Original Research Paper

A New Extension of the Lomax Distribution with Statistical Properties and Applications to Failure and Service Times Data Sets

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Abstract: In this work, we introduce and study a new alternative Lomax model. The maximum likelihood method is used to estimate the unknown model parameters. We show empirically the importance and wide flexibility of the new model in modeling two types of failure times data sets. The new model is much better than the gamma Lomax, exponentiated Lomax, beta Lomax and Lomax models so the new model is a good alternative to these models.

Keywords: Lomax Model, Odd Lindley-G Family, Estimation

Introduction and Justification

A random variable (r.v.) Z has the exponentiated Lomax (EL_x) distribution with three parameters α (power parameter), λ and β, if its Cumulative Distribution Function (CDF) is given by:

$$\Pi_{\alpha,\lambda,\beta}(z)|_{(z>0)} = \left[1-\left(1+z\beta^{-1}\right)^{-\lambda}\right]^{\alpha}, \quad (1)$$

where, $\alpha > 0$, $\lambda > 0$ and $\beta > 0$ are the shape parameters. Then the corresponding Probability Density Function (PDF) of (1) is:

$$\pi_{\alpha,\lambda,\beta}(z)|_{(z>0)} = \alpha\lambda\beta^{-1}\left(1+z\beta^{-1}\right)^{-(1+\lambda)}\left[1-\left(1+z\beta^{-1}\right)^{-\lambda}\right]^{\alpha-1}, \quad (2)$$

when $\alpha = 1$ we get the Lomax (L_x) or the Pareto type II (PaII) model with:

$$G_{\lambda,\beta}(z) = 1-\left(1+z\beta^{-1}\right)^{-\lambda} \text{ and } g_{\lambda,\beta}(z) = \lambda\beta^{-1}\left(1+z\beta^{-1}\right)^{-(1+\lambda)}.$$

The L_x model (Lomax, 1954) was originally pioneered for modeling business failure data. The L_x distribution has been found a wide application in many fields such as engineering, biological sciences, actuarial science, size of cities, income studies, wealth inequality, medical and reliability modeling. It has been applied for modeling data obtained from income and wealth (Harris, 1968; Atkinson and Harrison, 1978), reliability and life testing (Hassan and Al-Ghamdi, 2009), Hirsch related statistics (Glanzel, 2008), for modeling gauge lengths data (Afify *et al*., 2015), for modeling bladder cancer patient's data and remission times data (Yousof *et al*., 2017).

The main goal of this article is to introduce a new L_x model using the Odd Lindley-G (OLi-G) family of distributions (Silva *et al*., 2017) with scale parameter a = 1, the PDF and CDF of the OLi-G family of distribution are respectively given by:

$$f_{\psi}\left(x;a|_{(a=1)}\right) = \frac{1}{2}\pi_{\psi}(x)\bar{\Pi}_{\psi}(x)^{-3}\exp\left[-\frac{\Pi_{\psi}(x)}{\bar{\Pi}_{\psi}(x)}\right], \quad (3)$$

and:

$$F_{\psi}(x;a|a=1) = 1-\left[2+\bar{\Pi}_{\psi}(x)\right]\left[2\bar{\Pi}_{\psi}(x)\right]^{-1}\exp\left[-\frac{\Pi_{\psi}(x)}{\bar{\Pi}_{\psi}(x)}\right], \quad (4)$$

where, $\Pi_{\psi}(x)$ is the baseline CDF, $\psi = (\alpha, \lambda, \beta)$ is the parameter vector of the baseline distribution and $\bar{\Pi}(x,\psi)$ = 1-$\Pi(x, \psi)$ is the Survival Function (SF) of the baseline distribution. To this end, we use Equations (1), (2) and (3) to obtain the new three-parameter OLiEL$_x$ PDF (for $x > 0$):

$$f_{\alpha,\lambda,\beta}(x) = \frac{1}{2}\alpha\lambda\beta^{-1}\left(1+x\beta^{-1}\right)^{-(1+\lambda)}\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha-1} \times\left\{1-\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha}\right\}^{-3}\exp\left\{-\frac{\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha}}{1-\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha}}\right\}, \quad (5)$$

the corresponding CDF to (5) is given by:

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$$F_{\alpha,\lambda,\beta}(x)=1-\left[\begin{array}{l}\left(2+\left\{1-\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha}\right\}\right)\\ \times\left(2\left\{1-\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha}\right\}\right)^{-1}\\ \times\exp\left\{-\dfrac{\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha}}{1-\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha}}\right\},\end{array}\right]. \tag{6}$$

when α = 1 we get the two-parameter OLiL$_x$ model (Silva *et al*., 2017). The Hazard Rate Function (HRF) of the OLiEL$_x$ distribution can be obtained by as:

$$\begin{aligned}h_{\alpha,\lambda,\beta}(x)&=\frac{1}{2}\alpha\lambda\beta^{-1}\left(1+x\beta^{-1}\right)^{-(1+\lambda)}\\ &\times\left(\frac{2+\left\{1-\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha}\right\}}{2\left\{1-\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha}\right\}}\right)^{-1}\\ &\times\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha-1}\left\{1-\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha}\right\}^{-3}.\end{aligned}$$

We draw the PDF and HRF plots of the OLiEL$_x$ distribution in Fig. 1 and 2 for selected parameters values. We see that its PDF can be unimodal and decreasing. Also, its HRF can be only increasing.

