and 0) and absolute (between items 1 and 0).

The rotation of the φ_{20} carried out by the second actuator, is and relative rotation (between elements 2 and 1) and absolute (between items 2 and 0) due to the arrangement of the system.

The rotation of the $\theta = \varphi_{32}$ carried out by the third actuator, is only a relative rotation (between elements 3 and 2) and the corresponding absolute rotation (between items 3 and 0) is a function of the $\theta = \varphi_{32}$ and φ_{20} (Aversa *et al.*, 2016a; 2016b; 2016c; 2016d; 2016e; 2016f).

The kinematic chain 2-3 (consisting of kinematic elements in movement 2 and 3) is a kinematic chain in plan, which fall within a single plan or in one composed of several plane parallel. It is a system kinematic special, which may be studied separately. Then the element 1 (which it drives the kinematic chain 2-3) will be considered as coupler, kinematic couplers engines $A(O_2)$ and $B(O_3)$ becoming, the first fixed coupler and the second mobile coupler, both being kinematic couplers C_5, of rotation.

For the determination of the extent of mobility of the kinematic chain (in plane) 2-3, shall apply to the structural formula given by the relationship (1), where m is the total number of moving parts of the kinematic chain plan; in our case m = 2 (as we are talking about the two kinematic elements in movement noted with 2 and 3 respectively) and the C_5 represents the number kinematic couplers of the fifth-class, in this case $C_5 = 2$ (as we are talking about the couplers A and B or O_2 and O_3):

$$M_3 = 3 \cdot m - 2 \cdot C_5 = 3 \cdot 2 - 2 \cdot 2 = 6 - 4 = 2 \qquad (1)$$

The kinematic chain 2-3 having to the degree of mobility 2, must be actuated by two motors. It is preferred that those two actuators to be two electric motors, direct current, or alternately.

Alternatively the drive of motion may be done and with engines hydraulic, pneumatic, sonic, etc. Structural diagram of the plan kinematic chain 2-3 (Fig. 2) resembles with its kinematic schema.

The driver element 2 is linked to the element 1 considered fixed, through the drive coupler O_2 and driver element 3 is linked to the mobile element 2 through the drive coupler O_3.
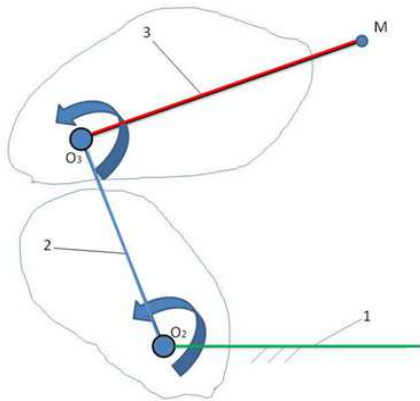
Fig. 2. Structural diagram of the kinematic chain plan 2-3 linked to item 1 considered fixed

As a result there is the kinematic chain open with two degree of mobility, carried out by the two actuators, i.e., the two electric motors, mounted on the motor kinematic couplers A and B or O_2 in question O_3 (Lorell *et al.*, 2003; Merlet, 2000; Miller, 2004; Petrescu *et al.*, 2009; Tsai, 2000).

Direct Kinematic of the Plan Chain 2-3

In Fig. 3 can be tracked cinematic diagram of the chain plan 2-3 open. In direct cinematic are known cinematic parameters φ_{20} and φ_{30} and must be determined by the analytical calculation the parameters x_M and y_M, which represents the scalar coordinates of the point *M* (endeffector *M*).

Are projected the vectors d_2 and d_3 on the Cartesian axis system considered fixed, *xOy* identical with the $x_2O_2y_2$, to obtain the system of equations scaling (2):

$$\begin{cases} x_{2M} \equiv x_M = x_{O_3} + x_{3M} \\ = d_2 \cdot \cos\phi_{20} + d_3 \cdot \cos\phi_{30} = d \cdot \cos\phi \\ y_{2M} \equiv y_M = y_{O_3} + y_{3M} \\ = d_2 \cdot \sin\phi_{20} + d_3 \cdot \sin\phi_{30} = d \cdot \sin\phi \end{cases} \quad (2)$$

After shall be determined in the form of Cartesian coordinates of the *M* using the relations given by the system (2), may be obtained immediately and the φ angle parameters using the relations estab-lished in the framework of the system (3):

$$\begin{cases} d^2 = x_M^2 + y_M^2 \\ d = \sqrt{x_M^2 + y_M^2} \\ \cos\phi = \dfrac{x_M}{d} = \dfrac{x_M}{\sqrt{x_M^2 + y_M^2}} \\ \sin\phi = \dfrac{y_M}{d} = \dfrac{y_M}{\sqrt{x_M^2 + y_M^2}} \\ \phi = sign(\sin\phi) \cdot \arccos(\cos\phi) \end{cases} \quad (3)$$
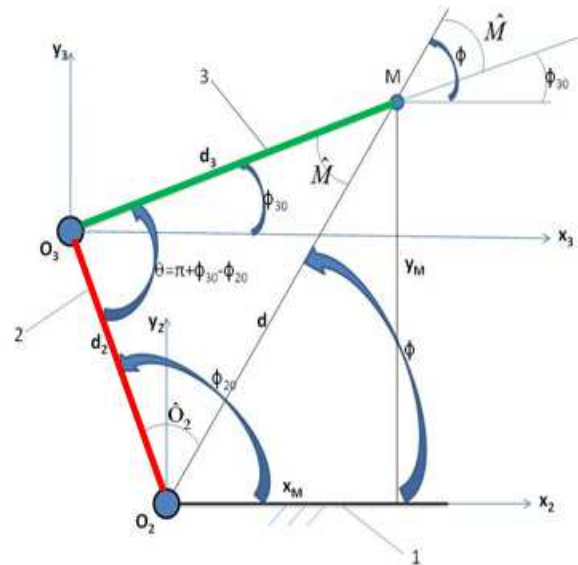

Fig. 3. Kinematic diagram of the kinematic chain plan 2-3 linked to item 1 considered fixed

The system (2) is written in a more concise manner in the form (4) which is derived with time to become the velocities system (5), which being derived with time generates in turn the system of accelerations (6):

$$\begin{cases} x_M = d_2 \cdot \cos\phi_{20} + d_3 \cdot \cos\phi_{30} \\ = d_2 \cdot \cos\phi_{20} + d_3 \cdot \cos(\theta + \phi_{20} - \pi) \\ y_M = d_2 \cdot \sin\phi_{20} + d_3 \cdot \sin\phi_{30} \\ = d_2 \cdot \sin\phi_{20} + d_3 \cdot \sin(\theta + \phi_{20} - \pi) \end{cases} \quad (4)$$

$$\begin{cases} v_M^x \equiv \dot{x}_M = -d_2 \cdot \sin\phi_{20} \cdot \omega_{20} - d_3 \cdot \sin\phi_{30} \cdot \omega_{30} \\ = -d_2 \cdot \sin\phi_{20} \cdot \omega_{20} - d_3 \cdot \sin\phi_{30} \cdot (\dot{\theta} + \omega_{20}) \\ v_M^y \equiv \dot{y}_M = d_2 \cdot \cos\phi_{20} \cdot \omega_{20} + d_3 \cdot \cos\phi_{30} \cdot \omega_{30} \\ = d_2 \cdot \cos\phi_{20} \cdot \omega_{20} + d_3 \cdot \cos\phi_{30} \cdot (\dot{\theta} + \omega_{20}) \end{cases} \quad (5)$$

$$\begin{cases} a_M^x \equiv \ddot{x}_M = -d_2 \cdot \cos\phi_{20} \cdot \omega_{20}^2 - d_3 \cdot \cos\phi_{30} \cdot \omega_{30}^2 \\ = -d_2 \cdot \cos\phi_{20} \cdot \omega_{20}^2 - d_3 \cdot \cos\phi_{30} \cdot (\dot{\theta} + \omega_{20})^2 \\ a_M^y \equiv \ddot{y}_M = -d_2 \cdot \sin\phi_{20} \cdot \omega_{20}^2 - d_3 \cdot \sin\phi_{30} \cdot \omega_{30}^2 \\ = -d_2 \cdot \sin\phi_{20} \cdot \omega_{20}^2 - d_3 \cdot \sin\phi_{30} \cdot (\dot{\theta} + \omega_{20})^2 \end{cases} \quad (6)$$

