Convergence Ratio Profile for Optimal Control Problems Governed by Ordinary Differential Equations with Matrix Coefficients

O. Olotu and S.A. Olorunsola

1The Federal University of Technology, Mathematical Science Department
P.M.B. 704, Akure, Ondo State, Nigeria
2The University Of Ado-Ekiti, Mathematical Science, Department
P.M.B.5383, Ado-Ekiti, Ekiti, State, Nigeria

Abstract: The geometric convergence ratio, the main focus of a discretized scheme for constrained quadratic control problem was examined. In order to allow for the numerical applications of the developed scheme, discretizing the time interval and using Euler’s scheme for its differential constraint obtained a finite dimensional approximation. Applying the penalty function method, an unconstrained problem was obtained on function minimization with bilinear form expression. This finally led to the construction of an operator. The Scheme was applied to a sampled problem and it exhibited geometric convergence ratio, \(a\), in the open interval \((0, 1)\) as depicted in column 6 of Table 1.

Key words: Quadratic, discretization, bilinear form expression, associated operator, convergence profile and geometric convergence

INTRODUCTION

In\(^{[1,2]}\), the scheme establishing the solution of optimal control problems constrained by evolution equation of the delay type with matrix coefficients was presented, without addressing the geometric convergence ratio profile. Here, a class of optimal control problems constrained by ordinary differential equation with matrix coefficients is considered. Discretization of the generalized problem is obtained by discretizing its objective function and using\(^{[3]}\) for its differential constraint. Using\(^{[4]}\), a penalty function method is applied to convert the constrained problem into an unconstrained formulation problem. With this formulation, an associated control operator was constructed as in\(^{[2]}\). Here, again, we state the constructed operator as a consequence of a theorem in this paper to allow for brevity and compactness of the paper. Consequently, consideration of the convergence profile and the geometric convergence ratio profile as they relate to such class of problems is highlighted by a sample problem for the confirmation of the success of the scheme.

Generalized problem 1

\[
\begin{align*}
\text{Min} & \int_0^Z \left( x(t)^T P x(t) + u(t)^T Q u(t) \right) dt \\
\text{subject to} & \\
\dot{x}(t) = A x(t) + C u(t), & x(0) = x_0, \quad 0 \leq t \leq Z, \\
\end{align*}
\]

where, \( A, C \) are \( n \times n \), symmetric matrices, \( x(t) \) and \( C \) are \( n \times m \) matrices not necessarily symmetric respectively.

MATERIALS AND METHODS

2.1 Discretization: By discretizing (1.1) and (1.2), we have

\[
X(t) = A x(t) + C u(t) \\
\Delta_k = A x_k(t) + C u_k(t) \\
X(0) = 0
\]

We then have the discretized generalized problem in the form:

\[
\begin{align*}
\text{Min} & J = \sum_{k=1}^n \Delta_k \left( x_k(t) + u_k(t) \right) P x_k(t) + u_k(t) \right) Q u_k(t) \\
\text{subject to} & \\
x(t_{k-1}) - x(t_k) / \Delta_k = A x_k(t_k) + C u_k(t_k) \\
x(0) = 0
\end{align*}
\]

Penalty method’s application: Using the standard penalty function\(^{[4]}\), we obtain the unconstrained problem 1

\[
\begin{align*}
\text{Min} J \left( x, u, \mu \right) = & \sum_{k=1}^n \left[ \Delta_k \left( x(t_k) + u(t_k) \right) Q u(t_k) \right] \\
\text{subject to} & \\
x(t_{k-1}) - x(t_k) / \Delta_k = A x_k(t_k) + C u_k(t_k) \\
\end{align*}
\]
Let \( z_k = \left( x_k(t_k), u_k(t_k) \right) \), and \( y_k(t_k) = x_{k+1}(t_k) \)

In (1.5), let

\[
MK = \mathbb{P} A_k + \mu + \mu A_k' A + 2\mu A_k
\]

\[
Nk = \mathcal{Q} A_k + \mu C_k A_k C_k A_k
\]

\[
Bk = 2C_k A_k' A + 2C_k A_k A_k A
\]

\[
P_k = -2\mu - 2\mu A_k
\]

\[
Ak = -2\mu C_k A_k
\]