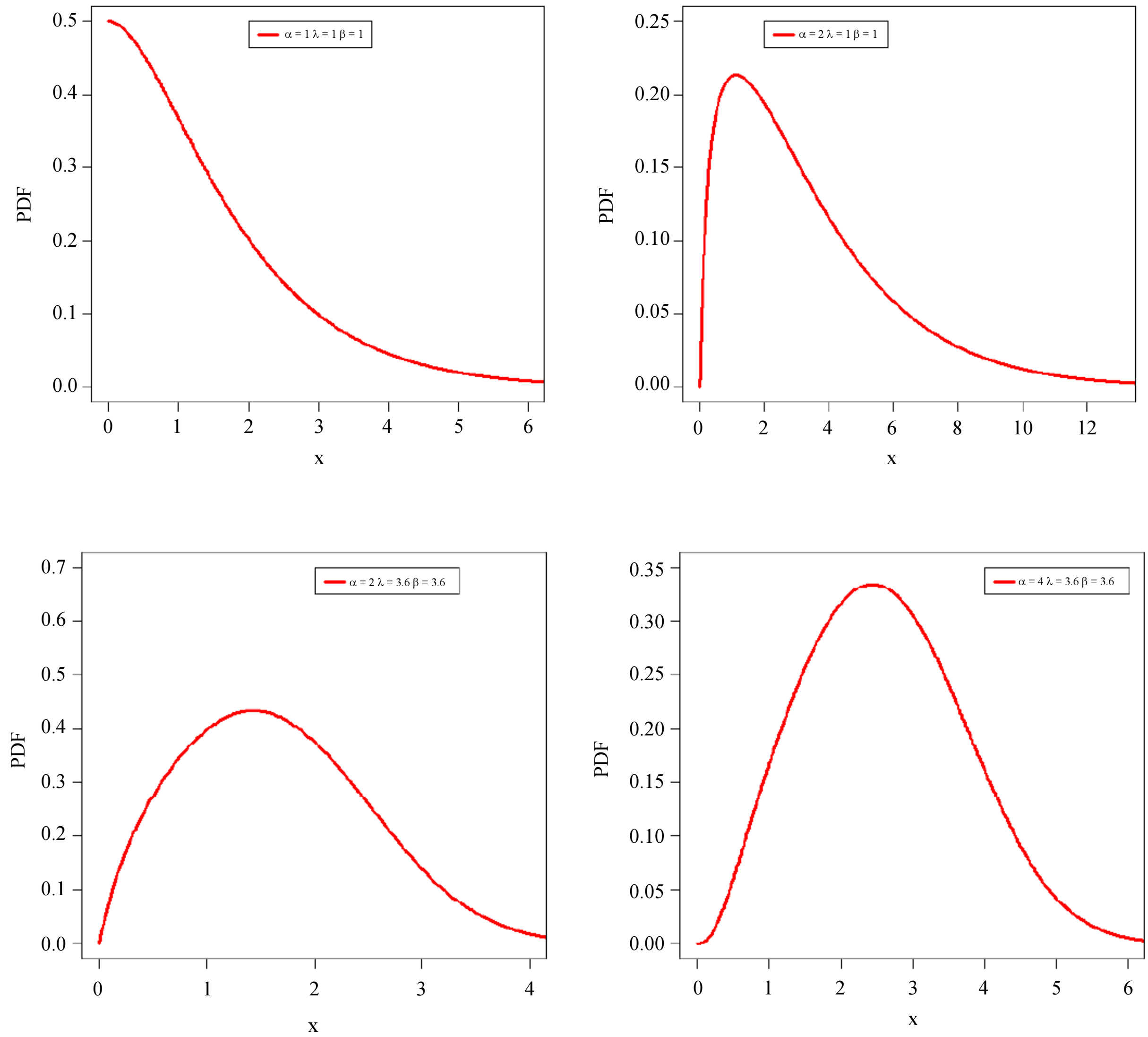

Fig. 1: Plots of the OLiELx PDF

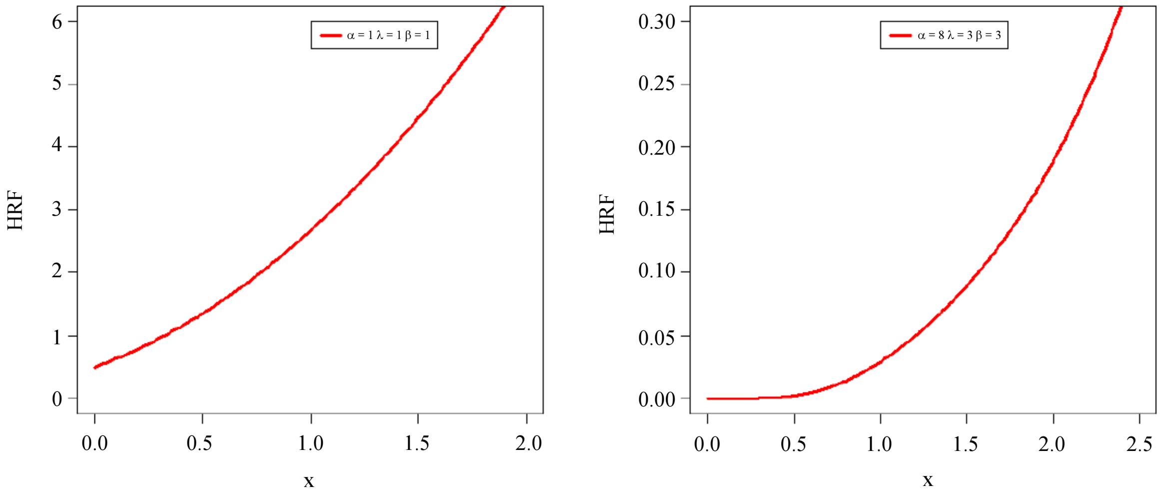

Fig. 2: Plots of the OLiEL$_x$ HRF

The major justification for the practicality of the new model is based on the enormous use of the EL$_x$ and L$_x$ models in modeling failure times data sets. The OLiEL$_x$ is a good alternative to the EL$_x$, the gamma Lomax, the beta Lomax, L$_x$ models as illustrated using two real data sets. The OLiEL$_x$ can also be considered as an appropriate model for fitting the right skewed and the unimodal data sets (see applications 1 and 2).

The rest of the paper is outlined as follows. In section 2, we derive some statistical properties for the new model. Maximum likelihood estimation of the model parameters is addressed in section 3. Section 4 gives the simulation studies. We provide two applications to real data sets to illustrate the importance of the new model in Section 4. Finally, we offer some concluding remarks in Section 5.

