Coupled Pettis-Hadamard Fractional Differential Systems with Retarded and Advanced Arguments

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Abstract: In this study, we present some results concerning the existence of weak solutions for some coupled systems of Hadamard fractional differential equations involving both retarded and advanced arguments. Our results are obtained by using fixed point theory and the technique of measure of weak noncompactness.

Keywords: Functional Differential Equation, Left-Sided Mixed Pettis Hadamard Integral of Fractional Order, Pettis Hadamard Fractional Derivative, Coupled System, Retarded, Advanced, Weak Solution, Fixed Point

Introduction

Fractional differential equations have recently been applied in various areas of engineering, mathematics, physics and bio-engineering and other applied sciences (Hilfer, 2000; Tarasov, 2010). For some fundamental results in the theory of fractional calculus and fractional differential equations we refer the reader to the monographs (Abbas et al., 2012; 2015; Ahmad and Ntouyas, 2015; Kilbas et al., 2006; Thiramanus et al., 2014; Zhou, 2014). There has been a significant development in fractional differential and integral equations with delay in recent years; (Abbas et al., 2012; 2015; Ahmad et al., 2017; Samko et al., 1993) and the references therein.

The measure of weak noncompactness is introduced by De Blasi (1977). The strong measure of noncompactness was considered in many papers; (Banas and Goebel, 1980; Akhmerov et al., 1992; Alvarez, 1985; Benchohra et al., 2008; Guo et al., 1996; Reich, 1973; 1972) and the references therein. Benchohra et al. (2008; O’Regan, 1999) authors considered some existence results by applying the techniques of the measure of noncompactness. Recently, several researchers obtained other results by application of the technique of measure of weak noncompactness; (Abbas et al., 2015; Benchohra et al., 2011; 2012) and the references therein.

In (Agarwal et al., 2016), the authors studied the existence and uniqueness of solutions for boundary value problems of Hadamard-type fractional functional differential equations and inclusions involving both retarded and advanced arguments. In (Abbas et al., 2017), the authors discuss the existence of weak solutions for the following boundary value problem for implicit Pettis Hadamard fractional differential equation:

\[
\begin{aligned}
\alpha < r < 1, \\
\phi_1(\xi, \psi_2(t)) = 0, \\
\psi_1(\xi, \omega_2(t)) = 0,
\end{aligned}
\]

where, \(\alpha, \beta > 0, r_i \in (1, 2], f_i : I \times C^{[-\alpha, -\beta]} \times C^{[-\beta, -\alpha]} \rightarrow E; i = 1.2\) are given continuous functions, \(\phi_i \in C^{[-\alpha, 1]}\) with \(\phi_i(1) = 0, \psi_i \in C^{[e, e + \beta]}\) and \(\psi_i(e) = 0, E\) is a real (or complex) Banach space with norm \(\|\cdot\|_E\) and dual \(E^*\), such
that \( E \) is the dual of a weakly compactly generated Banach space \( X \), \( C([-\alpha, \beta]) \) is the space of continuous functions from \([-\alpha, \beta]\) to \( E \) and \( ^{\mu} D_{t}^{\alpha} \) is the Pettis-Hadamard fractional derivative of order \( r_{i} \); \( i = 1, 2 \).

We denote by \( u_{i} \), the element of \( C([-\alpha, \beta]) \) defined by:

\[
 u_{i}(s) = u(t + s); t \in [-\alpha, \beta].
\]

This paper initiates the study of coupled systems of Hadamard fractional differential equations under weak topologies.

**Preliminaries**

Let \( C(I) \) be the Banach space of all continuous functions \( w \) from \( I \) into \( E \) with the supremum (uniform) norm:

\[
\|w\|_{\infty} = \sup_{t \in I} |w(t)|.
\]

As usual, we denote by \( AC(I) \) the space of absolutely continuous functions from \( I \) into \( E \). Also, by \( C^{1}([-\alpha, e + \beta, E]) := C([-\alpha, e + \beta, E]) \times \mathcal{C}([-\alpha, e + \beta, E]) \), we denote the product Banach space with the norm:

\[
\|(u, v)\|_{C^{1}([-\alpha, e + \beta, E])} = \|(u)\|_{C([-\alpha, e + \beta, E])} + \|(v)\|_{\mathcal{C}([-\alpha, e + \beta, E])}.
\]

Let \( (E, w) = (E, \sigma(E, E')) \) be the Banach space \( E \) with its weak topology.

