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Sequences of Closed Operators and Correctness

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Abstract: In applications and in mathematical physics equations it is very important for mathematical models corresponding to the given problem to be correct given. In this research we will study the relationship between the sequence of closed operators $A_n \to A$ and the correctness of the equation Ax = y. Also we will introduce the criterion for correctness.

Key words: Kvazi method, closed operator, correctness, Banach-Shtengaus theorem

INTRODUCTION

The operator $A: U \to V$ may be defined as the limit of some sequence of operators $A_n: U \to V$. For example the differential operator $\frac{d}{dt}: C^1[a,b] \to C[a,b]$ is defined as the limit of the sequence of the operators $\left\{A_n f(t) = (f(t+\frac{1}{n}) - f(t))\, n\right\}_{n=1}^{\infty}$, where $A_n: C^1[a,b] \to C[a,b]\ \forall n \in N$. And in many places in Applied and theoretical mathematics the properties of the operator A implies directly from the properties of some sequence of operators $\{A_n\}_1^{\infty}$ with some restrictions.

There arise the following question: Is it possible to know the important property as correctness of the equation $Ax = y$ knowing properties of the sequence of operators that converges to the operator A?

Also during present research (Radyno *et al*., 1993a; 1993b; Tkan and Ramadan, 1992; Ramadan and Tkan, 1992; Ramadan and Jehad, 2000; Ramadan, 1993; 1997; 1998; 1999; 2007a; 2007b; 2007c) in many places were arisen questions about relationship between a special sequences of operators and their limits. For example relationship between the sequences $\{F(\varphi_n f)\}$, $\{\varphi_n f\}$ and f, where F is the Fourier Transform, and f-be a generalized function (distribution).

Definition 4: The linear operator $A: U \to V$ is called closed if and only if the sequence $\{x_n\}_{n=1}^{\infty} \subset D(A)$, and $(x_n, A(x_n)) \to (x, y)$, then $x \in D(A)$ and $y = Ax$ (the graph of the operator A is closed in XxY).

In (Ramadan and Jehad, 2000) we proved the following theorems:

Let X, Y-be Banach spaces; and $A, A_n: X \to Y$ are closed operators such that $D(A) = D(A_n)$ and $\lim_{n\to\infty} A_n x = Ax\ \ \forall x \in D(A)$ (A_n strongly converges to A).

And suppose that all operators $A_n^{-1}: Y \to X$ exist and continuous and there is a positive constant C>0 such that for all n and for all $y \in Y$, $\|A_n^{-1}\| \le C$ then:

Theorem 1: For each $y \in R(A)$ there is a unique $x \in D(A)$ such that $Ax = y$.

Theorem2: For each $y \in R(A)$, $A_n^{-1}y \to A^{-1}y$, where $Ax = y$, then the operator $A^{-1}y$ is defined and bounded on $\overline{R(A)}$.

Theorem 3: If $A_n x_n = y_n$ and $y_n \to y$, then $x_n \to x$, where $Ax = y$.

Theorem 4: If $A_n x_n = y_n$ and $x_n \to x$, $\|A_n\| \le C$, $Ax = y$ then $y_n \to y$.

Preliminaries: In this study we will use the following conventional notations and definitions:

L(X,Y)	The set of all linear continuous operators $A: X \to Y$
D(A)	The domain of the operator A
R(A)	The range of the operator A
A^{-1}	The inverse of the operator A

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Ker(A) The kernel of the operator A
||A|| The norm of the operator A

Definition 1: Let $(V, \| \|_V)$, $(F, \| \|_F)$ are two Normed spaces and $A: F \to V$ be an operator. If $v \in V$ be known element, then the equation $Az = v$ called correct given if the following condition hold:

- Existence of solution: For each element $v \in V$ there is a solution $z \in F$ of the equation $Az = v$
- Uniqueness of solution : The solution which exists is unique
- Stability: For each $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that if $\| v_1 - v_2 \|_F \le \delta(\varepsilon)$, then $\| z_1 - z_2 \|_V \le \varepsilon$, where $Az_1 = v_1$, $Az_2 = v_2$

Example: The Cauchy-Laplace equation:

$$\begin{cases} \Delta u(x,y) = 0 \\ u(x,0) = f(x) \\ \dfrac{\partial u}{\partial y} \Big|_{y=0} = \varphi(x) \end{cases}$$

Is incorrect since if:

$$\begin{cases} \Delta u(x,y) = 0 \\ u(x,0) = 0 = f_1(x) \\ \dfrac{\partial u}{\partial y} \Big|_{y=0} = 0 = \varphi_1(x) \end{cases}$$

then the solution is $u_1(x,y) \equiv 0$.
and if:

$$\begin{cases} \Delta u(x,y) = 0 \\ u(x,0) = 0 = f_2(x) \\ \dfrac{\partial u}{\partial y} \Big|_{y=0} = \dfrac{1}{a} \sin ax = \varphi_2(x) \end{cases}$$

then the solution is $u_1(x,y) = \dfrac{1}{a^2} \sin ax \sinh ay$.