Statistical Properties

Shapes

The critical points of the OLiEL$_x$ density function are the roots of the equation:

$$
\begin{aligned}
0 &= 3\alpha\lambda\beta^{-1}\left(1+x\beta^{-1}\right)^{-(1+\lambda)}\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha-1}\left\{1-\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha}\right\}^{-1} \\
&+\alpha\lambda\beta^{-1}\left(1+x\beta^{-1}\right)^{-(1+\lambda)}\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha-1}\left\{1-\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha}\right\}^{-2} \\
&+\left(\begin{array}{l}\beta(\alpha\lambda)^{-1}\left(1+x\beta^{-1}\right)^{1+\lambda}\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{1-\alpha} \\ \times\dfrac{d}{dx}\left\{\alpha\lambda\beta^{-1}\left(1+x\beta^{-1}\right)^{-(1+\lambda)}\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha-1}\right\}\end{array}\right).
\end{aligned}
$$

The critical points of the of the HRF of the OLiEL$_x$ are obtained from the following equation:

$$
\begin{aligned}
0 &= \alpha\lambda\beta^{-1}\left(1+x\beta^{-1}\right)^{-(1+\lambda)}\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha-1}\left\{2-\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha}\right\} \\
&+2\alpha\lambda\beta^{-1}\left(1+x\beta^{-1}\right)^{-(1+\lambda)}\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha-1}\left\{1-\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha}\right\}^{-2} \\
&+\left(\begin{array}{l}(\alpha\lambda)^{-1}\beta\left(1+x\beta^{-1}\right)^{1+\lambda}\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{1-\alpha} \\ \dfrac{d}{dx}\left\{\alpha\lambda\beta^{-1}\left(1+x\beta^{-1}\right)^{-(1+\lambda)}\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha-1}\right\}\end{array}\right).
\end{aligned}
$$

We can examine the last two Equations to determine the local maximums and minimums and inflexion points via most computer algebra systems.

Quantile Function

Let X be a r.v. with CDF $F_{\lambda,\beta}(x)$. For any $u \in (0.1)$, the u^{th} Quantile Function (QF) $Q(u)$ of the r.v. X is the solution of $u = F(Q(u))$ for all $Q(u) > 0$, from Equation (6), we get:

$$
(u-1)2\exp(2) = -\frac{2-G(Q(u))}{1-G(Q(u))}\exp\left\{-\frac{2-G(Q(u))}{1-G(Q(u))}\right\},
$$

where:

$$
-\frac{2-G(Q(u))}{1-G(Q(u))},
$$

is the Lambert $W(\cdot)$ function of the real argument 2 $(u-1)$ exp (2): From Silva *et al*. (2017), we can write the following equation for QF of the new model as:

$$
Q(u) = \beta\left[\left(1-\left\{1-a\left[1+W\left(2(u-1)\exp(2)\right)\right]^{-1}\right\}^{\frac{1}{\alpha}}\right)^{-\frac{1}{\lambda}}-1\right],
$$

where, $W(\cdot)$ is Lambert function.

Useful Expansions

Starting with the PDF in (5) and using the power series for the exponential function:

$$\exp\left\{-\frac{\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha}}{1-\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha}}\right\},$$

we have:

$$f_{\alpha,\lambda,\beta}(x)=\frac{1}{2}\overbrace{\alpha,\lambda,\beta^{-1}\left(1+x\beta^{-1}\right)^{-(1+\lambda)}\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha-1}}^{\bar{\pi}_{\alpha,\lambda,\beta}(x)}$$
$$\times\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\overbrace{\left\{\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha}\right\}^{k}}^{\left[\Pi_{\alpha,\lambda,\beta}(x)\right]^{k}}$$
$$\underbrace{\times\left\{1-\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha}\right\}^{-(3+k)}}_{\left[1-\Pi_{\alpha,\lambda,\beta}(x)\right]^{-(3+k)}},$$

Using the generalized binomial expansion, we can write:

$$\left\{1-\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha}\right\}^{-(k+3)}=\sum_{m=0}^{\infty}\frac{\Gamma(3+m+k)}{m!\Gamma(3+k)}\left\{\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha}\right\}^{m},$$

then the OLiEL$_x$ density function can be expressed as an infinite mixture of exponentiated-L$_x$ (EL$_x$) density functions:

$$f_{\alpha,\lambda,\beta}(x)=\sum_{m,k=0}^{\infty}v_{m,k}\pi_{[(1+m+k)\alpha+1],\lambda,\beta}(x), \tag{7}$$

where:

$$v_{m,k}=\frac{(-1)^k\Gamma(3+m+k)}{m!k!(1+\lambda)\left[(1+m+k)\alpha+1\right]\Gamma(3+k)}$$

and:

$$\pi_{[(1+m+k)\alpha+1],\lambda,\beta}(x)=\left[(1+m+k)\alpha+1\right]$$
$$\underbrace{\times\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{-(1+m+k)\alpha}}_{\Pi_{(1+m+k)\alpha,\lambda,\beta}(x)}$$
$$\underbrace{\times\alpha\lambda\beta^{-1}\left(1+x\beta^{-1}\right)^{-(1+\lambda)}\left[1-\left(1+x\beta^{-1}\right)^{-\lambda}\right]^{\alpha-1}}_{\pi_{\alpha,\lambda,\beta}(x)}$$

is the EL$_x$ density with power parameter $[(1+m+k)\alpha+1]$. Similarly:

$$F_{\alpha,\lambda,\beta}(x)=\sum_{m,k=0}^{\infty}v_{m,k}\Pi_{[(1+m+k)\alpha+1],\lambda,\beta}(x), \tag{8}$$

Moments

The $r^{(th)}$ ordinary moment of X is given by:

$$\mu_r'=E\left(X^r\right)=\sum_{m,k=0}^{\infty}v_{m,k}\int_0^{\infty}x^r\pi_{[(1+m+k)\alpha+1],\lambda,\beta}(x)dx,$$

then we obtain:

$$\mu_r'=\sum_{m,k=0}^{\infty}\sum_{w=0}^{r}v_{m,k,w}^{(r,[(1+m+k)\alpha+1])}B\begin{pmatrix}\left[(1+m+k)\alpha+1\right],\\ 1+(w-r)/\lambda\end{pmatrix}|_{(r\leq\lambda)} \tag{9}$$

where:

$$B(c;d)=\int_0^1 u^{c-1}(1-u)^{d-1}du=\Gamma(c)\Gamma(d)/\Gamma(c+d)|_{(c,d\notin(0,-1,-2,\ldots))}$$

is the complete beta function and:

$$v_{m,k,w}^{(r,[(1+m+k)\alpha+1])}=v_{m,k}\left(\left[(1+m+k)\alpha+1\right]\right)\beta^r(-1)^w\binom{r}{w}.$$