**Definition 2.1**

A Banach space \( X \) is called Weakly Compactly Generated (WCG, in short) if it contains a weakly compact set whose linear span is dense in \( X \).

**Definition 2.2**

A function \( h: E \rightarrow E \) is said to be weakly sequentially continuous if \( h \) takes each weakly convergent sequence in \( E \) to a weakly convergent sequence in \( E \) (i.e., for any \( (u_{n}) \) in \( E \) with \( u_{n} \rightarrow u \) in \( (E, w) \) then \( h(u_{n}) \rightarrow h(u) \) in \( (E, w) \)).

**Definition 2.3.**

Pettis (1938) the function \( u: I \rightarrow E \) is said to be Pettis integrable on \( I \) if and only if there is an element \( u_{I} \in E \) corresponding to each \( J \subset I \) such that \( \phi(u_{J}) = \frac{1}{|J|} \int_{J} \phi(u(s)) ds \) for all \( \phi \in E' \), where the integral on the right hand side is assumed to exist in the sense of Lebesgue, (by definition, \( u_{J} = \frac{1}{|J|} \int_{J} u(s) ds \)).

Let \( P(I, E) \) be the space of all \( E \)-valued Pettis integrable functions on \( I \) and denote by \( L^{1}(I, E) \), the Banach space of measurable and Bochner integrable functions \( u: I \rightarrow E \). Define the normed space \( P_{1}(I, E) \) by:

\[
P_{1}(I, E) = \{ u \in P(I, E): \phi(u) \in L^{1}(I, E); \text{ for every } \phi \in E' \},
\]

with the norm:

\[
\|u\|_{P_{1}} = \int_{I} \|\phi(u(x))\|_{E} d\lambda(x),
\]

where, \( \lambda \) stands for a Lebesgue measure on \( I \).

The following result is due to Pettis (1938, Theorem 3.4 and Corollary 3.41).

**Proposition 2.4.**

Pettis (1938) if \( u \in P_{1}(I, E) \) and \( h \) is a measurable and essentially bounded \( E \)-valued function, then \( uh \in P_{1}(I, E) \).

For all what follows, the sign \( \|^\| \) denotes the Pettis integral.

Let us recall some definitions and properties of Hadamard fractional integration and differentiation. We refer to (Hadamard, 1892; Kilbas et al., 2006) for a more detailed analysis.

**Definition 2.5.**

Hadamard (1892; Kilbas et al., 2006) The Hadamard fractional integral of order \( q > 0 \) for a function \( g \in L_{1}(I, E) \), is defined as:

\[
\left( ^{\mu} I_{t}^{\alpha}g \right)(x) = \frac{1}{\Gamma(q)} \int_{0}^{x} \left( \ln \frac{x}{s} \right)^{\alpha-1} \frac{g(s)}{s} ds,
\]

where, \( \Gamma(\cdot) \) is the (Euler's) Gamma function defined by:

\[
\Gamma(\xi) = \int_{0}^{\infty} r^{\xi-1}e^{-r} dr; \xi > 0
\]

Provided the integral exists.

Let \( g \in P_{1}(I, E) \). For every \( \phi \in E' \), we have:

\[
\phi\left( ^{\mu} I_{t}^{\alpha}g \right)(t) = \left( ^{\mu} I_{t}^{\alpha}\phi g \right)(t); \text{ for a.e. } t \in I.
\]

Analogously to the Riemann-Liouville fractional calculus, the Hadamard fractional derivative is defined in terms of the Hadamard fractional integral in the following way. Set:

\[
\delta = x \frac{d}{dx}, n = [q] + 1,
\]

where, \([q]\) is the integer part of \( q > 0 \) and:

\[
AC_{\delta}^{\alpha} = \{ u: I \rightarrow E: \delta^{n-1}\left[ u(x) \right] \in AC(I) \}.
\]

**Definition 2.6.**

Hadamard (1892; Kilbas et al., 2006) the Hadamard fractional derivative of order \( q \) of a function \( w \in AC_{\delta}^{\alpha} \) is defined as:
\[ \left( \frac{d}{dt} \right)^{q} u(t) = \delta^q \left( \frac{d}{dt} \right)^{q} v(t), \]

\[ = \frac{1}{\Gamma(n-q)} \left( \int_{0}^{t} \left( \log \frac{t}{s} \right)^{q-1} \frac{h(s)}{s} ds. \right. \]

