The $q$-Riccati Algebra

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Abstract: For $q \in (0, 1)$, we introduce the $q$-Riccati Lie algebra. Using the $q$-derivative (or Jackson derivative), we give a representation of this Lie algebra.

Keywords: $q$-Derivative, $q$-Riccati Lie Algebra

Introduction

In the mathematical field of representation theory, the representation of a Lie algebra is a way of writing a Lie algebra as a set of matrices (or endomorphisms of a vector space) in such a way that the Lie bracket is given by the commutator. More precisely, a representation of a Lie algebra $g$ is a linear transformation:

$$\psi : g \to M(V)$$

where, $M(V)$ is the set of all linear transformations of a vector space $V$. In particular, if $V = \mathbb{R}^n$, then $M(V)$ is the set of $n \times n$ square matrices. The map $\psi$ is required to be a map of Lie algebras so that:

$$\psi([A,B]) = \psi(A)\psi(B) - \psi(B)\psi(A)$$

for all $A, B \in g$. Note that the expression $AB$ only makes sense as a matrix product in a representation. For example, if $A$ and $B$ are antisymmetric matrices, then $AB - BA$ is skew-symmetric, but $AB$ may not be antisymmetric. The possible irreducible representations of complex Lie algebras are determined by the classification of the semi simple Lie algebras. Any irreducible representation $V$ of a complex Lie algebra $g$ is the tensor product $V = V_0 \otimes L$, where $V_0$ is an irreducible representation of the quotient $g_{ad}/\text{Rad}(g)$ of the algebra $g$ and its Lie algebra radical and $L$ is a one-dimensional representation. In the study of representations of a Lie algebra, a particular ring, called the universal enveloping algebra, associated with the Lie algebra plays an important role. The Riccati algebra is a finite-dimensional linear space that is closed under commutator, that is $R$ is a Lie algebra.

In recent years the $q$-deformation of the Heisenburg commutation relation has drawn attention. Leeuwen and Maassen (1995) and many of other researcher like (Altoum, 2018a; 2018b; Rguigui, 2015a; 2015b; 2016a; 2016b; 2018a; 2018b; Altoum et al., 2017), the purpose is to study the probability distribution of a non-commutative random variable $a + a^*$, where $a$ is a bounded operator on some Hilbert space satisfying:

$$aa^* - qa^* a = 1,$$  

(1)

for some $q \in [-1, 1)$. The calculation is inspired by the case, $q = 0$, where $a$ and $a^*$ turn out to be the left and right shift on $l^2(\mathbb{N})$: In this case $a$ and $a^*$ can be quite nicely represented as operators on the Hardy class $H^2$ of all analytic functions on the unit disk with $L^2$ limits toward the boundary. Subsequently, they find a measure $\mu_q$, $q \in [0, 1)$, on the complex plane that replaces the Lebesgue measure on the unit circle in the above: $\mu_q$ is concentrated on a family of concentric circle, the largest of which has the radius $\frac{1}{\sqrt{1-q}}$. Their representation space (Leeuwen and Maassen, 1995) will be $S^2(\Omega_q, \mu_q)$, the completion of the analytic functions on
\(\mathcal{D}_q = \left\{ z \in \mathbb{C} | |z|^2 < \frac{1}{(1-q)} \right\}\) with respect to the inner product defined by \(\mu_q\). In this space annihilation operator \(a\) is represented by a \(q\) difference operator \(D_q\). As \(q\) tends to 1, \(\mu_q\) will tend to the Gauss measure on \(\mathbb{C}\) and \(D_q\) becomes differentiation. We recall some basic notations of the language of \(q\)-calculus (Abdi, 1962; Adams, 1929; Gasper and Rahman, 1990; Jackson, 1910; Leeuwen and Maassen, 1995). For \(q \in (0, 1)\) and analytic \(f: \mathbb{C} \to \mathbb{C}\) define operators \(Z\) and \(D_q\) as follows (Gasper and Rahman, 1990; Jackson, 1910; Leeuwen and Maassen, 1995):

\[
(Zf)(z) = \frac{f(z) - f(qz)}{z(1-q)}, \ z \neq 0
\]

\[
(D_qf)(z) = \begin{cases}
  f(z) - f(qz), & z \neq 0 \\
  f'(0) & \text{otherwise}
\end{cases}
\]

In this paper, we introduce the \(q\)-Riccati Algebra. This paper is organized as follows: In Section 1, we present preliminaries include \(q\)-calculus. In Section 2, we introduce the \(q\)-Riccati algebra. In section 3, we give a representation of this algebra.