Remark: The rotation speeds of the actuators were considered to be constant (relations 7):

$$\dot{\phi}_{20} = \omega_{20} = ct; \quad \dot{\theta} = ct \Rightarrow si \quad \omega_{30} = ct.$$
$$\varepsilon_{20} = \ddot{\theta} = \varepsilon_{30} = 0. \quad (7)$$

The relations (3) shall be derived and they and get the velocities (8) and the accelerations (9):

$$\begin{cases} d^2 = x_M^2 + y_M^2 \\ 2 \cdot d \cdot \dot{d} = 2 \cdot x_M \cdot \dot{x}_M + 2 \cdot y_M \cdot \dot{y}_M \\ d \cdot \dot{d} = x_M \cdot \dot{x}_M + y_M \cdot \dot{y}_M \\ \dot{d} = \dfrac{x_M \cdot \dot{x}_M + y_M \cdot \dot{y}_M}{d} \\ d \cdot \cos\phi = x_M \\ d \cdot \sin\phi = y_M \\ \dot{d} \cdot \cos\phi - d \cdot \sin\phi \cdot \dot{\phi} = \dot{x}_M \mid \cdot(-\sin\phi) \\ \dot{d} \cdot \sin\phi + d \cdot \cos\phi \cdot \dot{\phi} = \dot{y}_M \mid \cdot(\cos\phi) \\ \overline{} \\ d \cdot \dot{\phi} = \dot{x}_M \cdot(-\sin\phi) + \dot{y}_M \cdot(\cos\phi) \\ \dot{\phi} = \dfrac{\dot{y}_M \cdot \cos\phi - \dot{x}_M \cdot \sin\phi}{d} \\ \overline{} \\ \dot{d} = \dfrac{x_M \cdot \dot{x}_M + y_M \cdot \dot{y}_M}{d} \end{cases} \quad (8)$$

$$\begin{cases} d^2 = x_M^2 + y_M^2 \\ 2 \cdot d \cdot \dot{d} = 2 \cdot x_M \cdot \dot{x}_M + 2 \cdot y_M \cdot \dot{y}_M \\ d \cdot \dot{d} = x_M \cdot \dot{x}_M + y_M \cdot \dot{y}_M \\ \dot{d}^2 + d \cdot \ddot{d} = \dot{x}_M^2 + x_M \cdot \ddot{x}_M + \dot{y}_M^2 + y_M \cdot \ddot{y}_M \\ \ddot{d} = \dfrac{\dot{x}_M^2 + x_M \cdot \ddot{x}_M + \dot{y}_M^2 + y_M \cdot \ddot{y}_M - \dot{d}^2}{d} \\ d \cdot \cos\phi = x_M \\ d \cdot \sin\phi = y_M \\ \dot{d} \cdot \cos\phi - d \cdot \sin\phi \cdot \dot{\phi} = \dot{x}_M \mid \cdot(-\sin\phi) \\ \dot{d} \cdot \sin\phi + d \cdot \cos\phi \cdot \dot{\phi} = \dot{y}_M \mid \cdot(\cos\phi) \\ \overline{} \\ d \cdot \dot{\phi} = -\dot{x}_M \cdot \sin\phi + \dot{y}_M \cdot \cos\phi \\ \dot{d} \cdot \dot{\phi} + d \cdot \ddot{\phi} = \ddot{y}_M \cdot \cos\phi - \dot{y}_M \cdot \sin\phi \cdot \dot{\phi} \\ -\ddot{x}_M \cdot \sin\phi - \dot{x}_M \cdot \cos\phi \cdot \dot{\phi} \\ \ddot{\phi} \\ = \dfrac{\ddot{y}_M \cdot \cos\phi - \ddot{x}_M \cdot \sin\phi - \dot{y}_M \cdot \sin\phi \cdot \dot{\phi} - \dot{x}_M \cdot \cos\phi \cdot \dot{\phi} - \dot{d} \cdot \dot{\phi}}{d} \\ \overline{} \\ \ddot{d} = \dfrac{\dot{x}_M^2 + x_M \cdot \ddot{x}_M + \dot{y}_M^2 + y_M \cdot \ddot{y}_M - \dot{d}^2}{d} \end{cases} \quad (9)$$

The following will determine the positions, speeds and accelerations, according to the positions of the scales point O_3. It starts from the scalar coordinates of the point O_3 (the relationships 10):

$$\begin{cases} x_{O_3} = d_2 \cdot \cos\phi_{20} \\ y_{O_3} = d_2 \cdot \sin\phi_{20} \end{cases} \quad (10)$$

Is then determined scaling speeds and accelerations of the point O_3, by differentiation of the system (10), in which shall be replaced after derivation the products d.cos or d.sin with the respective positions, x_{O3} or y_{O3}, which become (in this way) variables (see relations 11 and 12):

$$\begin{cases} \dot{x}_{O_3} = -d_2 \cdot \sin\phi_{20} \cdot \omega_{20} = -y_{O_3} \cdot \omega_{20} \\ \dot{y}_{O_3} = d_2 \cdot \cos\phi_{20} \cdot \omega_{20} = x_{O_3} \cdot \omega_{20} \end{cases} \quad (11)$$

$$\begin{cases} \ddot{x}_{O_3} = -d_2 \cdot \cos\phi_{20} \cdot \omega_{20}^2 = -x_{O_3} \cdot \omega_{20}^2 \\ \ddot{y}_{O_3} = -d_2 \cdot \sin\phi_{20} \cdot \omega_{20}^2 = -y_{O_3} \cdot \omega_{20}^2 \end{cases} \quad (12)$$

Had been put in evidence in this mode the scalar speeds and accelerations of point O_3 according to their original positions (scaling) and absolute angular speed of the element 2.

The angular speed was considered to be constant. The technique of the determination of the velocities and acceleration depending on the positions, is extremely useful in the study of the dynamics of the system, of the vibrations and noise caused by the respective system.

This technique is common in the study of vibration system. Known the vibrations of scalar positions of point O_3 and then easily determine the vibration velocities and accelerations that point and other points of the system all the functions of positions scaling known point O_3.

Also by this technique can calculate local noise levels at various points in the system and the overall level of noise generated, with a precision good enough, compared with experimental measurements obtained with adequate equipment.

Absolute velocity of O_3 point (speed module) is given by relation (13) and the acceleration of O_3 can be written with expression (14):

$$\begin{cases} v_{O_3} = \sqrt{\dot{x}_{O_3}^2 + \dot{y}_{O_3}^2} \\ = \sqrt{d_2^2 \cdot \omega_{20}^2 \cdot \sin^2\phi_{20} + d_2^2 \cdot \omega_{20}^2 \cdot \cos^2\phi_{20}} \\ = \sqrt{d_2^2 \cdot \omega_{20}^2} = d_2 \cdot \omega_{20} \end{cases} \quad (13)$$

$$\begin{cases} a_{O_3} = \sqrt{\ddot{x}_{O_3}^2 + \ddot{y}_{O_3}^2} \\ = \sqrt{d_2^2 \cdot \omega_{20}^4 \cdot \cos^2\phi_{20} + d_2^2 \cdot \omega_{20}^4 \cdot \sin^2\phi_{20}} \\ = \sqrt{d_2^2 \cdot \omega_{20}^4} = d_2 \cdot \omega_{20}^2 \end{cases} \quad (14)$$