**Corollary 2.7.**

Kilbas et al. (2006) let \( q > 0 \) and \( n = [q] + 1. \) The equality \( D^q h(t) = 0 \) is valid if and only if:

\[ h(t) = e^{t}, \quad c_i (\log t)^{q-i} \text{ for each } t \in I, \]

where, \( c_i \in \mathbb{R} \) for each \( i = 1, \ldots, n \) are arbitrary constants.

We need the following auxiliary Lemma:

**Lemma 2.8.**

Agarwal et al. (2016) let \( 1 < r < 2, \) \( \phi \in C([1-\alpha, 1], E) \) with \( \phi(1) = 0, \phi \in C(e, e + \beta), E \) with \( \phi = 0 \) and \( \alpha : I \rightarrow E \) be a continuous function. The linear problem:

\[
\begin{align*}
D^\alpha u(t) &= \sigma(t); \quad t \in I, \\
\int_{0}^{t} u(s) ds &= \phi(t); \quad t \in [1-\alpha, 1], \\
u(t) &= \psi(t); \quad t \in [e, e + \beta],
\end{align*}
\]

has the following unique solution:

\[
u(t) = \begin{cases} 
\phi(t), & \text{if } t \in [1-\alpha, 1], \\
\int_{0}^{t} G(t, s) s^{-\alpha} ds, & \text{if } t \in I, \\
\psi(t), & \text{if } t \in [e, e + \beta],
\end{cases}
\]

where:

\[ G(t, s) = \Gamma^{\alpha}(r) \begin{cases} 
(\log t)^{r-1} (1 - \log s)^{r-1}, & 1 \leq s \leq t \leq e, \\
(\log t)^{r-1} (1 - \log s)^{r-1}, & 1 \leq t \leq s \leq e.
\end{cases} \tag{3}\]

**Definition 2.9.**

De Blasi (1977) let \( E \) be a Banach space, \( \Omega \) be the bounded subsets of \( E \) and \( B \) be the unit ball of \( E. \) The De Blasi measure of weak noncompactness is the map \( \mu : \Omega \rightarrow [0, \infty] \) defined by \( \mu(X) = \inf\{ \varepsilon > 0: \text{There exists a weakly compact subset } \Omega \subseteq E, \lambda \subseteq B, \| \Omega \| = \varepsilon \}. \)

The De Blasi measure of weak noncompactness satisfies the following properties:

- \( \mu(B) \leq \mu(A) \leq \mu(B) \)
- \( \mu(A) = 0 \Leftrightarrow A \) is weakly relatively compact
- \( \mu(A \cup B) = \max\{ \mu(A), \mu(B) \} \)
- \( \mu(\overline{A}) = \mu(A), (\overline{A} \) denotes the weak closure of \( A) \)
- \( \mu(A + B) \leq \mu(A) + \mu(B) \)
- \( \mu(\lambda A) = |\lambda| \mu(A) \)

- \( \mu(\text{conv}(A)) = \mu(A) \)
- \( \mu(\bigcup \lambda A) = h \mu(A) \)

The next result follows directly from the Hahn-Banach theorem.

**Proposition 2.10.**

Let \( E \) be a normed space and \( x_0 \in E \) with \( x_0 \neq 0. \) Then, there exists \( \varphi \in E^* \) with \( \| \varphi \| = 1 \) and \( \varphi(x_0) = \| x_0 \|. \)

For a given set \( V \) of functions \( v : I \rightarrow E \) let us denote by:

\[ V(t) = \{ v(t) : v \in V ; t \in I \}, \]

and:

\[ V(I) = \{ v(t) : v \in V ; t \in I \}. \]

**Lemma 2.11.**

Guo et al. (1996) let \( H \subseteq C \) be a bounded and equicontinuous. Then the function \( t \rightarrow \mu(H(t)) \) is continuous on \( I \) and:

\[ \mu(H) = \int_{0}^{1} \mu(H) ds, \]

and:

\[ \mu \left( \int_{0}^{1} u(s) ds \right) \leq \int_{0}^{1} \mu(H(s)) ds, \]

where, \( H(t) = \{ u(t) : u \in H; t \in I \} \) and \( \mu \) is the De Blasi measure of weak noncompactness defined on the bounded sets of \( C. \)