**Representation of the \(q\)-Riccati Algebra**

Let \(q \in (0, 1)\). Then, we define the \(q\)-Riccati Lie algebra as follows:

\[
\mathcal{R}_q = \langle A, B, C, D \rangle
\]

such that:

1. \([A, B] = AD\).
2. \([A, C] = [2]_q CD\).
3. \([B, C] = qCD\).
4. \([A, D] = 0\).
5. \([B, D] = (1-q)BD\).
6. \([C, D] = (1-q)[2]_q CD\).

**Representation of the \(q\)-Riccati Algebra**

Let \(M_{0,q}, M_{1,q}\) and \(M_{2,q}\) given by:

\[
\begin{align*}
M_{0,q} &= D_q \\
M_{1,q} &= XD_q \\
M_{2,q} &= X^2D_q
\end{align*}
\]

where, \(D_q\) and \(X\) are defined as follows:

\[
D_qf(x) = \frac{f(x) - f(qx)}{x(1-q)}
\]

\[
Xf(x) = xf(x).
\]

**Proposition 3.1**

For \(q \in (0, 1)\) we have:

i) \(\left[M_{0,q}, M_{1,q}\right] = M_{0,q}H_q\)

ii) \(\left[M_{0,q}, M_{2,q}\right] = [2]_q M_{1,q}H_q\)

iii) \(\left[M_{1,q}, M_{2,q}\right] = qM_{2,q}H_q\)

where, \(H_q\) is given by \(H_qf(x) = f(qx)\)

**Proof**

We have:

\[
\left[M_{0,q}, M_{1,q}\right] = \left[D_q, XD_q\right]
\]

\[
= D_qXD_q - XD_qD_q
\]

But:

\[
D_qXD_qf(x) = D_q\left(\frac{x f(x) - f(qx)}{x(1-q)}\right)
\]

\[
= \frac{1}{1-q} D_q\left( f(x) - f(qx)\right)
\]

\[
= \frac{1}{1-q} \frac{f(x) - f(qx) - f(qx) + f(q^2x)}{x(1-q)}
\]

\[
= \frac{1}{1-q} \frac{f(x) - 2f(qx) + f(q^2x)}{x(1-q)}
\]

and:

\[
XD_qD_qf(x) = xD_q\left(\frac{x f(x) - f(qx)}{x(1-q)}\right)
\]

\[
= \frac{x}{1-q} \left( \frac{f(x) - f(qx) - f(qx) + f(q^2x)}{x(1-q)} \right)
\]

\[
= \frac{1}{1-q} \frac{qf(x) - qf(qx) - f(qx) + f(q^2x)}{qx(1-q)}
\]

Then, we obtain:

\[
\left[M_{0,q}, M_{1,q}\right]f(x) = \frac{f(qx)(1-q) - (1-q)f(q^2x)}{qx(1-q)^2}
\]

\[
= \frac{f(qx) - f(q^2x)}{qx(1-q)}
\]

\[
= D_qf(qx)
\]

\[
= D_qH_qf(x)
\]

But:
\[ D_q x^2 D_q f(x) = x D_q \left( \frac{x^2 f(x) - f(qx)}{x(1-q)} \right) \]

\[ = \frac{1}{1-q} D_q (xf(x) - xf(qx)) \]

\[ = \frac{1}{1-q} \left( \frac{xf(x) - xf(qx)}{x(1-q)} - xqf(qx) - xqf(q^2x) \right) \]

\[ = \frac{1}{(1-q)} \left( f(x) - (1+q) f(qx) + qf(q^2x) \right) \]

Similarly, we get:

\[ X^2 D_q^2 f(x) = x^2 D_q \left( \frac{f(x) - f(qx)}{x(1-q)} \right) \]

\[ = \frac{x^2}{1-q} \left( \frac{qf(x) - qf(qx) - f(qx) + f(q^2x)}{x(1-q)} \right) \]

\[ = \frac{1}{q(1-q)} \left( qf(x) - (1+q) f(qx) + f(q^2x) \right) \]

