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Adapted Newton-Kantorovich Methods for Nonlinear Integral Equations

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Abstract: For this work, the main idea is to make an adapted modification to the Newton-Kantorovich method destined to solve a nonlinear integral equations, so that by this technical method we obtain a simple application to this solution. Moreover, we compare the numerical results obtained by this method against ones obtained by another authors. This comparison showed the efficiency of this method.

Keywords: Nonlinear Integral Equations, Newton-Kantorovich Method, Successive Approximations

Introduction

Nonlinear integral equation is an important branch in contemporary mathematics and arises many applied areas which include engineering problems, such as mechanics, physics, astronomy, biology, economics, potential theory and electrostatics (Golberg, 1990; Nadir and Gagui, 2014). Those equations are classified into Fredholm and Volterra equations following the upper bound of the region for the integral part is constant or variable. Many different methods are used to obtain the numerical solution of the nonlinear integral equations (Nadir and Rahmoune, 2007; Polyanin and Manzhirov, 2008):

$$\varphi(s)-\int_{\Omega}k\left(s,t,\varphi(t)\right)dt=f(s),\quad s,t\in\Omega \tag{1}$$

where, the functions $f(s)$ and $k(s, t, \varphi)$ are given and continuous functions in Ω and $D(k) = \Omega \times \Omega \times I$, $(I\in R)$, respectively, the function $\varphi(t)$ is to be determined as continuous function $\Omega \to I$.

For the solution of the nonlinear integral equations we present specifics conditions for the existence of this ones:

- $k(s, t, \varphi(t))$: $\Omega \times \Omega \times I \to R$ is continuous and bounded in this domain
- $k(s, t, \varphi)$: Is Lipschitzian for the third variable. In other words there exists $L > 0$ such that:

$$\left|k\left(s,t,\varphi_1\right)-k\left(s,t,\varphi_2\right)\right|\le L\left|\varphi_1-\varphi_2\right|\quad s,t\in\Omega\ \varphi_1,\varphi_2\in I$$

- f: $\Omega \to R$ is continuous
- $\dfrac{1}{L(b-a)}>1$

Depending on $\Omega = [a, s]$ or $\Omega = [a, b]$ the Equation 1 is a nonlinear Volterra or Fredholm integral equation, respectively.

We apply the Newton-Kantorovich method to the general nonlinear integral equation:

$$P(\varphi)=\varphi(s)-\int_{\Omega}k\left(s,t,\varphi(t)\right)dt-f(s)$$

It is known that, the fact where the function $k(s, t, \varphi)$ is continuous and Lipshitzian for the third variable then, $P(\varphi)$ is Fréchet differentiable mapping of a Banach spaces $C(\Omega)$ into itself for all $f(s)$; $\varphi(s) \in C(\Omega)$, say:

$$\begin{aligned}&GD(P;\varphi)h(s)=P'(\varphi)h(s)=h(s)\\&-\int_{\Omega}k_{\varphi}\left(s,t,\varphi(t)\right)h(t)dt\end{aligned} \tag{2}$$

where, $k_\varphi(s, t, \varphi(t))$ designates the derivative $\dfrac{\partial k}{\partial\varphi}\left(s,t,\varphi(t)\right)$.

For the resolution of the functional equation $P(\varphi) = 0$ where P is Fréchet differentiable on a convex set of a Banach space $C(\Omega)$, Kantorovich imitates the Newton method for the equation of the tangent line given by the first two terms of Taylor's formula, written as the method of successive approximation:

$$P(\varphi_{n+1}) = P(\varphi_n) + P'(\varphi_n)(\varphi_{n+1} - \varphi_n) = 0$$

Or equivalently:

$$P'(\varphi_n)(\varphi_{n+1} - \varphi_n) = -P(\varphi_n) \tag{3}$$

The explicit form to the Equation 3 is given as:

$$\begin{aligned}&\varphi_n(s) - \int_\Omega k(s,t,\varphi_n(t))dt - f(s) + (\varphi_{n+1}(s) - \varphi_n(s)) \\ &-\int_\Omega k_\varphi(s,t,\varphi_n(t))(\varphi_{n+1}(t) - \varphi_n(t))dt = 0\end{aligned}$$

Or still:

$$\begin{aligned}\varphi_{n+1}(s) &= f(s) - \int_\Omega k(s,t,\varphi_n(t))dt \\ &+\int_\Omega k_\varphi(s,t,\varphi_n(t))(\varphi_{n+1}(t) - \varphi_n(t))dt\end{aligned} \tag{4}$$

