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A Comparative Analysis on the Performance of the Convoluted Exponential Distribution and the Exponential Distribution in terms of Flexibility

[1]Enahoro Alfred Owoloko, [1]Pelumi Emmanuel Oguntunde and [2]Adebowale Olusola Adejumo

[1]*Department of Mathematics, Covenant University, Ogun State, Nigeria*
[2]*Department of Statistics, University of Ilorin, Kwara State, Nigeria*

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Corresponding Author:
Pelumi Emmanuel Oguntunde
Department of Mathematics, Covenant University, Ogun State, Nigeria
Email: peluemman@yahoo.com

Abstract: In this article, the convoluted exponential distribution which was derived as the sum of two independent exponentially distributed random variables was compared with the exponential distribution in terms of flexibility when applied to four real data sets. The idea is to verify if the convoluted exponential distribution would perform better than the exponential distribution in modeling real life situations. Some other basic statistical properties of the convoluted exponential distribution were also identified.

Keywords: Convolution, Distributions, Exponential, Properties

Introduction

The concept of convolution is a very useful topic in the theory of statistics. As a result, a number of researchers have worked on the sum of Independent and Identically Distributed (IID) random variables. For instance, Sun (2011) defined and studied the convoluted beta Weibull distribution, Shittu *et al*. (2012) proposed and studied the convoluted beta exponential distribution, Oguntunde *et al*. (2014) studied the convoluted exponential distribution. Meanwhile, applications to real data sets were not examined in all these researches.

Let Z denote a random variable, it has the Probability Density Function (PDF) of the convoluted beta Weibull distribution (the sum of two independent beta Weibull variates) given by:

$$f(z) = \frac{\beta_1\beta_2}{\beta_1\gamma_2 - \beta_2\gamma_1}\left(e^{-\frac{\beta_2}{\gamma_2}z} - e^{-\frac{\beta_1}{\gamma_1}z}\right) \tag{1}$$

The corresponding Cumulative Density Function (CDF) is given by:

$$F(z) = \frac{\beta_2\gamma_1}{\beta_1\gamma_2 - \beta_2\gamma_1}e^{-\frac{\beta_1}{\gamma_1}z} - \frac{\beta_1\gamma_2}{\beta_1\gamma_2 - \beta_2\gamma_1}e^{-\frac{\beta_2}{\gamma_2}z} + 1 \tag{2}$$

For $z, \beta_1, \beta_2, \gamma_1, \gamma_2, > 0$ and $\beta_1\gamma_2 \neq \beta_2\gamma_1$ where, β_1, β_2, γ_1 and γ_2 are scale parameters.

Details about how Equation 1 and 2 were derived are rigorously explained in Sun (2011).

In the same way, the PDF of the convoluted beta exponential distribution is given by:

$$f(z) = \frac{(b_1b_2\lambda_1\lambda_2)}{(b_2\lambda_2 - b_1\lambda_1)}e^{-b_2\lambda_2 z}\left[1 - e^{-(b_2\lambda_2 - b_1\lambda_1)z}\right] \tag{3}$$

The corresponding CDF is given by:

$$F(z) = \frac{(b_1b_2\lambda_1\lambda_2)}{(b_1\lambda_1 - b_2\lambda_2)}\left[\frac{1}{b_2\lambda_2}\left(1 - e^{-b_2\lambda_2 z}\right) - \frac{1}{b_1\lambda_1}\left(1 - e^{-b_1\lambda_1 z}\right)\right] \tag{4}$$

For b_1, b_2, λ_1, λ_2, $z > 0$

For details about the construction of Equation 3 and 4, readers are referred to Shittu *et al*. (2012).

Another interesting part of the concept of convolution is when the sum of independent random variables from different distributions is considered.

The interest of this research is to further explore the convoluted exponential distribution defined in Oguntunde *et al*. (2014) by assessing its flexibility over the exponential distribution using four real data sets. The rest of this paper is structured as follows; details about the convoluted exponential distribution (including existing and new properties) are provided in section 2, real life applications are discussed in section 3, followed by a concluding remark.

Convoluted Exponential Distribution: Existing and More Properties

In this section, the PDF, CDF and basic properties of the convoluted exponential distribution are highlighted as available in Oguntunde *et al*. (2014). Also, some other new properties are given.