In the following will determine the cinematic scalar parameters of the point M, endeffector, depending and on the parameters of the position of points O_3 and M (the systems of relations 15-17):

$$
\begin{cases}
x_M = x_{O_3} + d_3 \cdot \cos\phi_{30} \\
y_M = y_{O_3} + d_3 \cdot \sin\phi_{30} \\
d_3 \cdot \cos\phi_{30} = x_M - x_{O_3} \\
d_3 \cdot \sin\phi_{30} = y_M - y_{O_3}
\end{cases}
\tag{15}
$$

$$
\begin{cases}
\dot{x}_M = \dot{x}_{O_3} - d_3 \cdot \sin\phi_{30} \cdot \dot{\phi}_{30} \\
= -y_{O_3} \cdot \omega_{20} + (y_{O_3} - y_M) \cdot (\omega_{20} + \dot{\theta}) \\
= y_{O_3} \cdot \dot{\theta} - y_M \cdot (\omega_{20} + \dot{\theta}) \\
= (y_{O_3} - y_M) \cdot \dot{\theta} - y_M \cdot \omega_{20} \\
\dot{y}_M = \dot{y}_{O_3} + d_3 \cdot \cos\phi_{30} \cdot \dot{\phi}_{30} \\
= x_{O_3} \cdot \omega_{20} + (x_M - x_{O_3}) \cdot (\omega_{20} + \dot{\theta}) \\
= x_M \cdot (\omega_{20} + \dot{\theta}) - x_{O_3} \cdot \dot{\theta} \\
= (x_M - x_{O_3}) \cdot \dot{\theta} + x_M \cdot \omega_{20} \\
\dot{y}_{O_3} - \dot{y}_M = -d_3 \cdot \cos\phi_{30} \cdot (\omega_{20} + \dot{\theta}) \\
\dot{x}_M - \dot{x}_{O_3} = -d_3 \cdot \sin\phi_{30} \cdot (\omega_{20} + \dot{\theta})
\end{cases}
\tag{16}
$$

$$
\begin{cases}
\ddot{x}_M = (\dot{y}_{O_3} - \dot{y}_M) \cdot \dot{\theta} - \dot{y}_M \cdot \omega_{20} \\
\ddot{y}_M = (\dot{x}_M - \dot{x}_{O_3}) \cdot \dot{\theta} + \dot{x}_M \cdot \omega_{20} \\
\dot{y}_{O_3} - \dot{y}_M = (x_{O_3} - x_M) \cdot (\omega_{20} + \dot{\theta}) \\
\dot{x}_M - \dot{x}_{O_3} = (y_{O_3} - y_M) \cdot (\omega_{20} + \dot{\theta}) \\
\ddot{x}_M = (x_{O_3} - x_M) \cdot \\
\cdot (\omega_{20} + \dot{\theta}) \cdot \dot{\theta} + (x_{O_3} - x_M) \cdot \dot{\theta} \cdot \omega_{20} - x_M \cdot \omega_{20}^2 \\
\ddot{y}_M = (y_{O_3} - y_M) \cdot \\
\cdot (\omega_{20} + \dot{\theta}) \cdot \dot{\theta} + (y_{O_3} - y_M) \cdot \dot{\theta} \cdot \omega_{20} - y_M \cdot \omega_{20}^2 \\
\ddot{x}_M = 2 \cdot (x_{O_3} - x_M) \cdot \dot{\theta} \cdot \omega_{20} \\
+ (x_{O_3} - x_M) \cdot \dot{\theta}^2 - x_M \cdot \omega_{20}^2 \\
\ddot{y}_M = 2 \cdot (y_{O_3} - y_M) \cdot \dot{\theta} \cdot \omega_{20} \\
+ (y_{O_3} - y_M) \cdot \dot{\theta}^2 - y_M \cdot \omega_{20}^2 \\
\ddot{x}_M = (x_{O_3} - x_M) \cdot (2 \cdot \dot{\theta} \cdot \omega_{20} + \dot{\theta}^2) - x_M \cdot \omega_{20}^2 \\
\ddot{y}_M = (y_{O_3} - y_M) \cdot (2 \cdot \dot{\theta} \cdot \omega_{20} + \dot{\theta}^2) - y_M \cdot \omega_{20}^2 \\
\ddot{x}_M = (x_{O_3} - x_M) \cdot (\omega_{20} + \dot{\theta})^2 - x_{O_3} \cdot \omega_{20}^2 \\
\ddot{y}_M = (y_{O_3} - y_M) \cdot (\omega_{20} + \dot{\theta})^2 - y_{O_3} \cdot \omega_{20}^2
\end{cases}
\tag{17}
$$

Inverse Kinematic of the Plan Chain 2-3

In Fig. 3 can be tracked cinematic diagram of the chain plan 2-3 open. In inverse cinematic are known the parameters x_M and y_M (which represent the scalar coordinates of the point M, endeffector M), imposed parameters and must be determined by the analytical calculations the cinematic parameters φ_{20} and φ_{30}. Determine first the intermediate parameters d and φ with relations (18):

$$
\begin{cases}
d^2 = x_M^2 + y_M^2; \quad d = \sqrt{x_M^2 + y_M^2} \\
\cos\phi = \dfrac{x_M}{d} = \dfrac{x_M}{\sqrt{x_M^2 + y_M^2}}; \\
\sin\phi = \dfrac{y_M}{d} = \dfrac{y_M}{\sqrt{x_M^2 + y_M^2}} \\
\phi = semn(\sin\phi) \cdot \arccos(\cos\phi)
\end{cases}
\tag{18}
$$

In triangle certain O_2O_3M know the lengths of the three sides, d^2, d^3 (constant) and d (variable) so that it can be determined depending on the lengths of the sides all other elements of the triangle, specifically its angles and trigonometric functions of their (particularly interest us sin and cos). You can use various methods and they will be presented further two (as the most representatives): Method trigonometric and geometric method.

Trigonometric Method, Determining Positions

Writing of positions scaling Equations 19:

$$
\begin{cases}
d_2 \cdot \cos\phi_{20} + d_3 \cdot \cos\phi_{30} = x_M \\
d_2 \cdot \sin\phi_{20} + d_3 \cdot \sin\phi_{30} = y_M \\
\cos^2\phi_{20} + \sin^2\phi_{20} = 1 \\
\cos^2\phi_{30} + \sin^2\phi_{30} = 1
\end{cases}
\tag{19}
$$

The problem with these two equations scaling, trigonometric, with two unknowns (φ_{20} and φ_{30}) is that they transcend (they are trigonometric equations, transcendental, where φ_{20} unknown does not appear directly but in the form $\cos\varphi_{20}$ and $\sin\varphi_{20}$, so in reality the two trigonometric equations no longer have only two unknowns but four: $\cos\varphi_{20}$, $\sin\varphi_{20}$, $\cos\varphi_{30}$ and $\sin\varphi_{30}$).

To solve the system we need more two equations, so that in the system (19) were also added more two trigonometric equations, exactly the core "golden" as they are more say, for the angle φ_{20} and separate for the φ_{30} angle. In order to solve the first two equations of the system (19) are written in the form (20).

$$
\begin{cases}
d_2 \cdot \cos\phi_{20} - x_M = -d_3 \cdot \cos\phi_{30} \\
d_2 \cdot \sin\phi_{20} - y_M = -d_3 \cdot \sin\phi_{30}
\end{cases}
\tag{20}
$$

Each equation of the system (20) are squared, then added together both equations (squared) to obtain the equation of the form (21):

$$
\begin{cases}
d_2^2 \cdot (\cos^2\phi_{20} + \sin^2\phi_{20}) + x_M^2 + y_M^2 \\
-2 \cdot d_2 \cdot x_M \cdot \cos\phi_{20} \\
-2 \cdot d_2 \cdot y_M \cdot \sin\phi_{20} = d_3^2 \cdot (\cos^2\phi_{30} + \sin^2\phi_{30})
\end{cases}
\tag{21}
$$

Now is the time to use the two "equations gold" written at the end trigonometric system (2), with which Equation 21 becomes simplified expression (22):

$$\begin{cases} d_2^2 + x_M^2 + y_M^2 - 2 \cdot d_2 \cdot x_M \cdot \cos\phi_{20} \\ -2 \cdot d_2 \cdot y_M \cdot \sin\phi_{20} = d_3^2 \end{cases} \tag{22}$$

Arrange the terms of the Equation 22 in the most convenient form (23).

$$\begin{cases} d_2^2 - d_3^2 + x_M^2 + y_M^2 \\ = 2 \cdot d_2 \cdot (x_M \cdot \cos\phi_{20} + y_M \cdot \sin\phi_{20}) \end{cases} \tag{23}$$

Divide the Equation 23 with $2.d_2$ and follows a new form (24):

$$x_M \cdot \cos\phi_{20} + y_M \cdot \sin\phi_{20} = \frac{d_2^2 - d_3^2 + x_M^2 + y_M^2}{2 \cdot d_2} \tag{24}$$