Now, (1.5) becomes

\[
\sum_{n=0}^{\infty} \left[ MK_n x_n(t_k) + N_k y_n(t_k) + y_n y_k(t_k) + Ak x_n(t_k) u_n(t_k) \right] = 0
\]

2.3. Theorem 1: The exact control operator \( G \) satisfying generalized problem 1 is given by

\[
GZ_k(t_k) = G_{11} G_{12} G_{21} G_{22} \begin{pmatrix} x_k(t_k) \\ u_k(t_k) \end{pmatrix}
\]

where

\[
G_{11} = e_0 \sinh(t_k) + u_{0,1}(0) \mathcal{P}_0 \mathcal{A}_0 \cosh(t_k) - \sinh(T) u_{0,1}(0) \mathcal{P}_1
\]

\[
G_{12} = \int_0^T \left( u_{1,1}(s) \mathcal{A}_1 \mathcal{P}_1 \mathcal{A}_0 \cosh(T-s) \right) ds
\]

\[
G_{21} = -\int_0^T \left( u_{1,1}(s) \mathcal{P}_1 \mathcal{A}_1 \sinh(T-s) \right) ds
\]

\[
G_{22} = e_0 \sinh(t_k) N_k
\]

\[
G_{i,j}(t_k) = \int_0^T \left( \mathcal{M}_i + \mu + 2\mathcal{R}_i, \mathcal{A}_i, (0) \right) \mathcal{R}_i, (0) \mathcal{M}_i, (0) \mathcal{A}_i \sinh(T-s) \right) ds
\]

2.4. Remark 1: The scheme converges at the 4th iteration for each penalty parameter constant \( \mu \) as depicted in Table 1 for the numerical calculation.

\[
\lim_{n \to \infty} z_n = z^*
\]

Then \( \{z_n\} \) is said to converge geometrically to \( z^* \) with a convergence ratio \( y \) as reported by [3]. Here, we recall the various steps of the conjugate gradient algorithm that generates the convergent sequence \( \{z_n\} \) of solutions of problem 1 according to [3]. The algorithm employs the explicit knowledge of the control operator \( G \) developed in theorem 1.

**Step 1:** Choose initial values for the conjugate descent algorithm;

\[
z_0(t_k) = \left( x_0(t_k), u_0(t_k) \right), \quad z_0(t_k) \in H
\]

Compute \( p_0 = -g_0 \)

While the remaining members are computed as follows;

**Step 2:** Update \( x_0 \) and \( u_0 \) such that

\[
x_{n+1}(t_k) = x_n(t_k) + \alpha_n p_{n,n}
\]

\[
u_{n+1}(t_k) = u_n(t_k) + \alpha_n p_{n,n}
\]

where \( \alpha_n, p_{n,n} \) are the step length and the descent direction respectively.

and \( \alpha = \frac{\langle g_{n,n}, b_{n,n} \rangle}{\langle p_{n,n}, g_{n,n} \rangle} \)

**Step 3:** Update gradient and descent directions with the updating rule

\[
g_{x,n+1} = g_{x,n} + \alpha_n g_{n,n}
\]

\[
g_{u,n+1} = g_{u,n} + \alpha_n g_{n,n}
\]

\[
p_{x,n+1} = -g_{x,n} + \beta_n p_{x,n}
\]

\[
p_{u,n+1} = -g_{u,n} + \beta_n p_{u,n}
\]

Where \( g_{x,n}, p_{x,n} \) are the gradient and descent direction at the \( (n+1) \) th iteration respectively and \( G \) is the control operator in theorem 1.

\[
\beta_n = \frac{\langle g_{n+1,n}, g_{n,n} \rangle}{\langle g_{n,n}, g_{n,n} \rangle} \quad \text{and} \quad p^{* n} = \left( p_{x,n}, p_{u,n} \right)
\]

are the transposes of \( g_{x,n} \) and \( p_{x,n} \) respectively.