Science Publications

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The PDF of the convoluted exponential distribution is given by:

$$f(z) = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1}\left(e^{-\lambda_1 z} - e^{-\lambda_2 z}\right) \tag{5}$$

The corresponding CDF is given by:

$$F(z) = 1 + \frac{\lambda_1}{\lambda_2 - \lambda_1} e^{-\lambda_2 z} - \frac{\lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 z} \tag{6}$$

For λ_1, λ_2, $z > 0$
where; λ_1 and λ_2 are scale parameters
The mean is given by:

$$E(Z) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \tag{7}$$

$$Var(Z) = \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} \tag{8}$$

$$Skewness = \frac{2\left[\left(\frac{1}{\lambda_1}\right)^3 + \left(\frac{1}{\lambda_2}\right)^3\right]}{\left[\left(\frac{1}{\lambda_1}\right)^2 + \left(\frac{1}{\lambda_2}\right)^2\right]^{3/2}} \tag{9}$$

$$Kurtosis = \frac{6\left[\left(\frac{1}{\lambda_1}\right)^4 + \left(\frac{1}{\lambda_2}\right)^4\right]}{\left[\left(\frac{1}{\lambda_1}\right)^2 + \left(\frac{1}{\lambda_2}\right)^2\right]^2} \tag{10}$$

The moment generating function is given by:

$$M_Z(t) = \frac{\lambda_1 \lambda_2}{(\lambda_1 - t)(\lambda_2 - t)} \tag{11}$$

Renyi Entropy

The Renyi entropy being one of the functions used in quantifying the uncertainty or randomness in a system is mathematically given by:

$$R(p) = \frac{1}{1-p} log\left[\int_{-\infty}^{\infty} f^p(z) dz\right] \tag{12}$$

For $p \neq 1$ and $p>0$
For the Convoluted Exponential distribution, the entropy is derived from:

$$\begin{aligned}
R(p) &= \frac{1}{1-p}\log\left[\int_0^\infty \left\{\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left(e^{-\lambda_1 z} - e^{-\lambda_2 z}\right)\right\}^p dz\right] \\
&= \frac{1}{1-p}\log\left[\int_0^\infty \frac{\lambda_1^p\lambda_2^p}{(\lambda_2-\lambda_1)^p}\left(e^{-\lambda_1 z} - e^{-\lambda_2 z}\right)^p dz\right] \\
&= \frac{1}{1-p}\log\left[\frac{\lambda_1^p\lambda_2^p}{(\lambda_2-\lambda_1)^p}\int_0^\infty \left(e^{-\lambda_1 z} - e^{-\lambda_2 z}\right)^p dz\right] \\
&= \frac{1}{1-p}\left\{\log\left[\frac{\lambda_1^p\lambda_2^p}{(\lambda_2-\lambda_1)^p}\right] + \left[\log\int_0^\infty \left(e^{-\lambda_1 z} - e^{-\lambda_2 z}\right)^p dz\right]\right\} \\
&= \frac{1}{1-p}\left\{\log\left(\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\right)^p + \log\int_0^\infty \left(e^{-\lambda_1 z} - e^{-\lambda_2 z}\right)^p dz\right\} \\
&= \frac{1}{1-p}\left\{p\log\left(\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\right) + \log\left[\int_0^\infty \sum_{i=0}^{p}\binom{p}{i}(-1)^i\left(e^{-\lambda_1 z}\right)^{p-1}\left(e^{-\lambda_2 z}\right)^i dz\right]\right\} \\
&= \frac{1}{1-p}\left\{p\log\left(\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\right) + \log\left[\sum_{i=0}^{p}\binom{p}{i}\int_0^\infty (-1)^i\left(e^{-\lambda_1 z(p-i)}\right)\left(e^{-\lambda_2 z i}\right) dz\right]\right\} \\
&= \frac{1}{1-p}\left\{p\log\left(\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\right) + \log\left[\sum_{i=0}^{p}\binom{p}{i}\int_0^\infty (-1)^i\left(e^{-(\lambda_1(p-i)+\lambda_2 i)z}\right) dz\right]\right\} \\
&= \frac{1}{1-p}\left\{p\log\left(\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\right) + \log\sum_{i=0}^{p}\binom{p}{i}\left[(-1)^{i+1}\frac{e^{-(\lambda_1(p-i)+\lambda_2 i)z}}{\lambda_1 p - \lambda_1 i + \lambda_2 i}\right]_0^\infty\right\} \\
&= \frac{1}{1-p}\left\{p\log\left(\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\right) + \log\left[\sum_{i=0}^{p}\binom{p}{i}(-1)^{i+1}\frac{1}{\lambda_1 p - \lambda_1 i + \lambda_2 i}(0-1)\right]\right\} \quad (13) \\
&= \frac{1}{1-p}\left\{p\log\left(\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\right) + \log\left[\sum_{i=0}^{p}\binom{p}{i}(-1)^i\frac{1}{\lambda_1 p + (\lambda_2-\lambda_1)i}\right]\right\}; p \neq 1
\end{aligned}$$

Emperical Study

The tables for the mean and variance of the convoluted exponential distribution are provided in Table 1 and 2 respectively.