For our purpose we will need the following fixed point theorem:

**Theorem 2.12.**

O’Regan (1998) let \( Q \) be a nonempty, closed, convex and equicontinuous subset of a metrizable locally convex vector space \( C(I) \) such that \( 0 \in Q. \) Suppose \( T : Q \rightarrow Q \) is weakly-sequentially continuous. If the implication:

\[ V = \text{conv} \{ 0 \cup T(V) \} \Rightarrow V \text{ is relatively weakly compact}, \tag{4} \]

holds for every subset \( V \subseteq Q, \) then the operator \( T \) has a fixed point.

**Existence of Weak Solutions**

**Definition 3.1.**

A coupled functions \( (u, v) \in C^1([1-\alpha, e + \beta], E) \) is said to be a weak solution of the system (2) if \( (u, v) \) satisfies the equations \( \left( \frac{d}{dt} \right)^{q} u(t) = f(t, u, v) \) and \( \left( \frac{d}{dt} \right)^{q} v(t) = g(t, u, v). \)
\[ f_{2}(t, u, v) \text{ on } I \text{ and the conditions } (u(t), v(t)) = (\phi_{1}(t), \phi_{2}(t)); \phi_{1}(1) = 0; i = 1, 2 \text{ on } [1-\alpha, 1] \text{ and } (u(t), v(t)) = (\psi_{1}(t), \psi_{2}(t)); \psi_{1}(e) = 0; i = 1, 2 \text{ on } [e, e + \beta]. \]

Let us introduce the following hypotheses:

(H1) For a.e. \( t \in I \), the functions \( v \rightarrow f_{1}(t, u, \cdot) \) and \( v \rightarrow f_{2}(t, v, \cdot) \) are weakly sequentially continuous for a.e. \( u, v \in C[1-\alpha, e + \beta] \), and \( f_{1}(t, u, v) \) are Pettis integrable a.e. on \( I \).

(H2) There exist \( p_{i} \in (0, \infty) \) such that for all \( \varphi \in E^{*} \):

\[
\left| \varphi \left( f_{i}(t, u, v) \right) \right| \leq p_{i}(t) \left| \varphi \right|_{E^{*}} + ||\varphi||_{E^{*}} + ||v||_{E^{*}} + ||v||_{E^{*}}
\]

for a.e. \( t \in I \) and each \( u, v \in C[1-\alpha, e + \beta] \).

(H3) For each bounded and measurable set \( B \subset C[1-\alpha, e + \beta] \) and for each \( t \in I \), we have:

\[
\mu(f_{1}(t, B), 0) \leq p_{i}(t) \mu(B), \text{ i.e. } \mu_{i}(f_{1}(t, B), 0) \leq p_{i}(t) \mu_{i}(B),
\]

where:

\[
(f_{1}(t, B), 0) = \left\{ f_{1}(t, v_{1}(t), v_{2}(t)), 0 \right\}; v_{1}, v_{2} \in B,
\]

and:

\[
(0, f_{2}(t, B)) = \left\{ (0, f_{2}(t, v_{1}(t), v_{2}(t))) \right\}; v_{1}, v_{2} \in B.
\]

Set:

\[
p_{i}^{*} = \min p_{i}(t); i = 1, 2.
\]

Now we are able to state and prove our main result throughout the following theorem.

**Theorem 3.2.**

Assume that (H1)-(H3) and the following condition hold:

\[
L > \frac{2p_{i}^{*}}{\Gamma(1 + \alpha)} + \frac{2p_{i}^{*}}{\Gamma(1 + \beta)} < 1,
\]

Then the coupled system (2) has at least one weak solution defined on \([1-\alpha, e + \beta] \).