Now,
We obtain the following relations,

\[ \begin{align*}
G_{p_1}(t_k) &= \begin{pmatrix} G_{11}P_{s,n}(t_k) + G_{12}P_{n,n}(t_k) \\ G_{21}P_{s,n}(t_k) + G_{22}P_{n,n}(t_k) \end{pmatrix} = \begin{pmatrix} G_{p_1}(t_k) \\ G_{p_2}(t_k) \end{pmatrix}
\end{align*} \]

Setting \( J_{s,k} = V_{s,k}J(x,(t_k),u,(t_k),\mu) \) and \( J_{s,k} = V_{s,k}J(x,(t_k),u,(t_k),\mu) \)
where

\[ J(x,(t_k),u,(t_k),\mu) = \int_{0}^{T} \left( c(t_k,u(t),z(t_k))dt \right) \]

\[ + \mu \left[ \int_{0}^{T} \left( \int_{0}^{T} \left( \| \int_{0}^{T} \left( \| \int_{0}^{T} \left( \| \right) \right) \right) \right) \right] \]

\[ \mathcal{V}_{s,k}J(x,(t_k),u,(t_k),\mu) \]

is the gradient of \( J(x,(t_k),u,(t_k),\mu) \) at the \( k \)th step.

We obtain the following relations,

\[ P_{s,k} = -T \int_{0}^{T} J_{s,k}(x,(t_k),u,(t_k),\mu)dt \]

or more generally

\[ P_{s,k} = -T \int_{0}^{T} J_{s,k}(x,(t_k),u,(t_k),\mu)dt \]

Assuming the following remarks, we state and prove the following theorem 2

**Theorem 2:** The sequence \( \{z_n(t)\} \) of solutions to problem 1 using the explicit knowledge of the control operator \( A \) in theorem 1 converges geometrically to \( \{z^*\} \) with ratio

\[ \lambda = 1 - \alpha, \quad \alpha = \frac{1}{\|z\|} \max \left\{ \left\| A_{z_n} \right\| \left\| \langle P_{z,n}, P_{z,n} \rangle \right\| ^{-1} \right\} \quad (1.12) \]

**Proof:**

Let \( F(z) = \langle z - z^*, A_z \rangle \), \( z^T(t) = \langle x(t), u(t), t \rangle \).

At the optimality condition, we have \( A_{z(n)}(t) = 0 \)

Let \( z, z_n \) be in \( H \). \( F(z_n) = \langle z_n - z^*, A(z_n - z^*) \rangle = \langle z_n, A_{z*} \rangle - \langle z^*, A_{z*} \rangle \) (1.13)

and

\[ F(z_{n+1}) = \langle z_n + \alpha_t P_{z,n}, z^* \rangle, A(z_n + \alpha_t P_{z,n} - z^*) \]

\[ = \langle z_n, A_{z*} \rangle + \alpha_t \langle z_n, P_{z,n} \rangle + \alpha_t \langle P_{z,n}, A_{z*} \rangle + \alpha_t^2 z^T P_{z,n} \rightarrow \langle z^*, A_{z*} \rangle - \alpha_t \langle z^*, A_{z*} \rangle \] (1.14)

From (1.13) and (1.14),

\[ F(z_n) - F(z_{n+1}) = \frac{\left\| A_{z_n} \right\| ^2 F(z_n)}{\left\langle P_{z,n}, A_{z,n} \right\rangle \left\langle A_{z,n}, z_n \right\rangle} \quad (1.15) \]

Since \( A \) is a self-adjoint operator \([2,7,8] \), \( A_{z^*}=0 \)

Again, \( \left\langle A_{z_n}, z_n \right\rangle = \left\langle A_{z_n}, z_0 \right\rangle + \sum_{k=1}^{n} \alpha \left\langle A_{z_n}, P_{z,n} \right\rangle \) \( \left\langle A_{z_n}, P_{z,n} \right\rangle = 0 \) \( \left\langle A_{z_n}, z_n \right\rangle \) \( \left\langle A_{z_n}, z_n \right\rangle = 0 \) \( \left\langle A_{z_n}, z_n \right\rangle \)