Observing Fig. 2, from system (18) may be explicit the relation (25):

$$x_M^2 + y_M^2 = d^2 \tag{25}$$

Insert (25) to (24) and amplified fraction the right to d so that the expression (24) takes the form (26) a more useful form:

$$x_M \cdot \cos\phi_{20} + y_M \cdot \sin\phi_{20} = \frac{d_2^2 + d^2 - d_3^2}{2 \cdot d_2 \cdot d} \cdot d \tag{26}$$

Now, one can enter the expression cosine angle O_2, according to some O_2O_3M triangle (27):

$$\cos\hat{O}_2 = \frac{d_2^2 + d^2 - d_3^2}{2 \cdot d_2 \cdot d} \tag{27}$$

Using the expression (27) Equation 26 becomes simplified (28):

$$x_M \cdot \cos\phi_{20} - d \cdot \cos\hat{O}_2 = -y_M \cdot \sin\phi_{20} \tag{28}$$

Needs to remove $\sin\phi_{20}$, for which we isolated the term in sin and rose squared Equation 28 for using the equation gold trigonometric angle ϕ_{20} to transform sin in cosine, equation becoming one of the second degree in $\cos\phi_{20}$. After squaring (28) becomes (29):

$$\begin{cases} x_M^2 \cdot \cos^2\phi_{20} + d^2 \cdot \cos^2\hat{O}_2 \\ -2 \cdot d \cdot x_M \cdot \cos\hat{O}_2 \cdot \cos\phi_{20} = y_M^2 \cdot \sin^2\phi_{20} \end{cases} \tag{29}$$

Use golden formula and expression (29) becomes (30) to be arranged conveniently by grouping terms of bringing to the form (31):

$$\begin{cases} x_M^2 \cdot \cos^2\phi_{20} + d^2 \cdot \cos^2\hat{O}_2 \\ -2 \cdot d \cdot x_M \cdot \cos\hat{O}_2 \cdot \cos\phi_{20} = y_M^2 - y_M^2 \cdot \cos^2\phi_{20} \end{cases} \tag{30}$$

$$\begin{cases} (x_M^2 + y_M^2) \cdot \cos^2\phi_{20} - 2 \cdot d \cdot x_M \cdot \cos\hat{O}_2 \cdot \cos\phi_{20} \\ -(y_M^2 - d^2 \cdot \cos^2\hat{O}_2) = 0 \end{cases} \tag{31}$$

Discriminant of Equation 31 in the second degree in cosine may be calculated by Equation 32:

$$\begin{cases} \Delta = d^2 \cdot x_M^2 \cdot \cos^2\hat{O}_2 + d^2 \cdot (y_M^2 - d^2 \cdot \cos^2\hat{O}_2) \\ = d^2 \cdot (x_M^2 \cdot \cos^2\hat{O}_2 + y_M^2 - d^2 \cdot \cos^2\hat{O}_2) \\ = d^2 \cdot (y_M^2 - y_M^2 \cdot \cos^2\hat{O}_2) \\ = d^2 \cdot y_M^2 \cdot (1 - \cos^2\hat{O}_2) = d^2 \cdot y_M^2 \cdot \sin^2\hat{O}_2 \end{cases} \tag{32}$$

Radical second order of discriminant is expressed as (33):

$$R = \sqrt{\Delta} = \sqrt{d^2 \cdot y_M^2 \cdot \sin^2\hat{O}_2} = d \cdot y_M \cdot \sin\hat{O}_2 \tag{33}$$

Solutions to Equation 31 in the second degree in cosine are written in the form (34):

$$\begin{cases} \cos\phi_{20_{1,2}} = \frac{d \cdot x_M \cdot \cos\hat{O}_2 \mp d \cdot y_M \cdot \sin\hat{O}_2}{d^2} \\ = \frac{x_M \cdot \cos\hat{O}_2 \mp y_M \cdot \sin\hat{O}_2}{d} \\ = \frac{x_M}{d} \cdot \cos\hat{O}_2 \mp \frac{y_M}{d} \cdot \sin\hat{O}_2 \end{cases} \tag{34}$$

Further, in solutions (34) are replaced reports with corresponding trigonometric functions of angle φ, expressions (34) gaining form (35):

$$\begin{cases} \cos\phi_{20_{1,2}} = \frac{x_M}{d} \cdot \cos\hat{O}_2 \mp \frac{y_M}{d} \cdot \sin\hat{O}_2 \\ = \cos\phi \cdot \cos\hat{O}_2 \mp \sin\phi \cdot \sin\hat{O}_2 \\ = \cos(\phi \pm \hat{O}_2) \\ \cos\phi_{20} = \cos(\phi \pm \hat{O}_2) \end{cases} \tag{35}$$

One turns now to Equation 29 that it will be order in form (36) with a view to resolving them in sin. Equation 36 rises to the square and using trigonometric equation golden of angle ϕ_{20}, is obtained form (37):

$$x_M \cdot \cos\phi_{20} = d \cdot \cos\hat{O}_2 - y_M \cdot \sin\phi_{20} \tag{36}$$

$$\begin{cases} x_M^2 \cdot \cos^2 \phi_{20} = d^2 \cdot \cos^2 \hat{O}_2 + y_M^2 \cdot \sin^2 \phi_{20} \\ -2 \cdot y_M \cdot d \cdot \cos \hat{O}_2 \cdot \sin \phi_{20} \\ x_M^2 - x_M^2 \cdot \sin^2 \phi_{20} = d^2 \cdot \cos^2 \hat{O}_2 + y_M^2 \cdot \sin^2 \phi_{20} \\ -2 \cdot y_M \cdot d \cdot \cos \hat{O}_2 \cdot \sin \phi_{20} \\ (x_M^2 + y_M^2) \cdot \sin^2 \phi_{20} - 2 \cdot y_M \cdot d \cdot \cos \hat{O}_2 \cdot \sin \phi_{20} \\ -(x_M^2 - d^2 \cdot \cos^2 \hat{O}_2) = 0 \\ d^2 \cdot \sin^2 \phi_{20} - 2 \cdot y_M \cdot d \cdot \cos \hat{O}_2 \cdot \sin \phi_{20} \\ -(x_M^2 - d^2 \cdot \cos^2 \hat{O}_2) = 0 \end{cases} \quad (37)$$

Discriminant Equation 37 in the second degree in cosine takes the form (38):

$$\begin{cases} \Delta = y_M^2 \cdot d^2 \cdot \cos^2 \hat{O}_2 + d^2 \cdot (x_M^2 - d^2 \cdot \cos^2 \hat{O}_2) \\ d^2 \cdot (x_M^2 + y_M^2 \cdot \cos^2 \hat{O}_2 - x_M^2 \cdot \cos^2 \hat{O}_2 - y_M^2 \cdot \cos^2 \hat{O}_2) \\ = d^2 \cdot (x_M^2 - x_M^2 \cdot \cos^2 \hat{O}_2) = d^2 \cdot x_M^2 \cdot \sin^2 \hat{O}_2 \end{cases} \quad (38)$$

Solutions to Equation 36 are written now as (39):

$$\begin{cases} \sin \phi_{20} = \dfrac{y_M \cdot d \cdot \cos \hat{O}_2 \pm x_M \cdot d \cdot \sin \hat{O}_2}{d^2} \\ = \dfrac{y_M \cdot \cos \hat{O}_2 \pm x_M \cdot \sin \hat{O}_2}{d} \\ = \dfrac{y_M}{d} \cdot \cos \hat{O}_2 \pm \dfrac{x_M}{d} \cdot \sin \hat{O}_2 \\ = \sin \phi \cdot \cos \hat{O}_2 \pm \cos \phi \cdot \sin \hat{O}_2 = \sin(\phi \pm \hat{O}_2) \\ \\ \sin \phi_{20} = \sin(\phi \pm \hat{O}_2) \end{cases} \quad (39)$$

From resulted relations (40) it can write the basic relationship (41):

$$\begin{cases} \cos \phi_{20} = \cos(\phi \pm \hat{O}_2) \\ \sin \phi_{20} = \sin(\phi \pm \hat{O}_2) \end{cases} \quad (40)$$

$$\phi_{20} = \phi \pm \hat{O}_2 \quad (41)$$

The procedure for determining the angle φ_{30}, starting again from the system (19), where the first two transcendental equations are rewritten as (42) in order to eliminate φ_{20} angle this time:

$$\begin{cases} d_2 \cdot \cos \phi_{20} = x_M - d_3 \cdot \cos \phi_{30} \\ d_2 \cdot \sin \phi_{20} = y_M - d_3 \cdot \sin \phi_{30} \end{cases} \quad (42)$$