Setting $r = 1, 2, 3$ and 4 in (9) we get:

$$E(X)=\mu_1'=\sum_{m,k=0}^{\infty}\sum_{w=0}^{r}v_{m,k,w}^{(1,[(1+m+k)\alpha+1])}B\left(\left[(1+m+k)\alpha+1\right]\right),1+(w-1)/\lambda|_{(1\leq\lambda)},$$

which is the mean of X:

$$E\left(X^2\right)=\mu_2'=\sum_{m,k=0}^{\infty}\sum_{w=0}^{r}v_{m,k,w}^{(2,[(1+m+k)\alpha+1])}B\left(\left[(1+m+k)\alpha+1\right]\right),1+(w-2)/\lambda|_{(2\leq\lambda)},$$

$$E\left(X^3\right)=\mu_3'=\sum_{m,k=0}^{\infty}\sum_{w=0}^{r}v_{m,k,w}^{(3,[(1+m+k)\alpha+1])}B\left(\left[(1+m+k)\alpha+1\right]\right),1+(w-3)/\lambda|_{(3\leq\lambda)},$$

$$E\left(X^4\right)=\mu_4'=\sum_{m,k=0}^{\infty}\sum_{w=0}^{r}v_{m,k,w}^{(4,[(1+m+k)\alpha+1])}B\left(\left[(1+m+k)\alpha+1\right]\right),1+(w-4)/\lambda|_{(4\leq\lambda)},$$

The last four equations can be used for obtaining the first four moments about the mean. In general the $r^{(\text{th})}$ central moment of X, say M_r, is:

$$\mu_r = E(X-\mu)^r = \sum_{h=0}^{r}(-1)^h \binom{r}{h} (\mu_1')^r \mu_{r-h}',$$

where, $\mu = E(X)$. The skewness $\left(\sqrt{\beta_1}\right)$ and kurtosis (β_2) measures also can be calculated from the ordinary moments using well-known relationships. For the skewness and kurtosis coefficients, we have:

$$\sqrt{\beta_1} = \sqrt{\frac{\mu_3^3}{\mu_2^3}} \text{ and } \beta_2 = \frac{\mu_4}{\mu_2^2},$$

respectively.

We prove numerically that the new distribution provides better fits to two real data sets than other four extended L_x models with two, three and four parameters (see Section 4). These two examples show that the new OLiEL$_x$ distribution is a good alternative for modeling failure times data. Further, the OLiEL$_x$ density can be right-skewed or symmetric (Fig. 1). Whereas the OLiEL$_x$ HRF can be monotonically increasing (Fig. 2). The $\sqrt{\beta_1}$ of the OLiEL$_x$ distribution can range in the interval (-2.106, 4.424), whereas the β_2 of the OLiEL$_x$ distribution varies only in the interval (-14, 38) (Table 1).

Generating Function

The moment generating function (mgf) $M_X(t) = E(e^{tX})$ can be derived from Equation (7) as follows:

$$M_X(t) = \sum_{m,k,r=0}^{\infty}\sum_{w=0}^{r}\frac{t^r}{r!} v_{m,k,w}^{(r,[(1+m+k)\alpha+1])} B\left(\left[(1+m+k)\alpha+1\right]\right),$$
$$1+(w-r)/\lambda\,|_{(r\le\lambda)},$$

Incomplete Moments and Mean Deviations

The $s^{(th)}$ incomplete moment, say $I_s(t)$, of X can be expressed from (7) as:

$$I_s(t) = \sum_{m,k=0}^{\infty}\sum_{w=0}^{r} v_{m,k,w}^{(s,[(1+m+k)\alpha+1])} B_t\left(m+k+1, 1+(w-s)/\lambda\right)|_{(s\le\lambda)},$$

where:

$$B_t(p;q) = \int_0^t u^{p-1}(1-u)^{q-1}\,du = \sum_{j=0}^{\infty}\frac{(1-q)_j}{j!(p+j)} t^{p+j}$$

is the incomplete beta function and:

$$(n)_k = n(n+1)\ldots(n+k-1)$$

is known as Pochhammer's symbol after the German mathematician Pochhammer [1841-1920].

The mean deviations about the mean:

$$MD_{(E(X))} = E\left(|X-E(X)|\right)$$
$$= -2I_1 E(X) + 2E(X)F\left(E(X)\right)$$

and about the median:

$$MD_{(Median(X))} = E\left(|X-Median(X)|\right)$$
$$= -2I_1\left(Median(X)\right) + E(X)$$

of X, $F(E(X))$ is easily calculated from (5) and $I_1(t)$ is the first incomplete moment given by the last Equation with $s = 1$. A general equation for $I_1(t)$ can be derived from $I_s(t)$ as:

$$I_1(t) = \sum_{m,k=0}^{\infty}\sum_{w=0}^{r} v_{m,k,w}^{(1,[(1+m+k)\alpha+1])} B_t\left(\left[(1+m+k)\alpha+1\right], 1+(w-1)/\lambda\right).$$

Moment of Residual and Reversed Residual Life

The $n^{(th)}$ moment of the residual life, say:

$$m_n(t) = E\left[(X-t)^n\,|_{(X>t,n=1,2,\ldots)}\right],$$

uniquely determines $F(x)$. The $n^{(th)}$ moment of the residual life of X is given by:

$$m_n(t) = \left[1-F_{\alpha,\lambda,\beta}(t)\right]^{-1}\int_t^{\infty}(x-t)^n\,dF_{\alpha,\lambda,\beta}(x),$$

therefore:

$$m_n(t) = \sum_{m,k=0}^{\infty}\sum_{w=0}^{s}\frac{a_{m,k,w}^{(n,[(1+m+k)\alpha+1])}}{1-F_{\lambda,\beta}(t)} B_t\left(\left[(1+m+k)\alpha+1\right], 1+(w-n)/\lambda\right)|_{(n\le\lambda)},$$

where:

$$a_{m,k,w}^{(n,[(1+m+k)\alpha+1])} = v_{m,k,w}^{(n,[(1+m+k)\alpha+1])}\sum_{d=0}^{n}(1-t)^n.$$

The $n^{(th)}$ moment of the reversed residual life, say:

$$M_n(t) = E\left[(t-X)^n\,|_{(X\le t,t>0,n=1,2,\ldots)}\right],$$

uniquely determines $F(x)$. We obtain:

$$M_n(t) = F_{\alpha,\lambda,\beta}(t)^{-1}\int_0^t (t-x)^n\,dF_{\alpha,\lambda,\beta}(x).$$

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Table 1: Mean, variance, skewness and kurtosis of the OLiEL$_x$ distribution with β = 1.5 and different values of α and λ