Which gives:

\[ \left[ M_{1,q} , M_{2,q} \right] f(x) = \frac{1}{q(1-q)} \left( (1+q)(-q+1) f(qx) + (q^2 - 1) f(q^2x) \right) \]

\[ = x(1+q) \frac{f(qx) - f(q^2x)}{q x} \]

\[ = x[2] q D_q f(qx) \]

\[ = [2] q X D_q H_q f(qx) \]

We have:

\[ \left[ M_{1,q} , M_{2,q} \right] f(x) = [X D_q, X^2 D_q] = X D_q X^2 D_q - X^2 D_q X D_q \]

\[ X D_q X^2 D_q f(x) = x D_q \left( \frac{xf(x) - xf(qx)}{(1-q)} \right) \]

\[ = \frac{x}{1-q} \left( \frac{xf(x) - xf(qx) - qsf(qx) + qsf(q^2x)}{x(1-q)} \right) \]

\[ = \frac{x}{(1-q)} \left( f(x) - (1+q) f(qx) + qf(q^2x) \right) \]

Similarly, we have:

\[ X^2 D_q X D_q f(x) = x^2 D_q \left( \frac{f(x) - f(qx)}{x(1-q)} \right) \]

\[ = \frac{x^2}{1-q} \left( \frac{f(x) - f(qx) - f(qx) - f(q^2x)}{x(1-q)} \right) \]

\[ = \frac{x}{q(1-q)} \left( f(x) - 2f(qx) + f(q^2x) \right) \]

Then, we get:

\[ \left[ M_{1,q} , M_{2,q} \right] f(x) = \frac{x}{q(1-q)} \left( (1-q) f(qx) - (q-1) f(q^2x) \right) \]

\[ = \frac{x}{(1-q)} \left( f(qx) - f(q^2x) \right) \]

\[ = q^2 D_q f(qx) \]

\[ = q X^2 D_q H_q f(x) \]

**Proposition 3.2**

For \( q \in (0, 1) \) we have:

i) \( [M_{0,q} , H_q] = 0 \).

ii) \( [M_{1,q} , H_q] = (1-q)M_{1,q} H_q \).

iii) \( [M_{2,q} , H_q] = (1-q)[2]_q M_{2,q} H_q \).

**Proof**

We have:

\[ \left[ D_q , H_q \right] f(x) = D_q H_q f(x) - H_q D_q f(x) \]

\[ = D_q f(qx) - H_q \frac{f(x) - f(qx)}{x(1-q)} \]

\[ = \frac{f(qx) - f(q^2x)}{q x(1-q)} \]

\[ = 0. \]

Then, we get:

\[ [M_{0,q} , H_q] = 0. \]

We have:

\[ \left[ X D_q , H_q \right] f(x) = X D_q H_q f(x) - H_q X D_q f(x) \]

\[ = x D_q f(qx) - H_q \left( x D_q f(x) \right) \]

\[ = x D_q f(qx) - q D_q f(qx) \]

\[ = (1-q) X D_q H_q f(x). \]
Then, we get:

\[
[M_{\alpha,q}, H_q] = (1-q)M_{\lambda,q}H_q.
\]

We have:

\[
\begin{align*}
X^2D_q H_q f(x) &= X^2D_q H_q f(x) - H_q \left( X^2D_q f(x) \right) \\
&= x^2D_q f(qx) - (qx)^2D_q f(x) \\
&= (1-q^2)X^2D_q H_q f(x) \\
&= (1-q)[2]_q X^2D_q H_q f(x).
\end{align*}
\]

Then, we obtain:

\[
[M_{\alpha,q}, H_q] = (1-q)[2]_q M_{\lambda,q}H_q.
\]

which complete the proof.

Now, we give the representation theorem of the q-Riccati algebra.

**Theorem 3.3**

Let \( \varphi: R_q \to g(\mathfrak{T}(\mathcal{D}_q; \mu_q)) \) a linear mapping such that:

\[
\begin{align*}
\varphi(A) &= M_{\alpha,q} \\
\varphi(B) &= M_{\lambda,q} \\
\varphi(C) &= M_{\lambda,q} \\
\varphi(D) &= H_q.
\end{align*}
\]

Then, \((\mathfrak{T}(\mathcal{D}_q; \mu_q), \varphi)\) is a representation of \( R_q \).

**Proof**

The proof follows from Proposition 3.1 and Proposition 3.2.

**Author’s Contributions**

All authors equally contributed in this work.

**Ethics**

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and no ethical issues involved.

**References**