Raises the two Equation 42 squared and added together, resulting in the equation of the form (43), which is arranged in the most convenient forms (44) and (45):

$$\begin{cases} d_2^2 = x_M^2 + y_M^2 + d_3^2 \\ -2 \cdot d_3 \cdot x_M \cdot \cos \phi_{30} - 2 \cdot d_3 \cdot y_M \cdot \sin \phi_{30} \end{cases} \quad (43)$$

$$x_M \cdot \cos \phi_{30} + y_M \cdot \sin \phi_{30} = d \cdot \frac{d^2 + d_3^2 - d_2^2}{2 \cdot d \cdot d_3} \quad (44)$$

$$x_M \cdot \cos \phi_{30} + y_M \cdot \sin \phi_{30} = d \cdot \cos \hat{M} \quad (45)$$

Need to determine first the cosine so that we isolate initially term in sin, Equation 45 with a special its form (46), which by squaring generates expression (47), an expression that arranges form (48):

$$x_M \cdot \cos \phi_{30} - d \cdot \cos \hat{M} = -y_M \cdot \sin \phi_{30} \quad (46)$$

$$\begin{cases} x_M^2 \cdot \cos^2 \phi_{30} + d^2 \cdot \cos^2 \hat{M} \\ -2 \cdot d \cdot x_M \cdot \cos \hat{M} \cdot \cos \phi_{30} = y_M^2 - y_M^2 \cdot \cos^2 \phi_{30} \end{cases} \quad (47)$$

$$\begin{cases} d^2 \cdot \cos^2 \phi_{30} - 2 \cdot d \cdot x_M \cdot \cos \hat{M} \cdot \cos \phi_{30} \\ -(y_M^2 - d^2 \cdot \cos^2 \hat{M}) = 0 \end{cases} \quad (48)$$

Equation 48 is a second degree equation in cosine with solutions given by expression (49):

$$\begin{cases} \cos \phi_{30} = \\ \dfrac{d \cdot x_M \cdot \cos \hat{M} \pm \sqrt{d^2 \cdot x_M^2 \cdot \cos^2 \hat{M} + d^2 \cdot (y_M^2 - d^2 \cdot \cos^2 \hat{M})}}{d^2} \\ = \dfrac{d \cdot x_M \cdot \cos \hat{M} \pm \sqrt{d^2 \cdot y_M^2 \cdot (1 - \cos^2 \hat{M})}}{d^2} \\ = \dfrac{d \cdot x_M \cdot \cos \hat{M} \pm d \cdot y_M \cdot \sin \hat{M}}{d^2} \\ = \dfrac{x_M}{d} \cdot \cos \hat{M} \pm \dfrac{y_M}{d} \cdot \sin \hat{M} \\ = \cos \phi \cdot \cos \hat{M} \pm \sin \phi \cdot \sin \hat{M} = \cos(\phi \mp \hat{M}) \\ \\ \cos \phi_{30} = \cos(\phi \mp \hat{M}) \end{cases} \quad (49)$$

Further write Equation 45 as (50), which is isolated this time cosine to eliminate it and then to determine the term sin. The expression (52) is an equation of the second degree in sin, admitting the solutions given by (53):

$$x_M \cdot \cos \phi_{30} = d \cdot \cos \hat{M} - y_M \cdot \sin \phi_{30} \quad (50)$$

Equation 50 amounts to the square and obtain the equation of the form (51) suitable to be arranged in the form (52):

$$\begin{cases} x_M^2 \cdot (1 - \sin^2 \phi_{30}) = d^2 \cdot \cos^2 \hat{M} \\ + y_M^2 \cdot \sin^2 \phi_{30} - 2 \cdot y_M \cdot d \cdot \cos \hat{M} \cdot \sin \phi_{30} \end{cases} \quad (51)$$

$$\begin{cases} d^2 \cdot \sin^2\phi_{30} - 2 \cdot y_M \cdot d \cdot \cos\hat{M} \cdot \sin\phi_{30} \\ -(x_M^2 - d^2 \cdot \cos^2\hat{M}) = 0 \end{cases} \tag{52}$$

$$\begin{cases} \sin\phi_{30} = \\ \dfrac{d \cdot y_M \cdot \cos\hat{M} \mp \sqrt{d^2 \cdot y_M^2 \cdot \cos^2\hat{M} + d^2 \cdot (x_M^2 - d^2 \cdot \cos^2\hat{M})}}{d^2} \\ = \dfrac{d \cdot y_M \cdot \cos\hat{M} \mp \sqrt{d^2 \cdot x_M^2 \cdot (1 - \cos^2\hat{M})}}{d^2} \\ = \dfrac{d \cdot y_M \cdot \cos\hat{M} \mp d \cdot x_M \cdot \sin\hat{M}}{d^2} \\ = \dfrac{y_M}{d} \cdot \cos\hat{M} \mp \dfrac{x_M}{d} \cdot \sin\hat{M} \\ = \sin\phi \cdot \cos\hat{M} \mp \cos\phi \cdot \sin\hat{M} = \sin(\phi \mp \hat{M}) \\ \\ \sin\phi_{30} = \sin(\phi \mp \hat{M}) \end{cases} \tag{53}$$

Then, are qualified only the relations (54) and (55):

$$\begin{cases} \cos\phi_{30} = \cos(\phi \mp \hat{M}) \\ \sin\phi_{30} = \sin(\phi \mp \hat{M}) \end{cases} \tag{54}$$

$$\phi_{30} = \phi \mp \hat{M} \tag{55}$$

Trigonometric Method, Determining Velocities

Starting from relations (56) required in the study of velocities in inverse kinematics:

$$\begin{cases} \dot\phi = \dfrac{\dot y_M \cdot \cos\phi - \dot x_M \cdot \sin\phi}{d} \\ \dot d = \dfrac{x_M \cdot \dot x_M + y_M \cdot \dot y_M}{d} \end{cases} \tag{56}$$

It starts from the relationship linking the cosine of the angle $\hat{O}_2$ of the triangle sides, which derives in relation to time and thus obtains the value $\dot{\hat{O}}_2$ written simply $\dot{O}_2$ (relations 57):

$$\begin{cases} 2 \cdot d_2 \cdot d \cdot \cos O_2 = d_2^2 - d_3^2 + d^2 \\ 2 \cdot d_2 \cdot \dot d \cdot \cos O_2 - 2 \cdot d_2 \cdot d \cdot \sin O_2 \cdot \dot O_2 \\ = 2 \cdot d \cdot \dot d \Rightarrow \dot O_2 = \dfrac{d_2 \cdot \dot d \cdot \cos O_2 - d \cdot \dot d}{d_2 \cdot d \cdot \sin O_2} \end{cases} \tag{57}$$

It derives the Equation 24 to give the angular velocity $\omega_{20} \equiv \dot\phi_{20}$ (relation 58):

$$\omega_{20} \equiv \dot\phi_{20} = \dot\phi \pm \dot O_2 \tag{58}$$

To determine ω_{20} (relationship 58) we need $\dot\phi$ to be calculated from (56) and $\dot O_2$ which is determined from (57). In turn $\dot O_2$ requires for its calculation $\dot d$ that calculates all from system (56). Inlet velocities $\dot x_M$ and $\dot y_M$ are known, are required as input, or choose convenient, or can be calculated based on criteria imposed. Similarly, one determines and the angular velocity $\omega_{30} \equiv \dot\phi_{30}$ (relation 59). It derives the equation (55) to obtain the angular velocity $\omega_{30} \equiv \dot\phi_{30}$ (expression 60). Then, $\dot\phi$ is calculated using the expression system known already (56) and $\dot M$ is determined from the system (59) and by the help of the system (56) which it causes and $\dot d$:

$$\begin{cases} 2 \cdot d_3 \cdot d \cdot \cos M = d_3^2 - d_2^2 + d^2 \\ 2 \cdot d_3 \cdot \dot d \cdot \cos M - 2 \cdot d_3 \cdot d \cdot \sin M \cdot \dot M \\ = 2 \cdot d \cdot \dot d \Rightarrow \dot M = \dfrac{d_3 \cdot \dot d \cdot \cos M - d \cdot \dot d}{d_3 \cdot d \cdot \sin M} \end{cases} \tag{59}$$

$$\omega_{30} \equiv \dot\phi_{30} = \dot\phi \mp \dot M \tag{60}$$

Trigonometric Method, Determining Accelerations

It starts from the relationships (61) required in the study accelerations in inverse kinematics. The relation of the system (57) derived a second time to time, gives the system (62):

$$\begin{cases} \ddot\phi = \dfrac{\ddot y_M \cdot \cos\phi - \ddot x_M \cdot \sin\phi}{d} \\ + \dfrac{-\dot y_M \cdot \sin\phi \cdot \dot\phi - \dot x_M \cdot \cos\phi \cdot \dot\phi - \dot d \cdot \dot\phi}{d} \\ \ddot d = \dfrac{\dot x_M^2 + x_M \cdot \ddot x_M + \dot y_M^2 + y_M \cdot \ddot y_M - \dot d^2}{d} \end{cases} \tag{61}$$

$$\begin{cases} 2 \cdot d_2 \cdot d \cdot \cos O_2 = d_2^2 - d_3^2 + d^2 \\ 2 \cdot d_2 \cdot \dot d \cdot \cos O_2 - 2 \cdot d_2 \cdot d \cdot \sin O_2 \cdot \dot O_2 = 2 \cdot d \cdot \dot d \\ \Rightarrow d_2 \cdot d \cdot \sin O_2 \cdot \dot O_2 = d_2 \cdot \dot d \cdot \cos O_2 - d \cdot \dot d \\ \ddot O_2 = \dfrac{\ddot d d_2 \cos O_2 - \ddot d d - 2\dot d d_2 \sin O_2 \cdot \dot O_2 - d d_2 \cos O_2 \cdot \dot O_2^2 - \dot d^2}{d_2 \cdot d \cdot \sin O_2} \end{cases} \tag{62}$$

The following is derives the expression (58) to obtains expression (63), which generates angular acceleration absolute $\varepsilon_2 \equiv \varepsilon_{20}$, that is calculated with $\ddot\phi$ out of the system (61) and with $\ddot O_2$ out of the system (45) and for the determination of $\ddot O_2$ longer needed $\ddot d$ removed all from relations (61):

$$\varepsilon_2 \equiv \varepsilon_{20} = \dot{\omega}_{20} \equiv \ddot{\phi}_{20} = \ddot{\phi} \pm \ddot{O}_2 \qquad (63)$$

Now it derives a second time (59) and obtains the system (64):

$$\begin{cases} 2 \cdot d_3 \cdot d \cdot \cos M = d_3^2 - d_2^2 + d^2 \\ 2 \cdot d_3 \cdot \dot{d} \cdot \cos M - 2 \cdot d_3 \cdot d \cdot \sin M \cdot \dot{M} = 2 \cdot d \cdot \dot{d} \\ \Rightarrow d_3 \cdot d \cdot \sin M \cdot \dot{M} = d_3 \cdot \dot{d} \cdot \cos M - d \cdot \dot{d} \\ \ddot{M} = \dfrac{\ddot{d}d_3 \cos M - \ddot{d}d - 2\dot{d}d_3 \sin M \cdot \dot{M} - dd_3 \cos M \cdot \dot{M}^2 - \dot{d}^2}{d_3 \cdot d \cdot \sin M} \end{cases} \qquad (64)$$

It derives again the time relation (60) and we obtain the expression (65) of the angular acceleration absolute $\varepsilon_3 \equiv \varepsilon_{30}$ which is determined with $\ddot{\phi}$ and $\ddot{M}$:

$$\varepsilon_3 \equiv \varepsilon_{30} = \dot{\omega}_{30} \equiv \ddot{\phi}_{30} = \ddot{\phi} \mp \ddot{M} \qquad (65)$$

Geometric Method, Determining Positions

It starts by writing positions equations, geometric (geometro-analytical) (66). Scaling coordinates (x_M, y_M) of point M (endeffector) are known and must be determined and scaling coordinates of the point O_3, which we denote by (x, y). The relationships of the system (66) are obtained by writing the geometro-analytical equations of the circles, the rays d_3 and d_2 respectively:

$$\begin{cases} (x - x_M)^2 + (y - y_M)^2 = d_3^2 \\ x^2 + y^2 = d_2^2 \end{cases} \qquad (66)$$

Release binomials of the first equation of the system, enter the second equation into the first, is also used and expression $d^2 = x_M^2 + y_M^2$, of amplify fraction with the factor convenient $d \cdot d_2$ and obtaining the final expression of system (67), which is written together with the equation of the second from system (66) in the new system (68) who needs to be addressed:

$$\begin{cases} x_M \cdot x + y_M \cdot y = \dfrac{d_2^2 + d^2 - d_3^2}{2} \\ x_M \cdot x + y_M \cdot y = d \cdot d_2 \cdot \dfrac{d_2^2 + d^2 - d_3^2}{2 \cdot d \cdot d_2} \\ x_M \cdot x + y_M \cdot y = d \cdot d_2 \cdot \cos O_2 \end{cases} \qquad (67)$$

$$\begin{cases} x_M \cdot x + y_M \cdot y = d \cdot d_2 \cdot \cos O_2 \\ x^2 + y^2 = d_2^2 \end{cases} \qquad (68)$$

From the first equation of the system (68) explains the value of y, which rose and squared (69):

$$\begin{cases} y = \dfrac{d \cdot d_2 \cdot \cos O_2 - x_M \cdot x}{y_M} \\ y^2 = \dfrac{d^2 \cdot d_2^2 \cdot \cos^2 O_2 + x_M^2 \cdot x^2 - 2 \cdot x_M \cdot d_2 \cdot d \cdot \cos O_2 \cdot x}{y_M^2} \end{cases} \qquad (69)$$

The phrase 'second (69) is introduced into a second relationship of (68) to give Equation 70, which are arranged conveniently in the form (71):

$$\begin{cases} y_M^2 \cdot x^2 + d^2 \cdot d_2^2 \cdot \cos^2 O_2 + x_M^2 \cdot x^2 \\ -2 \cdot x_M \cdot d_2 \cdot d \cdot \cos O_2 \cdot x - y_M^2 \cdot d_2^2 = 0 \end{cases} \qquad (70)$$

$$\begin{cases} d^2 \cdot x^2 - 2 \cdot x_M \cdot d_2 \cdot d \cdot \cos O_2 \cdot x \\ -d_2^2 \cdot (y_M^2 - d^2 \cdot \cos^2 O_2) = 0 \end{cases} \qquad (71)$$

Equation 71 is an equation of the second degree in x, which allows real solutions (72):

$$\begin{cases} x = \dfrac{x_M \cdot d_2 \cdot d \cdot \cos O_2}{d^2} \mp \\ \qquad \dfrac{\sqrt{x_M^2 \cdot d_2^2 \cdot d^2 \cdot \cos O_2 + d^2 \cdot d_2^2 \cdot (y_M^2 - d^2 \cdot \cos^2 O_2)}}{d^2} \\ = \dfrac{x_M \cdot d_2 \cdot d \cdot \cos O_2 \mp d_2 \cdot d \cdot y_M \cdot \sqrt{1 - \cos^2 O_2}}{d^2} \\ = \dfrac{x_M \cdot d_2 \cdot \cos O_2 \mp d_2 \cdot y_M \cdot \sqrt{\sin^2 O_2}}{d} \\ = \dfrac{x_M \cdot d_2 \cdot \cos O_2 \mp d_2 \cdot y_M \cdot \sin O_2}{d} \\ = d_2 \cdot \left(\dfrac{x_M}{d} \cdot \cos O_2 \mp \dfrac{y_M}{d} \cdot \sin O_2 \right) \\ = d_2 \cdot (\cos \phi \cdot \cos O_2 \mp \sin \phi \cdot \sin O_2) \\ = d_2 \cdot \cos(\phi \pm O_2) \\ \\ x = d_2 \cdot \cos(\phi \pm O_2) \end{cases} \qquad (72)$$