It can be observed from Table 1 that the mean of the Convoluted Exponential distribution decreases as the parameter increases and vice versa.

Table 2 reveals that the variance of the Convoluted Exponential distribution decreases as the value of the parameters increases and vice versa.

Application

The convoluted exponential distribution and the exponential distribution are both applied to four real data sets and the distribution corresponding to the best fit is selected using the Akaike Information Criteria (AIC) and the Log-likelihood.

Table 1. Mean of the convoluted exponential distribution at different parameter values

	$\lambda_2 = 1$	$\lambda_2 = 2$	$\lambda_2 = 3$	$\lambda_2 = 4$	$\lambda_2 = 5$	$\lambda_2 = 6$	$\lambda_2 = 7$	$\lambda_2 = 8$	$\lambda_2 = 9$	$\lambda_2 = 10$
$\lambda_1 = 1$	2.000	1.500	1.333	1.250	1.200	1.167	1.143	1.125	1.111	1.100
$\lambda_1 = 2$	1.500	1.000	0.833	0.750	0.700	0.667	0.643	0.625	0.611	0.600
$\lambda_1 = 3$	1.333	0.833	0.667	0.583	0.533	0.500	0.476	0.458	0.444	0.433
$\lambda_1 = 4$	1.250	0.750	0.583	0.500	0.450	0.417	0.393	0.375	0.361	0.350
$\lambda_1 = 5$	1.200	0.700	0.533	0.450	0.400	0.367	0.343	0.325	0.311	0.300
$\lambda_1 = 6$	1.167	0.67	0.500	0.417	0.367	0.333	0.310	0.292	0.278	0.267
$\lambda_1 = 7$	1.143	0.643	0.476	0.393	0.343	0.310	0.286	0.268	0.254	0.243
$\lambda_1 = 8$	1.125	0.625	0.458	0.375	0.325	0.292	0.268	0.250	0.236	0.225
$\lambda_1 = 9$	1.111	0.611	0.444	0.361	0.311	0.278	0.254	0.236	0.222	0.211
$\lambda_1 = 10$	1.100	0.600	0.433	0.350	0.300	0.267	0.243	0.225	0.211	0.200

Table 2. Variance of the convoluted exponential distribution at different parameter values

	$\lambda_2 = 1$	$\lambda_2 = 2$	$\lambda_2 = 3$	$\lambda_2 = 4$	$\lambda_2 = 5$	$\lambda_2 = 6$	$\lambda_2 = 7$	$\lambda_2 = 8$	$\lambda_2 = 9$	$\lambda_2 = 10$
$\lambda_1 = 1$	2.000	1.250	1.111	1.062	1.040	1.028	1.020	1.016	1.012	1.010
$\lambda_1 = 2$	1.250	0.500	0.361	0.313	0.290	0.278	0.270	0.266	0.262	0.260
$\lambda_1 = 3$	1.111	0.361	0.222	0.174	0.151	0.139	0.132	0.127	0.123	0.121
$\lambda_1 = 4$	1.063	0.313	0.174	0.125	0.103	0.090	0.083	0.078	0.075	0.073
$\lambda_1 = 5$	1.040	0.290	0.151	0.103	0.080	0.068	0.060	0.056	0.052	0.050
$\lambda_1 = 6$	1.028	0.278	0.139	0.090	0.068	0.056	0.048	0.043	0.040	0.038
$\lambda_1 = 7$	1.020	0.270	0.132	0.083	0.060	0.048	0.041	0.036	0.033	0.030
$\lambda_1 = 8$	1.016	0.266	0.127	0.078	0.056	0.043	0.036	0.031	0.028	0.026
$\lambda_1 = 9$	1.012	0.262	0.123	0.075	0.052	0.040	0.033	0.028	0.025	0.022
$\lambda_1 = 10$	1.010	0.260	0.121	0.073	0.050	0.038	0.030	0.026	0.022	0.020