Now consider:

$$\|f_1 - f_2\|_C = \sup_x |f_1(x) - f_2(x)| = 0$$

$$\|\varphi_1 - \varphi_2\|_C = \sup_x |\varphi_1(x) - \varphi_2(x)| = \frac{1}{a}$$

But:

$$\|u_1 - u_2\|_C = \sup_x |u_1(x,y) - u_2(x,y)|$$
$$= \sup_x \left| \frac{1}{a^2} \sin ax \sinh ax \right| = \frac{1}{a^2} \sinh ay$$

Now let X, Y-be any two Normed spaces:

Definition 2: The sequence of linear continuous operators $A_n \subset L(X,Y)$ is strongly converges to operator $A \subset L(X,Y)$ if and only if for each $x \in X$ $\|A_n x - Ax\| \to 0$;

Definition 3: The sequence of linear continuous operators $\{A_n\}_{n=1}^{\infty} \subset L(X,Y)$ is said to be uniformly convergent to the operator $A \in L(X,Y)$ if and only if $\|A_n - A\| \to 0$.

Remark: It is easy to check that from the uniformly convergent implies the strongly convergent, but the converse is not true (Antonevich and Radyno, 2006)

MATERIALS AND METHODS

Theorem 5: Let X, Y-be Normed spaces; and $A, A_n : X \to Y$ are closed operators such that $D(A) = \bigcap_n D(A_n) \ne \{0\}$ and A_n strongly converges to A and let $A^{-1}{}_n \in L(Y,X)$ and $\|A^{-1}{}_n\| \le C$, then the equation Ax = y is correct given.

Proof:

- Suppose $Ax^* = 0$, $x^* \in D(A)$. Since $\lim_{n\to\infty} A_n x^* = Ax^*$, then for each $\varepsilon > 0$ $\exists n^*$: $\|A_n x^*\| \le \varepsilon$ $\forall n > n^*$. Consider $\|x^*\| = \|A_n^{-1} A_n x^*\| \le C \|A_n x^*\| \le C\varepsilon$, $n \ge n^* \Rightarrow \|x^*\| = 0 \Rightarrow$ $x^* = 0 \Rightarrow \ker A = \{0\} \Rightarrow$ for each $y \in R(A)$ there is a unique $x \in D(A)$ such that $Ax = y$
- To prove the stability of the equation $Ax = y$ we take tow points $x_1, x_2 \in D(A)$ where $Ax_1 = y_1$, $Ax_2 = y_2$. Now let $\| y_1 - y_2 \| \le \delta$, consider $\| x_1 - x_2 \| = \| A_n^{-1} A_n x_1 - A_n^{-1} A_n x_2 \| \le C \| y_1 - y_2 \| \le C\delta \Rightarrow$ $\forall \varepsilon > 0$ $\exists \delta = \dfrac{\varepsilon}{C} > 0$ such that if $\| y_1 - y_2 \| \le \delta_\varepsilon$ $\Rightarrow \| x_1 - x_2 \| \le \varepsilon$

Corollary: In conditions of Theorem 5 If we replace the equality $D(A) = \bigcap_n D(A_n) \neq \{0\}$ by $D(A) \supset \bigcap_n D(A_n) \neq \{0\}$, then we can say that the equation $Ax = y$ is correct given in sense of Tychonof and the set $\bigcap_n D(A_n)$ is the domain of Correctness.

Proof: We get the proof by modifying the argument in the proof of Theorem 5.

Theorem 6: (Criterion for correctness): Let X and Y be Banach spaces and if $A, A_n : X \to Y$, $D(A) = \bigcap_n D(A_n) \neq \{0\}$ and A_n strongly converges to A and $A^{-1}{}_n \in L(Y, X)$, then the equation $Ax = y$ is correct given if and only if $\|A^{-1}{}_n\| \leq C$.