λ	α	Mean	Variance	$\sqrt{\beta_1}$	β_2
0.5	0.5	2.872888	17.1728	3.784633	29.29589
	1	10.5	222.7499	4.02216	33.08947
	2	38.44636	3137.958	4.223212	36.22753
	5	222.8358	112857.4	4.370628	38.57605
	10	864.9294	1755134	4.423885	-14.07279
	20	3404.382	27681971	-2.105726	2.763589
2.5	0.5	0.2794296	0.0629023	1.179014	4.319735
	1	0.6025587	0.1720848	0.7994452	3.377934
	2	1.104976	0.3774538	0.597283	3.064221
	5	2.109887	0.9002299	0.4757101	2.940227
	10	3.198071	1.638229	0.4364157	2.910089
	20	4.656197	2.915106	0.4172134	2.896965
5	0.5	0.1299453	0.01242288	1.007976	3.682173
	1.5	0.3752224	0.04227459	0.4161971	2.681674
	2.5	0.5382854	0.06152505	0.2757932	2.607453
	5	0.8070189	0.09249494	0.1645138	2.596024
	10	1.129978	0.1303217	0.1072898	2.606683
	20	1.509419	0.177695	0.07841155	2.616292

Then, the $n^{(th)}$ moment of the reversed residual life of X becomes:

$$M_n(t) = \sum_{m,k=0}^{\infty}\sum_{w=0}^{s}\frac{c_{m,k,w}^{(n,[(1+m+k)\alpha+1])}}{F_{\alpha,\lambda,\beta}(t)}B_t\begin{pmatrix}[(1+m+k)\alpha+1],\\ 1+(w-n)/\lambda\end{pmatrix}\Big|_{(n\le\lambda)},$$

where:

$$c_{m,k,w}^{(n,[(1+m+k)\alpha+1])} = v_{m,k,w}^{(n,[(1+m+k)\alpha+1])}\sum_{d=0}^{n}(-1)^d\binom{n}{d}t^{n-d}.$$

Order Statistics

Let $X_1,\ldots,X_n$ be a random sample from the OLiEL$_x$ model of distributions and let $X_{1:n},\ldots,X_{n:n}$ be the corresponding order statistics. The PDF of the $i^{(th)}$ order statistic, say $X_{i:n}$, can be expressed as:

$$f_{\alpha,\lambda,\beta}^{(i:n)}(x) = \left[B(i,n-i+1)\right]^{-1} f_{\alpha,\lambda,\beta}(x)F_{\alpha,\lambda,\beta}(x)^{i-1}\left[1-F_{\alpha,\lambda,\beta}(x)\right]^{n-i}, \quad (10)$$

where, $B(\cdot,\cdot)$ is the beta function. Substituting (5) and (6) in Equation (10), we obtain:

$$f_{\alpha,\lambda,\beta}^{(i:n)}(x) = \sum_{m,p=0}^{\infty}\sum_{j=0}^{k+n-i}\Upsilon_{m,p,j}\pi_{[(j+m+p)\alpha+1],\lambda,\beta}(x),$$

where:

$$\Upsilon_{m,p,j} = \sum_{k=0}^{i=1}\frac{(-1)^{k+m}\lambda^{j+m+2}(1+\lambda)^{-(j+1)}}{m!B(i,n-i+1)[(j+m+p)\alpha+1]}\binom{(j+m+p)\alpha}{j+m}\binom{k+n-1}{j}\binom{i-1}{k}.$$

Then, the q^{th} moment of $X_{i:n}$ is given by ($\forall\lambda>q$):

$$E\left(X_{i:n}^q\right) = \sum_{m,p=0}^{\infty}\sum_{j=0}^{k+n-i}\sum_{w=0}^{r}\Upsilon_{m,k,j,w}^{(q,[(j+m+p-1)\alpha+1])} B\left(\left[(j+m+p-1)\alpha+1\right],1+(w-q)/\lambda\right),$$

where:

$$\Upsilon_{m,k,j,w}^{(q,[(j+m+p-1)\alpha+1])}\Upsilon_{m,p,j}\left[(j+m+p)\alpha+1\right]\beta^q(-1)^w\binom{q}{w}.$$

Estimation

The Maximum Likelihood Estimators (MLEs) enjoy desirable properties and can be used for constructing confidence intervals and regions and also in test statistics. Let $x_1,\ldots,x_n$ be a random sample from OLiEL$_x$ distribution with parameters α, λ and β. Let $\Psi = (\alpha, \lambda, \beta)^{\mathrm{T}}$ be the 3×1 parameter vector. For determining the MLE of Ψ, we have the log-likelihood function:

$$\begin{aligned}\ell = \ell(\Psi) &= -n\log 2 + n\log\alpha + n\log\lambda \\ &-n\log\beta - (1+\lambda)\sum_{i=1}^{n}\log\left(1+x_i\beta^{-1}\right) \\ &+(\alpha-1)\sum_{i=1}^{n}\log\left[1-\left(1+x_i\beta^{-1}\right)^{-\lambda}\right] \\ &-3\sum_{i=1}^{n}\log\left\{1-\left[1-\left(1+x_i\beta^{-1}\right)^{-\lambda}\right]^{\alpha}\right\} \\ &-\sum_{i=1}^{n}\left(\left[1-\left(1+x_i\beta^{-1}\right)^{-\lambda}\right]^{\alpha}/\left\{1-\left[1-\left(1+x_i\beta^{-1}\right)^{-\lambda}\right]^{\alpha}\right\}\right).\end{aligned} \quad (11)$$

The components of the score vector:

$$U(\Psi) = \frac{\partial \ell(\Psi)}{\partial \Psi} = \left(\frac{\partial \ell(\Psi)}{\partial \alpha}, \frac{\partial \ell(\Psi)}{\partial \lambda}, \frac{\partial \ell(\Psi)}{\partial \beta} \right)^{\mathrm{T}},$$

can be easily derived.

Simulation Studies

We consider a random sample of size n = 20, 50, 100, 150, 500 and 1000 from the OliEL$_x$ density corresponding to particular choices of the parameters:

- I: $\alpha = 0.5$, $\lambda = 1.5$, $\beta = 0.5$
- I: $\alpha = 0.5$, $\lambda = 3$, $\beta = 1.5$

The results are presented in Table 2. We provide the MLEs, biases (Bias) and Mean Square of Errors (MSEs) for the estimates of all the parameters under both the methods of estimation. The results show that the maximum likelihood estimation performs well. In general, the biases and MSEs of the parameters are reasonably small. The biases and MSEs decrease as the sample size increases. The results suggest that the ML method can be used to estimate the parameters of the OLiEL$_x$ model.