It further determines the unknown y by introducing value x obtained from (72) in the first relationship of the system (69). One obtains the expression (73):

$$\begin{cases} y = \dfrac{d \cdot d_2 \cdot \cos O_2 - x_M \cdot d_2 \cdot \left(\dfrac{x_M}{d} \cdot \cos O_2 \mp \dfrac{y_M}{d} \cdot \sin O_2 \right)}{y_M} \\ = \dfrac{d_2 \cdot \left((x_M^2 + y_M^2) \cdot \cos O_2 - x_M^2 \cdot \cos O_2 \pm x_M \cdot y_M \cdot \sin O_2 \right)}{d \cdot y_M} \\ = d_2 \cdot \left(\dfrac{y_M}{d} \cdot \cos O_2 \pm \dfrac{x_M}{d} \cdot \sin O_2 \right) \\ = d_2 \cdot (\sin \phi \cdot \cos O_2 \pm \cos \phi \cdot \sin O_2) = d_2 \cdot \sin(\phi \pm O_2) \end{cases} \qquad (73)$$

From (72) and (73) retain only the latest expression concentrated both in system (74):

$$\begin{cases} x = d_2 \cdot \cos(\phi \pm O_2) \\ y = d_2 \cdot \sin(\phi \pm O_2) \end{cases} \tag{74}$$

From Fig. 3 may write Equation 75:

$$\begin{cases} x = d_2 \cdot \cos\phi_{20} \\ y = d_2 \cdot \sin\phi_{20} \end{cases} \tag{75}$$

Comparing systems (74) and (75) the resulting system (76), from that is deducting direct relation (77):

$$\begin{cases} \cos\phi_{20} = \cos(\phi \pm O_2) \\ \sin\phi_{20} = \sin(\phi \pm O_2) \end{cases} \tag{76}$$

$$\phi_{20} = \phi \pm O_2 \tag{77}$$

Geometric Method, Determining Velocities

Starting from the positions system (66), which is derived a function of time and obtains the velocities system (78). The system (78) is rewritten in simplified form (79):

$$\begin{cases} 2 \cdot (x - x_M) \cdot (\dot{x} - \dot{x}_M) + 2 \cdot (y - y_M) \cdot (\dot{y} - \dot{y}_M) = 0 \\ 2 \cdot x \cdot \dot{x} + 2 \cdot y \cdot \dot{y} = 0 \end{cases} \tag{78}$$

$$\begin{cases} (x - x_M) \cdot \dot{x} + (y - y_M) \cdot \dot{y} \\ = (x - x_M) \cdot \dot{x}_M + (y - y_M) \cdot \dot{y}_M \\ x \cdot \dot{x} + y \cdot \dot{y} = 0 \end{cases} \tag{79}$$

In (79) break out the brackets and get the system (80):

$$\begin{cases} x \cdot \dot{x} + y \cdot \dot{y} - (x_M \cdot \dot{x} + y_M \cdot \dot{y}) \\ = (x - x_M) \cdot \dot{x}_M + (y - y_M) \cdot \dot{y}_M \\ x \cdot \dot{x} + y \cdot \dot{y} = 0 \end{cases} \tag{80}$$

The second relationship of system (80) is inserted in the first, then the first expression is multiplied by (-1), so the system is simplified, receiving the form (81):

$$\begin{cases} x_M \cdot \dot{x} + y_M \cdot \dot{y} = (x_M - x) \cdot \dot{x}_M + (y_M - y) \cdot \dot{y}_M \\ x \cdot \dot{x} + y \cdot \dot{y} = 0 \end{cases} \tag{81}$$

The system (81) is solved in two steps.

In the first step is multiplied the first relation of the system (81) by (*y*) and the second by (-*y*$_M$), after which the result expressions are gathered member by member to provide the following relation (82) in which it is explained $\dot{x}$:

$$\dot{x} = \frac{y \cdot [(x_M - x) \cdot \dot{x}_M + (y_M - y) \cdot \dot{y}_M]}{x_M \cdot y - y_M \cdot x} \tag{82}$$

In step two we want to achieve $\dot{y}$ for which multiplies the first relationship of the system (81) by (*x*) and the second with (-*x*$_M$), gather relations obtained member with member and explains $\dot{y}$ resulting relationship (83):

$$\dot{y} = \frac{-x \cdot [(x_M - x) \cdot \dot{x}_M + (y_M - y) \cdot \dot{y}_M]}{x_M \cdot y - y_M \cdot x} \tag{83}$$

Relationships (82) and (83) are written confined within the system (84):

$$\begin{cases} \dot{x} = y \cdot h \\ \dot{y} = -x \cdot h \\ \\ h = \dfrac{(x_M - x) \cdot \dot{x}_M + (y_M - y) \cdot \dot{y}_M}{x_M \cdot y - y_M \cdot x} \end{cases} \tag{84}$$

Geometric Method, Determining Accelerations

Starting from the velocities system (84), which is derived a function of time and obtains acceleration system (85). The system (85) is rewritten as (86):

$$\begin{cases} \ddot{x} = \dot{y} \cdot h + y \cdot \dot{h} = -x \cdot h^2 + y \cdot \dot{h} \\ \ddot{y} = -\dot{x} \cdot h - x \cdot \dot{h} = -y \cdot h^2 - x \cdot \dot{h} \\ h \cdot (x_M \cdot y - y_M \cdot x) \\ = (x_M - x) \cdot \dot{x}_M + (y_M - y) \cdot \dot{y}_M \\ \dot{h} \cdot (x_M \cdot y - y_M \cdot x) \\ + h \cdot (\dot{x}_M \cdot y + x_M \cdot \dot{y} - \dot{y}_M \cdot x - y_M \cdot \dot{x}) \\ = (\dot{x}_M - \dot{x}) \cdot \dot{x}_M + (x_M - x) \cdot \ddot{x}_M \\ + (\dot{y}_M - \dot{y}) \cdot \dot{y}_M + (y_M - y) \cdot \ddot{y}_M \\ \\ \dot{h} = \dfrac{(\dot{x}_M - \dot{x}) \cdot \dot{x}_M + (x_M - x) \cdot \ddot{x}_M}{x_M \cdot y - y_M \cdot x} \\ + \dfrac{(\dot{y}_M - \dot{y}) \cdot \dot{y}_M + (y_M - y) \cdot \ddot{y}_M}{x_M \cdot y - y_M \cdot x} \\ - h \cdot \dfrac{\dot{x}_M \cdot y + x_M \cdot \dot{y} - \dot{y}_M \cdot x - y_M \cdot \dot{x}}{x_M \cdot y - y_M \cdot x} \end{cases} \tag{85}$$

$$\begin{cases} \ddot{x} = \dot{y} \cdot h + y \cdot \dot{h} = -x \cdot h^2 + y \cdot \dot{h} \\ \ddot{y} = -\dot{x} \cdot h - x \cdot \dot{h} = -y \cdot h^2 - x \cdot \dot{h} \\ \dot{h} = \dfrac{(\dot{x}_M - \dot{x} - y \cdot h) \cdot \dot{x}_M + (\dot{y}_M - \dot{y} + x \cdot h) \cdot \dot{y}_M}{x_M \cdot y - y_M \cdot x} \\ + \dfrac{(x_M - x) \cdot \ddot{x}_M + (y_M - y) \cdot \ddot{y}_M + y_M \cdot \dot{x} \cdot h - x_M \cdot \dot{y} \cdot h}{x_M \cdot y - y_M \cdot x} \end{cases} \tag{86}$$

Geometric Method, Determining Angular Velocities and Accelerations

Once determined velocities and accelerations O_3 point, we can move on to determining the absolute angular velocities and angular accelerations of the system. It starts from the system (75), which is derived a function of time and get the system (87):

$$\begin{cases} \dot{x} = -d_2 \cdot \sin\phi_{20} \cdot \dot{\phi}_{20} \\ \dot{y} = d_2 \cdot \cos\phi_{20} \cdot \dot{\phi}_{20} \end{cases} \tag{87}$$

For correct solution of the system (87) mounts the first relationship to the (-sinφ$_{20}$) and the second with (cosφ$_{20}$), then collect both relationships obtained (member by member) and the explanation of the $\dot{\phi}_{20}$, obtained searched expression (88):

$$\omega_2 \equiv \omega_{20} \equiv \dot{\phi}_{20} = \frac{\dot{y} \cdot \cos\phi_{20} - \dot{x} \cdot \sin\phi_{20}}{d_2} \tag{88}$$