Data Set I

The first data set represents the height of 100 female athletes (measured in centimeters); it is one of the thirteen variables in the Australian athletes' data reported in Cook and Weisberg (1994). The data was collected at the Australian Institute of Sport. It has been previously used and analyzed by (Jamalizadeh *et al*., 2011; Choudhury and Abdul Matin, 2011; Al-Aqtash *et al*., 2014). The data is as follows:

148.9, 149.0, 156.0, 156.9, 157.9, 158.9, 162.0, 162.0, 162.5, 163.0, 163.9, 165.0, 166.1, 166.7, 167.3, 167.9, 168.0, 168.6, 169.1, 169.8, 169.9, 170.0, 170.0, 170.3, 170.8, 171.1, 171.4, 171.4, 171.6, 171.7, 172.0, 172.2, 172.3, 172.5, 172.6, 172.7, 173.0, 173.3, 173.3, 173.5, 173.6, 173.7, 173.8, 174.0, 174.0, 174.0, 174.1, 174.1, 174.4, 175.0, 175.0, 175.0, 175.3, 175.6, 176.0, 176.0, 176.0, 176.0, 176.8, 177.0, 177.3, 177.3, 177.5, 177.5, 177.8, 177.9, 178.0, 178.2, 178.7, 178.9, 179.3, 179.5, 179.6, 179.6, 179.7, 179.7, 179.8, 179.9, 180.2, 180.2, 180.5, 180.5, 180.9, 181.0, 181.3, 182.1, 182.7, 183.0, 183.3, 183.3, 184.6, 184.7, 185.0, 185.2, 186.2, 186.3, 188.7, 189.7, 193.4, 195.9.

The data is summarized in Table 3 and the performances of the competing distributions are given in Table 4.

Data Set II

The second data set was reported by (Bjerkedal, 1960) and it has also been studied by (Tahir *et al*., 2014). It represents the survival times of 72 guinea pigs (in days) infected with virulent tubercle. The data is as follows:

10, 33, 44, 56, 59, 72, 74, 77, 92, 93, 96, 100, 100, 102, 105, 107, 107, 108, 108, 108, 109, 112, 113, 115, 116, 120, 121, 122, 122, 124, 130, 134, 136, 139, 144, 146, 153, 159, 160, 163, 163, 168, 171, 172, 176, 183, 195, 196, 197, 202, 213, 215, 216, 222, 230,231, 240, 245, 251, 253, 254, 255, 278, 293, 327, 342, 347, 361, 402, 432, 458, 555.

The data is summarized in Table 5 and the performances of the competing distributions are given in Table 6.

Data Set III

The third data set has been previously studied by (Quesenberry and Kent, 1982; Pal and Tiensuwan, 2014). It represents the time to failure of a ployster/viscose yarn in a textile experiment at 2.3% strain level. The data is as follows:

86, 146, 251, 653, 98, 249, 400, 292, 131, 169, 175, 176, 76, 264, 15, 364, 195, 262, 88, 264, 157, 220, 42, 321, 180, 198, 38, 20, 61, 121, 282, 224, 149, 180, 325, 250, 196, 90, 229, 166, 38, 337, 65, 151, 341, 40, 40, 135, 597, 246, 211, 180, 93, 315, 353, 571, 124, 279, 81, 186, 497, 182, 423, 185, 229, 400, 338, 290, 398, 71, 246, 185, 188, 568, 55, 55, 61, 244, 20, 284, 393, 396, 203, 829, 239, 236, 286, 194, 277, 143, 198, 264, 105, 203, 124, 137, 135, 350, 193, 188.

The data is summarized in Table 7 and the performances of the competing distributions are given in Table 8.

Table 3. Summary of data on height of 100 female athletes

Min.	Q1	Q2	Q3	Mean	Variance	Skewness	Kurtosis	Max.
148.9	171.0	175.0	179.7	174.6	67.9339	-0.5598	4.1967	195.9

Table 4. Performance of distributions with standard errors in parentheses using DATA I

Distributions	Estimates	t-statistic	p-value	Log-likelihood	AIC
Exponential	$\hat{\lambda} = 0.0057276(0.0005729)$	9.998	$< 2\times10^{-16}$	-616.2463	1234.493
Convoluted exponential	$\hat{\lambda}_1 = 0.005728(0.0005729)$	9.998	$< 2\times10^{-16}$	-616.2449	1236.49
	$\hat{\lambda}_2 = 391.7(2.966)$	132.07	$< 2\times10^{-16}$		