Real Data Modeling

In this section, we provide two applications to two real data sets to prove the importance and flexibility of the OLiEL$_x$ distribution. We compare the fit of the OLiEL$_x$ with competitve L$_x$ models namely: the EL$_x$ model (Gupta *et al*., 1998), the gamma Lomax (GaL$_x$) model (Cordeiro *et al*., 2015), the beta Lomax (BL$_x$) model (Lemonte and Cordeiro, 2013) and L$_x$ model. The CDFs of these distributions are, respectively, given by (for $x > 0$ and α, β, λ, $a > 0$):

$$F_{\alpha,\beta,\lambda}(x) = \left[1 - \left(1 + x\beta^{-1}\right)^{-\lambda}\right]^{\alpha},$$

$$F_{\alpha,\beta,\lambda}(x) = \Gamma^{-1}(\alpha)\Gamma\left(\alpha; \lambda \log\left[1 + x\beta^{-1}\right]\right)$$

and:

$$F_{\alpha,\beta,\theta,\lambda}(x) = \frac{1}{B(\alpha,\theta)} B\left(1 - \left(1 + x\beta^{-1}\right)^{-\lambda}; \alpha, \theta\right),$$

where, $\Gamma(\cdot)$ is the gamma function, $\Gamma(\cdot;\cdot)$ is the incomplete gamma function, $B(\cdot,\cdot)$ is the complete beta function and $B(\cdot;\cdot\ \cdot,\cdot)$ is the incomplete beta function.

Data Set I

The first real data set represents the data on failure times of 84 aircraft wind-shield given in Murthy *et al*. (2004). The data are: 0.0400, 1.8660, 2.3850, 3.4430, 0.3010, 1.8760, 2.4810, 3, 4.035, 1.281, 2.0850, 1.98100, 2.6610, 4.449, 1.6190, 2.224, 2.890, 4.121, 1.3030, 2.089, 3.7790, 3.4670, 0.3090, 1.8990, 2.610, 3.4780, 0.5570,1.248, 2.0100, 2.688, 3.9240, 1.2810, 2.038, 2.820, 2.902, 4.167, 1.4320, 2.097, 2.934, 4.2400, 0.94300, 1.9120, 2.632, 3.5950, 1.0700, 1.91400, 2.2230, 3.1140, 1.9110, 2.6250, 3.5780, 3.699, 1.12400, 2.6460, 1.480, 2.135, 1.5060, 2.190, 3.000, 4.3050, 1.568, 2.1940, 2.962, 4.2550, 1.5050, 2.1540, 2.9640, 4.2780, 3.103, 4.376, 1.615, 3.1170, 4.485, 1.652, 2.2290, 3.166, 4.570, 1.652, 2.3000, 3.344, 4.602, 1.7570, 2.324, 3.3760, 4.6630. The Total Time Test (TTT) plot (Aarset, 1987) for data set I is presented in Fig. 3. From Fig. 3 we note that the empirical HRF of data sets is increasing.

Table 2: The MLEs and Bias and MSE values for the OLiELx

	$\hat{\alpha}$		$\hat{\lambda}$		$\hat{\beta}$	
n	Bias	MSE	Bias	MSE	Bias	MSE
20	0.186	0.04	-0.421	0.19	0.308	0.108
50	0.183	0.036	-0.426	0.186	0.304	0.097
100	0.182	0.035	-0.431	0.189	0.304	0.095
150	0.183	0.034	-0.429	0.185	0.304	0.094
500	0.183	0.034	-0.032	0.187	0.305	0.093
1000	0.081	0.034	-0.032	0.187	0.005	0.093
20	0.519	0.052	-0.975	0.992	1.91	0.907
50	0.208	0.049	-0.996	1.009	0.901	0.844
100	0.117	0.049	-0.995	0.997	0.906	0.36
150	0.102	0.048	-0.994	0.994	0.911	0.84
500	0.008	0.048	-0.547	1.001	0.833	0.911
1000	0.008	0.048	-0.1	0.001	0.91	0.83

Data Set II

The second real data set represents the data on service times of 63 aircraft windshield given in Murthy *et al*. (2004). The data are: 0.046, 1.436, 2.4640, 4.8810, 1.262, 2.5430,2.592, 0.140, 1.492, 2.600, 0.150, 0.2800, 2.878, 0.487, 1.9630, 2.950, 0.6220, 1.978, 3.0030, 0.389, 1.9200, 3.622, 1.0850, 2.163, 0.9000, 2.053, 3.1020, 1.580, 2.670, 0.248, 1.7190, 2.717, 0.952, 2.065, 3.3040, 0.9960, 2.117, 3.483, 1.003, 2.1370, 3.500, 1.0100, 2.141, 1.794, 2.819, 0.3130, 1.915, 2.820, 3.665, 1.0920, 2.183, 3.695, 1.1520, 2.2400, 4.015, 1.183, 2.3410, 4.628, 1.2440, 2.435, 4.806, 1.2490, 5.140. The TTT plot for data set II is presented in Fig. 4. From Fig. 4 we note that the empirical HRF of data sets is also increasing.

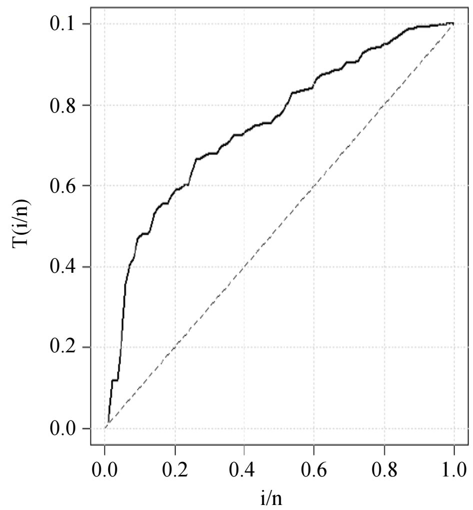

Fig. 3: TTT plots for data set I

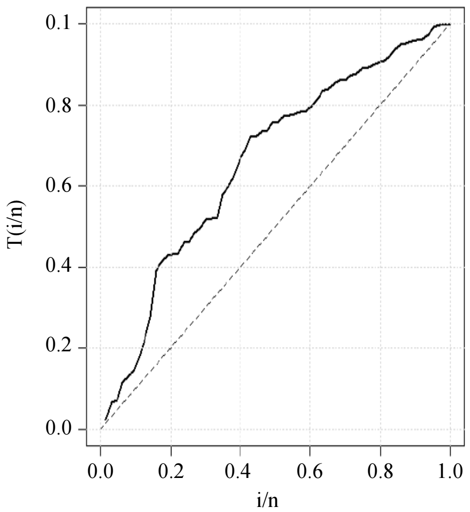

Fig. 4: TTT plots for data set II

These data sets were recently studied by Tahir *et al*. (2015). The unit for measurement is 1000 h for both data sets. In order to compare the distributions, the estimated log-likelihood values $\hat{\ell}$, Akaike Information Criteria (AIC), Cramer von Mises (W^*) and Anderson-Darling (A^*) goodness of fit statistics were calculated for all models. The statistics W^* and A^* are described in detail in Chen and Balakrishnan (1995). The W^* and A^* statistics are given by:

$$W^* = (1+1/2n)\left[\frac{1}{12n} + \sum_{j=1}^{n} \omega_j\right],$$

and:

$$A^* = c^{(n)}\left(n + n^{-1}\sum_{j=1}^{n} c_j\right),$$

where:

$$\omega_j = \left[z_i - (2j-1)/(2n)\right]^2,$$
$$c^{(n)} = 1 + \left[9/\left(4n^2\right)\right] + \left[3/(4n)\right],$$

and:

$$c_j = (2j-1)\log\left[z_i\left(1 - z_{n-j+1}\right)\right],$$

where, $z_i = F(y_j)$ and the y_j'*s* values are the ordered observations. The smaller these statistics are, the better the fit is. In general, it can be chosen as the best model which has the smaller values of the AIC, W^* and A^* statistics and the larger values of $\hat{\ell}$. The required computations are obtained by using the "maxLik" and "goftest" sub-routines in R-software. The analysis results of both these applications are listed in Table 3-6. These results show that the new distribution has the lowest AIC, W^* and A^* values and, has the biggest estimated $-\hat{\ell}$ among all the fitted models. Hence, it could be chosen as the best model under these criteria.

From Table 5 and 6, the proposed OLiEL$_x$ lifetime model is much better than GaL$_x$, BL$_x$, EL$_x$ and L$_x$ models so the new model is a good alternative to these models in modeling aircraft windshield data.

Figure 5 gives the fitted PDF plot for the two data sets. Figure 6 gives the P-P plot for the two data sets. Figure 7 gives the fitted HRF plot for the two data sets. From Fig. 5-7 we note that the proposed model give adequate fit to the used data sets.

The values in all tables are calculated using the *R* program. We prove empirically that the new distribution provides better fits in two applications to real data sets than other competitive extended L_x distributions. The OLiEL$_x$ distribution can be a good alternative for modeling failure times.

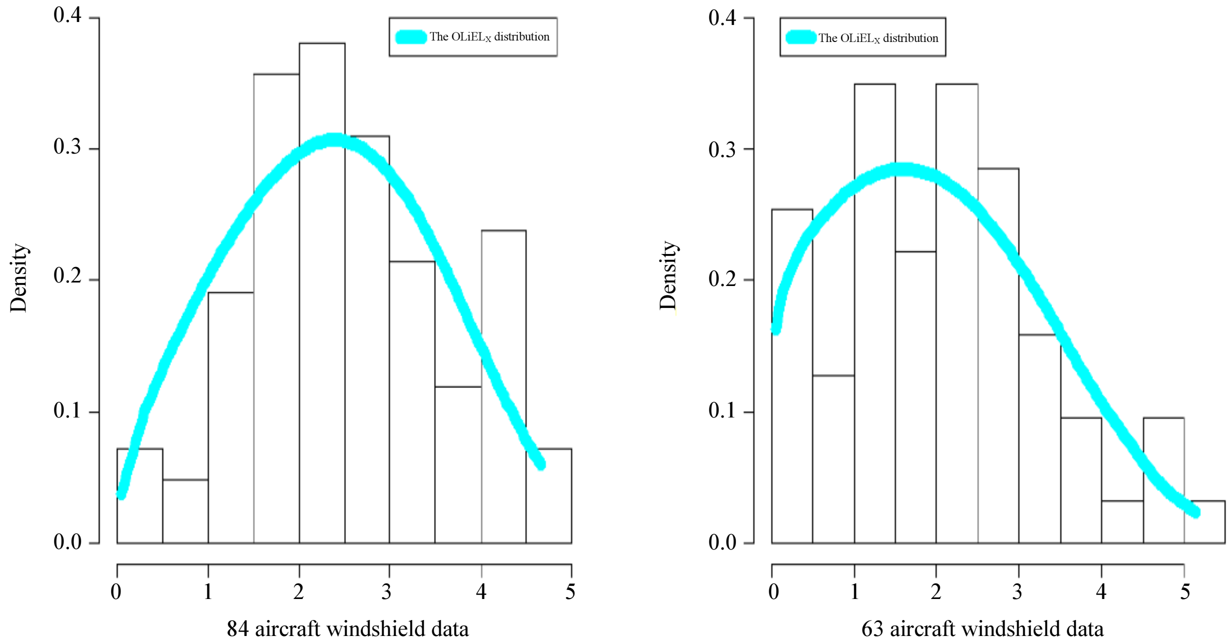

Fig. 5: The fitted PDF plot for data set I (left panel), the fitted PDF and CDF plot for data set II (right panel)