**Proof**

Define the operators \( N_{1}, N_{2} : C[1-\alpha, e + \beta] \rightarrow C[1-\alpha, e + \beta] \) by:

\[
\begin{aligned}
(N_{1}, u)(t) &= \left\{ \phi_{1}(t); t \in [1-\alpha, 1] \right\}, \\
(N_{2}, v)(t) &= \left\{ \psi_{2}(t); t \in [e, e + \beta] \right\}.
\end{aligned}
\]

and:

\[
\begin{aligned}
(N_{1}, u)(t) &= \left\{ \int_{t}^{1-\alpha} G_{1}(t, s) \frac{f_{1}(s, u(s), v(s))}{s} ds; t \in I \right\}, \\
(N_{2}, v)(t) &= \left\{ \int_{e}^{e + \beta} G_{2}(t, s) \frac{f_{2}(s, v(s), 0)}{s} ds; t \in I \right\}.
\end{aligned}
\]

We shall show that the operator \( N \) satisfies all the assumptions of Theorem 2.12. The proof will be given in several steps.

**Step 1.** \( N \) maps \( Q \) into itself.

Let \((u, v) \in Q; t \in I \) and assume that \((N(u, v))(t) \neq (0, 0) \). Then there exists \( \psi_{i} \in E^{*} \); \( i = 1, 2 \) such that for each \( t \in I \), we have:

\[
\psi_{i}(t) = \int_{t}^{1-\alpha} G_{1}(t, s)\frac{f_{1}(s, u(s), v(s))}{s} ds + \int_{e}^{e + \beta} G_{2}(t, s)\frac{f_{2}(s, v(s), 0)}{s} ds.
\]
$\| \left( \mathbf{N}(u,v) \right)(t) \|_e = \left\{ \left[ \left( \mathbf{N}_1(u)(t), \phi_2 \left( \left( \mathbf{N}_2 y \right)(t) \right) \right) \right] \right\}.

Thus:

$$\| \left( \mathbf{N}_1(u) \right)(t) \|_e = \left\{ \left[ \left( \mathbf{G}_1 \left( t, s \right) f_1 \left( s, u, v \right) \right) ds \right] \right\}.$$ 

If $t \in [1-\alpha, 1]$, then:

$$\| \mathbf{N}_1(u)(t) \|_e \leq \phi_1 \left( \left[ \mathbf{G}_1 \left( t, s \right) f_1 \left( s, u, v \right) ds \right] \right).$$

Also, if $t \in [\epsilon, \epsilon + \beta]$, then:

$$\| \mathbf{N}_1(u)(t) \|_e \leq \psi_1 \left( \left[ \mathbf{G}_1 \left( t, s \right) f_1 \left( s, u, v \right) ds \right] \right).$$

For each $t \in I$ and any $i = 1, 2$, we have:

$$\| \left( \mathbf{G}_i \left( t, s \right) \right) \|_e \leq \frac{1}{\pi^{1/2}} \int_{\Gamma} \left[ (\log t)^{i-1} \frac{1}{1 + \frac{1}{2}} + (\log t)^{i-2} \frac{1}{1 + \frac{1}{2}} \right] ds = \frac{1}{\pi^{1/2}}.$$

Thus, for each $t \in I$, we have:

$$\| \left( \mathbf{N}_1(u)(t) \right) \|_e \leq \frac{2 \rho_i^*}{\Gamma(1 + \eta_i)} \leq R_i.$$  

Hence, for each $t \in [1-\alpha, \epsilon + \beta]$, we have:

$$\| \left( \mathbf{N}_1(u)(t) \right) \|_e \leq R,$$

Also, for each $t \in [1-\alpha, \epsilon + \beta]$, we obtain:

$$\| \left( \mathbf{N}_2 y(t) \right) \|_e \leq R.$$  

Therefore, for each $t \in [1-\alpha, \epsilon + \beta]$ and all $(u, v) \in Q$, we get:

$$\| \left( \mathbf{N}(u,v) \right)(t) \|_e \leq R_1 + R_2 = R.$$ 

Next, let $t_1, t_2 \in I$ such that $t_1 < t_2$ and let $(u, v) \in Q$, with:

$$\left( \mathbf{N}(u,v) \right)(t_1) - \left( \mathbf{N}(u,v) \right)(t_2) \neq (0, 0).$$

Then there exists $\phi_i \in E^*$, $i = 1, 2$ with $\| \phi_i \| = 1$ such that:

$$\| \left( \mathbf{N}_1(u)(t_2) - \left( \mathbf{N}_1(u)(t_1) \right) \|_e = \phi_i \left( \left( \mathbf{N}_1(u)(t_2) - \left( \mathbf{N}_1(u)(t_1) \right) \right).$$