Proof:

- Suppose $Ax = y$ is correct $\Rightarrow$ the operator A^{-1} exists and bounded. A_n strongly converges to A means $\forall \varepsilon > 0 \exists n_0 \ \forall n > n_0$ $\|A_n x - Ax\| = \|y_n - y\| \leq \varepsilon$. Let $A_n x = y_n$, $Ax = y \Rightarrow A_n^{-1} y_n = x$ and $A^{-1} y = x$. Consider $\|A_n^{-1} y - A^{-1} y\| =$ $\|A_n^{-1} y + A_n^{-1} y_n - A_n^{-1} y_n - A^{-1} y\| \leq \|A_n^{-1} y - A_n^{-1} y_n\| +$ $\|A_n^{-1} y_n - A^{-1} y\| = \|A_n^{-1} y - A_n^{-1} y_n\| \leq \|A_n^{-1}\| \, \|y - y_n\| \to 0$, that is $A_n^{-1} \to A^{-1}$ strongly, $\|A_n^{-1}\| \leq C_y$. Now by Banach-Shtengaus theorem (Antonevich and Radyno, 2006) $\|A^{-1}{}_n\| \leq C$
- Suppose that $\|A^{-1}{}_n\| \leq C$, then by using Theorem 5 we conclude that $Ax = y$ is correct

RESULTS AND DISCUSSION

We can use the proved criterion to discuss the correctness of the mathematical models corresponding to given problem. For example if we use the Kvazi Method (Lattes and Lions, 1970) to solve the model:

$$A : \begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = 0 & t < T \\ u(x, T) = \chi(x) \end{cases}$$

$$D(A) = \left\{ u \in L_2(R) \ : \frac{\partial^2 u}{\partial x^2} \in L_2(R) \right\} = H^2(R)$$

then we construct the following sequence:

$$A_\varepsilon : \begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \dfrac{\partial^2 u_\varepsilon}{\partial x^2} - \varepsilon \dfrac{\partial^4 u_\varepsilon}{\partial x^4} = 0 & t < T \\ u_\varepsilon(x, T) = \chi(x) \\ \dfrac{\partial^2 u_\varepsilon}{\partial x^2}(x, t) \in L_2(R) \\ \dfrac{\partial^4 u_\varepsilon}{\partial x^4}(x, t) \in L_2(R) \end{cases}$$

Solving this system by using Fourier Transform $\hat{u}$ (of x):

$$A : \begin{cases} \dfrac{\partial \hat{u}}{\partial t} + (\xi^2 - \varepsilon \xi^4) \hat{u} = 0 & t < T \\ \hat{u}(\xi, T) = \hat{\chi}(\xi) \end{cases}$$

We get $\hat{u}_\varepsilon(t, \xi) = e^{-(\varepsilon \xi^4 - \xi^2)(T - t)} \hat{\chi}(\xi)$. Consider $\|A_\varepsilon^{-1}\|^2 = \|\hat{u}_\varepsilon\|^2_{L_2(R)} = \int_R e^{-2(\varepsilon \xi^4 - \xi^2)} \left[\hat{\chi}(\xi) \right]^2 d\xi \leq C_\varepsilon^2 \int_R \left| \hat{\chi}(\xi) \right|^2 d\xi$ for $\varepsilon \xi^4 - \xi^2 \geq 0 \Rightarrow -\frac{1}{\sqrt{\varepsilon}} \leq \xi \leq \frac{1}{\sqrt{\varepsilon}}$. Now if we find the maximum of the function $f(\xi) = e^{-2(\varepsilon \xi^4 - \xi^2)}$, then $\frac{d}{d\xi} f(\xi) = (-8\varepsilon \xi^3 + 4\xi) e^{-2(\varepsilon \xi^4 - \xi^2)} = 0$ when $\xi = 0$, $\xi = \pm \frac{1}{\sqrt{2\varepsilon}}$. Since $f(0) = f(\pm \frac{1}{\sqrt{\varepsilon}}) = 1$ and $f(\pm \frac{1}{\sqrt{2\varepsilon}}) = e^{\frac{1}{2\varepsilon}} \sim C_\varepsilon^2$ which means that there is no C such that $\|A_\varepsilon^{-1}\| \leq C$, and so the equation is incorrect given.

CONCLUSION

The above proved results play a very important rule for solving and discussions mathematical models corresponding to the given problems in mathematical physics equations. And the above proved criterion can be used as an indirect method for explaining if the given model is correct or not.

Since from the uniformly convergent implies the strongly convergent, then all results stay true if we replace in the conditions of above proved theorems the strongly convergent by uniformly convergent.

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