Substituting, (1.16) in (1.15), we have,

\[ F(z_n) - F(z_{n+1}) = \frac{\left\| A_{z_n} \right\| ^2 F(z_n)}{\left\langle P_{z,n}, A_{z,n} \right\rangle \left\langle A_{z,n}, z_n \right\rangle} \]

So,

\[ F(z_{n+1}) = \left[1 - \frac{\left\| A_{z_n} \right\| ^2}{\left\langle P_{z,n}, A_{z,n} \right\rangle \left\langle A_{z,n}, z_n \right\rangle} \right] F(z_n) \quad (1.17) \]

By (1.16a), \( F(z_n) \geq m \left\| z_n - z^* \right\|^2 \),

So that (1.17) becomes

\[ \left\| z_{n+1} - z^* \right\|^2 \left\| z_n - z^* \right\|^2 \leq F(z_{n+1}) - F(z_n) \leq 1 - \frac{\left\| A_{z_n} \right\| ^2}{\left\langle P_{z,n}, A_{z,n} \right\rangle \left\langle A_{z,n}, z_n \right\rangle} \]

Thus end the analytic proof of the geometric convergence ratio.

**DATA AND ANALYSIS**

**Hypothetical example:** Now, we shall consider an example to test the efficiency of the developed scheme.

**Example 1:**

\[ \min_{t} \int_{0}^{1} \left( x(t)^T P(t)x(t) + u(t)^T Qu(t) \right) dt \]
Table 1: Convergence profile and geometric ratio profile for example

<table>
<thead>
<tr>
<th>Penalty constant</th>
<th>Number of iterations</th>
<th>Objective function</th>
<th>Constraint satisfaction</th>
<th>Penalty function</th>
<th>Geometric ratio</th>
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<td>4</td>
<td>5</td>
<td>6</td>
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\[
\frac{1}{\lambda}(I-A\lambda^t C) = (I-C\lambda^t A) x_1 = x_1^0 \quad 0 \leq \lambda \leq 1, \quad (1.18)
\]

where, \( x(t), u(t) \in \mathbb{R}^n \), \( x_1(t), u(t) \) denote the transposes of \( x(t) \) and \( u(t) \) respectively

\[ P, Q \text{ are 2 by 2 symmetric matrices} \]
\[ A \text{ and } C \text{ are 2 by 2 and } B \text{ by 2 matrices not necessarily symmetric respectively.} \]

and

\[ P = \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}, Q = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, C = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}, u_1 - u_5 

RESULTS

As seen in Table 1, column 3, the scheme shows a good convergence profile, particularly at the 3rd iteration for \( \mu=0.0001 \) and at the 4th iteration for \( 0.0002 \leq \mu \leq 0.0005 \), where the iterates are 3.568452, 1.931005, 1.850631, 1.8650631, 1.879398 respectively. However, the first cycle at the 3rd iteration, is not comparable enough, since its iterate value 3.568452 is comparatively higher than any of the other cycles.

Column 6 shows the geometric convergence ratio profile for each \( \mu \) per cycle. It is seen that \( \frac{1}{\lambda} - \frac{1}{\lambda(1-\xi)} \leq 1 \), for \( n \) large, \( \xi \)

depicted in the convergence ratio column 6 of Table 1.

For \( \mu=0.0001 \), the geometric ratio convergence starting at 0.9483081 and ending at 0.82418, shows values lying between 0 and 1. This characterizes a geometric ratio convergence. Similarly, this same trend holds for \( 0.002 \leq \mu \leq 0.0005 \) in column 6 of Table 1.

DISCUSSION

In this study, the scheme has demonstrated its objective having its geometric ratio convergence established between 0 and 1 as seen in Table 1. Hence, its success and reliability as it relates to problems of this class in terms of convergence profile and geometric convergence ratio via this sampled problem is being demonstrated.

REFERENCES