And:

$$\| \left( \mathbf{N}_2 y(t_2) - \left( \mathbf{N}_2 y(t_1) \right) \|_e = \phi_2 \left( \left( \mathbf{N}_2 y(t_2) - \left( \mathbf{N}_2 y(t_1) \right) \right).$$

Thus:

$$\| \left( \mathbf{N}_1(u)(t_2) - \left( \mathbf{N}_1(u)(t_1) \right) \|_e = \phi_1 \left( \left( \mathbf{N}_1(u)(t_2) - \left( \mathbf{N}_1(u)(t_1) \right) \right).$$

This gives:

$$\| \left( \mathbf{N}_1(u)(t_2) - \left( \mathbf{N}_1(u)(t_1) \right) \|_e = \phi_1 \left( \left[ \mathbf{G}_1 \left( t_2, s \right) - \mathbf{G}_1 \left( t_1, s \right) \right] f_1 \left( s, u, v \right) ds \right).$$

Also, we can obtain:

$$\| \left( \mathbf{N}_2 y(t_2) - \left( \mathbf{N}_2 y(t_1) \right) \|_e = \phi_2 \left( \left[ \mathbf{G}_2 \left( t_2, s \right) - \mathbf{G}_2 \left( t_1, s \right) \right] f_2 \left( s, u, v \right) ds \right).$$

Hence $M(Q) \subset Q$.

**Step 2.** $N$ is weakly-sequentially continuous.

Let $(u_n, v_n)$ be a sequence in $Q$ and let $(u_n(t), v_n(t)) \to (u(t), v(t))$ in $(E, \omega) \times (E, \omega)$ for each $t \in [1-\alpha, \epsilon + \beta]$. Fix $t \in [1-\alpha, \epsilon + \beta]$, since the functions $f_i$, $i = 1, 2$ satisfy the assumption $(H_i)$, we have $f(t, u_n, v_n)$ converge weakly uniformly to $f(t, u, v)$. Hence the Lebesgue dominated convergence theorem for Pettis integral implies $(N(u_n), v_n(t))$ converges weakly uniformly to $(N(u), v(t))$ in $(E, \omega) \times (E, \omega)$, for each $t \in [1-\alpha, \epsilon + \beta]$. Thus, $N(u_n, v_n) \to N(u, v)$. Hence, $N : Q \to Q$ is weakly-sequentially continuous.

**Step 3.** The implication (4) holds.

Let $V$ be a subset of $Q$ such that $V = \overline{\text{conv}} \left( \mathbf{N}(V) \cup \{0, 0\} \right)$. Obviously:

$$V(t) = \overline{\text{conv}} \left( \mathbf{N}(V) \cup \{0, 0\} \right), \text{ for all } t \in [1-\alpha, \epsilon + \beta].$$

Further, as $V$ is bounded and equicontinuous, by Lemma 3 in (Bugajewski and Szul, 1993), the function $t \mapsto \mu(V(t))$ is continuous on $[1-\alpha, \epsilon + \beta]$.

It is clear that for any $t \in [1-\alpha, 1] \cup [\epsilon, \epsilon + \beta]$, the set $V$ is weakly relatively compact. Next, from $(H_3)$, $(H_4)$, Lemma 2.11 and the properties of the measure $\mu$, for any $t \in I$, we have:
\[ \mu(V(t)) \leq \mu((NV)(t) \cup \{(0,0)\}) \]
\[ \leq \mu((NV)(t)) = \mu\left(\left\{ (N_1v_1)(t), (N_2v_2)(t); (v_1, v_2) \in V \right\}\right) \]
\[ \leq \int |G_1(t,s)| \mu\left(\left\{ (f_1(s,v_1(s),v_2(s)), G_2(t,s)f_2(s,v_1(s),v_2(s)); (v_1, v_2) \in V \right\}\right) ds \]
\[ \leq \int |G_1(t,s)| \mu\left(\left\{ (f_1(s,v_1(s),v_2(s)), (0, f_2(s,v_1(s),v_2(s))); (v_1, v_2) \in V \right\}\right) ds \]
\[ + \int |G_2(t,s)| \mu\left(\left\{ (0, f_2(s,v_1(s),v_2(s))); (v_1, v_2) \in V \right\}\right) ds \]
\[ \leq \int |G_1(t,s)| p_1(s) \mu\left(\left\{ (v_1, v_2); (v_1, v_2) \in V \right\}\right) ds \]
\[ + \int |G_2(t,s)| p_2(s) \mu\left(\left\{ (v_1, v_2); (v_1, v_2) \in V \right\}\right) ds \]
\[ \leq \left( p_1 \int |G_1(t,s)| ds + p_2 \int |G_2(t,s)| ds \right) \mu(V(t)) \]
\[ \leq \left( \frac{2p_1}{\Gamma(1+\rho_1)} + \frac{2p_2}{\Gamma(1+\rho_2)} \right) \mu(V(t)) \]
\[ = L_{\rho_1,\rho_2} \mu(V(t)). \]