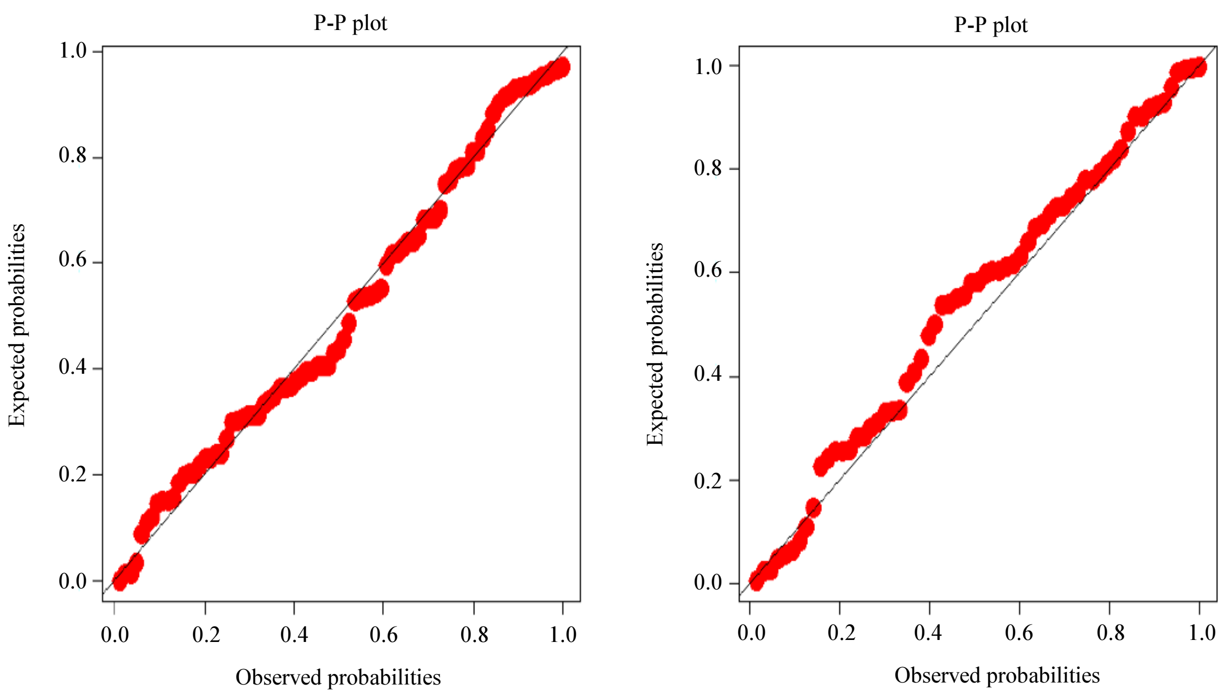

Fig. 6: The P-P plot for data set I (left panel), the P-P plot for data set II (right panel)

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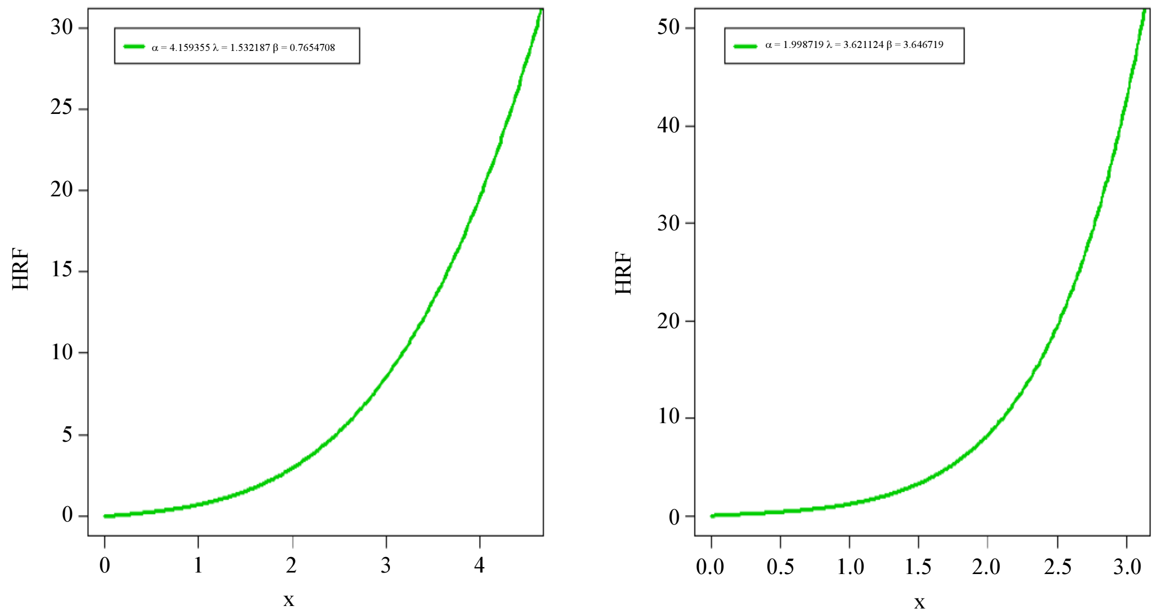

Fig. 7: The fitted HRF plot for data set I (left panel), the fitted PDF and HRF plot for data set II (right panel)

Table 3: MLEs, standard erros (in parentheses) for data set I

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$
$OLiEL_x$	0.1593	0.7655		0.7322
	(0.3712)	(0.04057)		(1.7779)
BL_x	3.6036	118.8374	33.638	4.8307
	(0.6187)	(63.714)	(9.238)	(429.00)
EL_x	3.6261	26257.6808		20074.51
	(0.6236)	(99.742)		(2041.83)
GL_x	3.5876	37029		52001
	(0.5133)	(81.1644)		(7955)
L_x		131789		51425
		(296.1200)		(5933.49)

Table 4: MLEs, standard erros (in parentheses) for data set II

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$
$OLiEL_x$	0.5987	0.6467		1.624
	(0.39)	(0.05)		(0.96)
BL_x	1.9218	169.5800	31.2595	4.9685
	(0.3185)	(339.21)	(316.8)	(50.53)
EL_x	1.9145	32881.9		22971.2
	(0.3483)	(162.22)		(3209.5)
GL_x	1.9073	39197.6		35842.4
	(0.3214)	(151.653)		(6945)
L_x		207019		99269
		(301.237)		(11863.5)

Table 5: $-\hat{\ell}$ and goodness-of-fits statistics for data set I

Model	$-\hat{\ell}$	AIC	W^*	A^*
$OLiEL_x$	126.923	259.846	0.0907	0.0875
BL_x	138.718	285.4354	1.4084	0.1680
EL_x	141.3997	288.7994	1.7435	0.2194
GL_x	138.404	282.809	1.3667	0.1619
L_x	164.990	333.9767	1.3976	0.1665

Table 6: $-\hat{\ell}$ and goodness-of-fits statistics for data set II.

Model	$-\hat{\ell}$	AIC	W^*	A^*
$OLiEL_x$	**98.10294**	**202.2059**	**0.4989**	**0.0488**
BL_x	102.9611	213.9223	1.1336	0.1872
EL_x	103.5468	213.9223	1.2331	0.2037
GL_x	102.8333	211.6664	1.1121	0.2038
L_x	109.2988	222.5976	1.1265	0.1861

Conclusion

In this article, we introduce and study a new alternative Lomax model. The maximum likelihood estimation method is used to estimate the unknown model parameters. We show empirically the importance and wide flexibility of the new model in modeling two types of failure times data sets. This model is much better than gamma Lomax, exponentiated Lomax, beta Lomax and Lomax models so the new model is a good alternative to these models. We hope that the new model will attract wider applications in engineering, reliability and other areas of research.

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Ethics

The author declares that there is no conflict of interests regarding the publication of this article.

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