Table 5. Summary of data on survival times of 72 guinea pigs

Min.	Q1	Q2	Q3	Mean	Variance	Skewness	Kurtosis	Max.
10.0	108.0	149.5	224.0	176.8	10705.1	1.3413	4.9885	555.0

Table 6. Performance of distributions with standard errors in parentheses using DATA II

Distributions	Estimates	t-statistic	p-value	Log-likelihood	AIC
Exponential	$\hat{\lambda} = 0.0056550(0.0006666)$	8.484	$< 2\times10^{-16}$	-444.615	891.2299
Convoluted exponential	$\hat{\lambda}_1 = 0.005655(0.0006666)$	8.484	$< 2\times10^{-16}$	-444.614	893.2281
	$\hat{\lambda}_2 = 0.04461(2.966)$	150.429	$< 2\times10^{-16}$		

Table 7. Summary of data on failure times of Yarn at 2.3% strain level

Min.	Q1	Q2	Q3	Mean	Variance	Skewness	Kurtosis	Max.
15.0	129.2	195.5	282.5	222.0	20914.38	1.3600	5.8601	829.0

Table 8. Performance of distributions with standard errors in parentheses using DATA III

Distributions	Estimates	t-statistic	p-value	Log-likelihood	AIC
Exponential	$\hat{\lambda} = 0.0045049(0.0004506)$	9.988	$< 2\times10^{-16}$	-640.2587	1282.517
Convoluted exponential	$\hat{\lambda}_1 = 0.004505(0.0004506)$	9.998	$< 2\times10^{-16}$	-640.2585	1284.517
	$\hat{\lambda}_2 = 2309(2.966)$	778.597	$< 2\times10^{-16}$		

Table 9. Summary of data on electronic components

Min.	Q1	Q2	Q3	Mean	Variance	Skewness	Kurtosis	Max.
0.030	0.775	1.795	2.900	1.936	2.062	0.603	2.720	5.090

Table 10. Performance of distributions with standard errors in parentheses using DATA IV

Distributions	Estimates	t-statistic	p-value	Log-likelihood	AIC
Exponential	$\hat{\lambda} = 0.5167(0.1155)$	4.472	7.75×10^{-6}	-33.20731	68.41463
Convoluted exponential	$\hat{\lambda}_1 = 0.5187(0.1160)$	4.471	7.78×10^{-6}	-33.14827	70.29655
	$\hat{\lambda}_2 = 131.4146(8.3888)$	15.665	$< 2\times10^{-16}$		

Data Set IV

The fourth data set has been previously used by (Teimouri and Gupta, 2013; Nasiru, 2015). It represents the lifetime of 20 electronic components. The data is as follows:

0.03, 0.22, 0.73, 1.25, 1.52, 1.8, 2.38, 2.87, 3.14, 4.72, 0.12, 0.35, 0.79, 1.41, 1.79, 1.94, 2.4, 2.99, 3.17, 5.09

The data is summarized in Table 9 and the performances of the competing distributions are given in Table 10.

Remark

Considering Table 4, 6, 8 and 10, the model with the lowest AIC or highest log-likelihood is considered to be the best fit. This means that the exponential distribution is considered the best fit and thereby highlighted.

Conclusion

A comparison between the convoluted exponential distribution and the exponential distribution has been successfully done in terms of real life applications. It was observed that the exponential distribution out-performed the convoluted exponential distribution considering the four applications provided in this research. The decisions and conclusion in this study is based on the log-likelihood and AIC values posed by the distributions under study. For all the four data sets, the AIC value of the Exponential distribution is the lowest while its log-likelihood values are higher than that of the Convoluted Exponential distribution. Nevertheless, the authors did not underrate the concept of convolution. Convolution still remains a relevant topic in the theory of statistics. Further research would involve comparing convoluted beta Weibull distribution derived by (Sun, 2011) with beta Weibull distribution derived by (Famoye *et al*., 2005) and comparing convoluted beta exponential distribution derived by (Shittu *et al*., 2012) with beta exponential distribution derived by (Nadarajah and Kotz, 2006) to assess their flexibilities in modeling real life data sets.

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Author's Contributions

Enahoro Alfred Owoloko: Being a Co-supervisor, he read through the manuscript, reviewed, corrected and approved it before sending to a journal outlet.

Pelumi Emmanuel Oguntunde: Being a research student, he conceived the idea and drafted the paper.

Adejumo Olusola Adejumo: Being a supervisor, he guided the student in the write-up, checked the analysis and codes used for correctness.

Ethics

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and no ethical issues involved.

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