Thus:

\[ L_{\rho_1,\rho_2} \mu(V(t)) \leq L_{\rho_1,\rho_2} \mu(V(t)). \]

Hence, the inequality (5) implies that \( L_{\rho_1,\rho_2} \mu(V(t)) = 0. \)

This shows \( \mu(V(t)) = 0; \) for each \( t \in I. \) Then from Theorem 2 in (Mitchell and Smith, 1977), we conclude that the set \( V \) is weakly relatively compact in \( C^2[-1, a, e + \beta]. \) Therefore, Theorem 2.12 implies that \( N \) has a fixed point which is a solution of the coupled system (2).

**An Example**

Let:

\[ E = F = \left\{ u = (u_1, u_2, \ldots, u_n); u_n < \infty \right\} \]

be the Banach space with the norm:

\[ \| u \| = \sup_{j \in N} |u_j|. \]

Consider the following coupled system of Hadamard fractional differential equations of the form:

\[ (u(t), v(t)) = (1 - e^{-t}, 0); t \in [-1, 1] \]
\[ \left( D^{\rho_1} u(t), D^{\rho_2} v(t) \right) = (f_1(t, u_1, v_1), g_1(t, u_1, v_1)); t \in [1, e], \right. \]
\[ (u(t), v(t)) = (-1 + nt, 0); t \in [e, 2e]. \]

where:

\[ f_c(t, u, v) \]
\[ = \frac{ct^\gamma e^{-6}}{1 + \| u \|_{C_{[-1, e]}} + \| v \|_{C_{[-1, e]}}} \left( e^{-t} + \frac{1}{e^{\gamma 5}} \right) u(t); t \in [1, e], \]

and:

\[ g_c(t, u, v) = \frac{ct^\gamma e^{-6}}{1 + \| u \|_{C_{[-1, e]}} + \| v \|_{C_{[-1, e]}}} \]

with:

\[ u = (u_1, u_2, \ldots, u_n), v = (v_1, v_2, \ldots, v_n), \] and \( c = \frac{e^t}{24} \Gamma \left( \frac{1}{2} \right). \]

Set:

\[ f(t, u, v), g(t, u, v) \]

Clearly, the functions \( f \) and \( g \) are continuous. For each \( u, v \in E \) and \( t \in [1, e], \) we have:

\[ \| f(t, u, v) \| \leq \frac{ct^\gamma e^{-6}}{1 + \| u \|_{C_{[-1, e]}} + \| v \|_{C_{[-1, e]}}} \left( e^{-t} + \frac{1}{e^{\gamma 5}} \right), \]

and:

\[ \| g(t, u, v) \| \leq \frac{ct^\gamma e^{-6}}{1 + \| u \|_{C_{[-1, e]}} + \| v \|_{C_{[-1, e]}}}. \]
Hence, the hypothesis $(H_3)$ is satisfied with $p^*_1 = p^*_2 = ce^{-d}$.

We shall show that condition (5) holds. Indeed:

$$\frac{2p^*_1}{\Gamma(1+r_1)} + \frac{2p^*_2}{\Gamma(1+r_2)} = c \frac{\Gamma(\frac{1}{r})}{\Gamma(\frac{1}{r})} = \frac{1}{2} < 1.$$  

Simple computations show that all conditions of Theorem 3.2 are satisfied. It follows that the coupled system (10) has at least one solution on $[-2, 2]$.  

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Ethics

The Authors declare there is not conflict of interest.

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