The velocities system (87) is again derived the time and to obtain the absolute angular accelerations system (89):

$$\begin{cases} \ddot{x} = -d_2 \cdot \cos\phi_{20} \cdot \dot{\phi}_{20}^2 - d_2 \cdot \sin\phi_{20} \cdot \ddot{\phi}_{20} \\ \ddot{y} = -d_2 \cdot \sin\phi_{20} \cdot \dot{\phi}_{20}^2 + d_2 \cdot \cos\phi_{20} \cdot \ddot{\phi}_{20} \end{cases} \tag{89}$$

For correct solution of the system (89) amplify the first its relationship to the (-sinφ$_{20}$) and the second with (cosφ$_{20}$) (see equations from system 90), then collect both relationships obtained (member by member) and the explanation of the $\dot{\phi}_{20}$, obtained searched expression (91):

$$\begin{cases} \ddot{x} = -d_2 \cdot \cos\phi_{20} \cdot \dot{\phi}_{20}^2 - d_2 \cdot \sin\phi_{20} \cdot \ddot{\phi}_{20} \quad | \quad \cdot(-\sin\phi_{20}) \\ \ddot{y} = -d_2 \cdot \sin\phi_{20} \cdot \dot{\phi}_{20}^2 + d_2 \cdot \cos\phi_{20} \cdot \ddot{\phi}_{20} \quad | \quad \cdot(\cos\phi_{20}) \end{cases} \tag{90}$$

$$\varepsilon_2 \equiv \varepsilon_{20} \equiv \dot{\omega}_{20} \equiv \ddot{\phi}_{20} = \frac{\ddot{y} \cdot \cos\phi_{20} - \ddot{x} \cdot \sin\phi_{20}}{d_2} \tag{91}$$

Remember the two relationships in the system (92):

$$\begin{cases} \omega_2 \equiv \omega_{20} \equiv \dot{\phi}_{20} = \frac{\dot{y} \cdot \cos\phi_{20} - \dot{x} \cdot \sin\phi_{20}}{d_2} \\ \varepsilon_2 \equiv \varepsilon_{20} \equiv \dot{\omega}_{20} \equiv \ddot{\phi}_{20} = \frac{\ddot{y} \cdot \cos\phi_{20} - \ddot{x} \cdot \sin\phi_{20}}{d_2} \end{cases} \tag{92}$$

Still express Equation 93, (Fig. 3):

$$\begin{cases} x_M - x = d_3 \cdot \cos\phi_{30} \\ y_M - y = d_3 \cdot \sin\phi_{30} \end{cases} \tag{93}$$

The relationships of the system (93) further are derived with the time and obtained the velocities equations, system (94):

$$\begin{cases} \dot{x}_M - \dot{x} = -d_3 \cdot \sin\phi_{30} \cdot \dot{\phi}_{30} \\ \dot{y}_M - \dot{y} = d_3 \cdot \cos\phi_{30} \cdot \dot{\phi}_{30} \end{cases} \tag{94}$$

For correct solution of the system (94) amplify the first its relationship to the (-sinφ$_{30}$) and the second with (cosφ$_{30}$), then collect both relationships obtained (member by member) and the explanation of the $\dot{\phi}_{30}$, obtained wanted expression (95):

$$\omega_3 \equiv \omega_{30} \equiv \dot{\phi}_{30} = \frac{(\dot{y}_M - \dot{y}) \cdot \cos\phi_{30} - (\dot{x}_M - \dot{x}) \cdot \sin\phi_{30}}{d_3} \tag{95}$$

It derives the then with the time, the velocities system (94) and the absolute angular accelerations Obtain system (96):

$$\begin{cases} \ddot{x}_M - \ddot{x} = -d_3 \cdot \cos\phi_{30} \cdot \dot{\phi}_{30}^2 - d_3 \cdot \sin\phi_{30} \cdot \ddot{\phi}_{30} \\ \ddot{y}_M - \ddot{y} = -d_3 \cdot \sin\phi_{30} \cdot \dot{\phi}_{30}^2 + d_3 \cdot \cos\phi_{30} \cdot \ddot{\phi}_{30} \end{cases} \tag{96}$$

For correct solution of the system (96) amplify the first its relationship to the (-sinφ$_{30}$) and the second with (cosφ$_{30}$), then collect both relationships obtained (member by member) and by the explanation of the $\ddot{\phi}_{30}$, obtained wanted expression (97):

$$\varepsilon_3 \equiv \varepsilon_{30} \equiv \dot{\omega}_{30} \equiv \ddot{\phi}_{30} = \frac{(\ddot{y}_M - \ddot{y}) \cdot \cos\phi_{30} - (\ddot{x}_M - \ddot{x}) \cdot \sin\phi_{30}}{d_3} \tag{97}$$

Keep in the system (98) found the two solutions and in system (99) centralize all four of them:

$$\begin{cases} \omega_3 \equiv \omega_{30} \equiv \dot{\phi}_{30} = \frac{(\dot{y}_M - \dot{y}) \cdot \cos\phi_{30} - (\dot{x}_M - \dot{x}) \cdot \sin\phi_{30}}{d_3} \\ \varepsilon_3 \equiv \varepsilon_{30} \equiv \dot{\omega}_{30} \equiv \ddot{\phi}_{30} = \frac{(\ddot{y}_M - \ddot{y}) \cdot \cos\phi_{30} - (\ddot{x}_M - \ddot{x}) \cdot \sin\phi_{30}}{d_3} \end{cases} \tag{98}$$

$$\begin{cases} \omega_2 \equiv \omega_{20} \equiv \dot{\phi}_{20} = \frac{\dot{y} \cdot \cos\phi_{20} - \dot{x} \cdot \sin\phi_{20}}{d_2} \\ \omega_3 \equiv \omega_{30} \equiv \dot{\phi}_{30} = \frac{(\dot{y}_M - \dot{y}) \cdot \cos\phi_{30} - (\dot{x}_M - \dot{x}) \cdot \sin\phi_{30}}{d_3} \\ \varepsilon_2 \equiv \varepsilon_{20} \equiv \dot{\omega}_{20} \equiv \ddot{\phi}_{20} = \frac{\ddot{y} \cdot \cos\phi_{20} - \ddot{x} \cdot \sin\phi_{20}}{d_2} \\ \varepsilon_3 \equiv \varepsilon_{30} \equiv \dot{\omega}_{30} \equiv \ddot{\phi}_{30} = \frac{(\ddot{y}_M - \ddot{y}) \cdot \cos\phi_{30} - (\ddot{x}_M - \ddot{x}) \cdot \sin\phi_{30}}{d_3} \end{cases} \tag{99}$$

Conclusion

Today, anthropomorphic structures are used more and more in almost all the fields of industrial. Robotic structures have emerged from the need for automation and robotics of the industrial processes. The first industrial robots were called upon by the heavy industry and in particular by the automobiles industry. The automotive industry not only has requested the appearance of the industrial robots but even their subsequent development.

The most used were and have remained, the robots anthropomorphic, because they are more easily designed, built, maintained, are easily to handle, more dynamics, robust, economic and in general they have a broadly working area. The structures of the solid anthropomorphic robots are made up of elements and the couples of rotation, to which can add on an occasional basis and one or more couplers with translational moving. The couplers of rotation have been proven their effectiveness by moving them easier, more dynamic, step by step and especially being the most reliable. In general the couplers of rotation are moving more easily and more continuous, are actuated better and easier, control is less expensive and more reliable and programming the movements of rotation is also much simpler and more efficient.

This paper presents and treats (in an original way) the specific elements of the structures of robotic solid mobile anthropomorphic type. Are "placed on the wallpaper", the geometry, kinematics and dynamics of the anthropomorphic robotic solid systems, in an original vision of the authors. Last part presents the inverse kinematics of anthropomorphic systems, with mechanical elements and points: Geometry, cinematic, positions, displacements, velocities and accelerations, by two methods (as the most representatives): First one the method trigonometric and second one the geometric method.

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Author's Contributions

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Ethics

This article is original. Authors declare that are not ethical issues that may arise after the publication of this manuscript.

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