In the Newton-Kantorovich method, we remark that, the kernels $k(s, t, \varphi_n(t))$ and $k_\varphi(s, t, \varphi_n(t))$ of the right-hand side of the Equation 4 are replaced by the ones $k(s, t, \varphi_0(t))$ and $k_\varphi(s, t, \varphi_0(t))$ where φ_0 represents the initial value so that, the Equation 4 becomes a linear integral equation. However, in our work we treat the Equation 4 by adapted a modification, where we replace the expression $(\varphi_{n+1}(t)- \varphi_n(t))$ in the right-hand side by the one $(\varphi_n(t)- \varphi_{n-1}(t))$ so that, the Equation 4 becomes:

$$\begin{aligned}\varphi_{n+1}(s) &= f(s) - \int_\Omega k(s,t,\varphi_n(t))dt \\ &+\int_\Omega k_\varphi(s,t,\varphi_n(t))(\varphi_n(t) - \varphi_{n-1}(t))dt\end{aligned} \tag{5}$$

The first approximation $\varphi_1(s)$ is obtained by substituting the initial approximation $\varphi_0(s) = f(s)$ into the right hand side of the integral equation, giving:

$$\begin{aligned}\varphi_n(s) &= f(s) - \int_\Omega k(s,t,f(s))dt \\ &+\int_\Omega k_\varphi(s,t,f(s))(f(s) - 0)dt\end{aligned}$$

And so on, higher iterates may be defined by Equation 4 where we approximate the two integrals presented in Equation 4 by one of the basic numerical integration formulas such as trapezoid method, Simpson methods, or Gauss methods.

Convergence and Applications Theorem

Let P be an operator defined on a Banach space E into a Banach space F and Fréchet differentiable for $\varphi \in \Omega$ an open convex set in E, satisfies the following conditions:

$$(A1)\ \ \|P'(\varphi) - P'(\psi)\| \le L\|\varphi - \psi\|,\ \varphi,\psi \in \Omega$$

$$(A2)\ \ \left\|\left[P'(\varphi_0)\right]^{-1}\right\| \le M,\ \ \varphi_0 \in \Omega$$

$$(A3)\ \ \left\|\left[P'(\varphi_0)\right]^{-1} P(\varphi_0)\right\| \le N,\ \ \varphi_0 \in \Omega$$

With the constants L,M and N satisfying $LM < 1$, $LMN \le \frac{1}{2}$ then there exists a domain $\Omega_1 = \left\{\varphi; \|\varphi - \varphi_0\| \le h = \frac{(\sqrt{2}-1)}{LM\sqrt{2}}\right\} \subset \Omega$ such that, the successive approximations:

$$\varphi_{n+1} = \varphi_n - \left[P'(\varphi_n)\right]^{-1} P(\varphi_n)$$

Are defined for all n, $\varphi_n \in \Omega_1$, $n = 1, 2, ...$ and converge to the exact solution $\varphi \in \Omega_1$ which satisfies $P(\varphi) = 0$: Further:

$$\|\varphi_n - \varphi\| \le \frac{C}{2^n(LMN)}, C \in \mathbb{R}_+^*, n = 1,2,3.....$$

Proof

Indeed, it is easy to see that:

$$\begin{aligned}&P(\varphi) - P(\psi) = \int_0^1 P'(\varphi + t(\varphi - \psi))(\varphi - \psi)dt \\ &= \int_0^1 (P'(\varphi + t(\varphi - \psi)) - P'(\varphi))(\varphi - \psi)dt \\ &+\int_0^1 P'(\varphi)(\varphi - \psi)dt \\ &\|P(\varphi) - P(\psi) - P'(\varphi)(\varphi - \psi)\| \\ &= \left\|\int_0^1 (P'(\varphi + t(\varphi - \psi)) - P'(\varphi))(\varphi - \psi)dt\right\| \\ &\|P(\varphi) - P(\psi) - P'(\varphi)(\varphi - \psi)\| \\ &\le \int_0^1 \|P'(\varphi + t(\varphi - \psi) - P'(\varphi))\|\|(\varphi - \psi)\|dt\end{aligned}$$

Using condition (A1), on Ω we obtain:

$$\|P(\varphi) - P(\psi) - P'(\varphi(\varphi - \psi))\| \le \frac{L}{2}\|(\varphi - \psi)\|^2 \tag{6}$$

Also, for $\varphi \in \Omega_1$, we get:

$$\|P'(\varphi) - P'(\varphi_0)\| \le L\|\varphi - \varphi_0\| \le Lh < \frac{1}{M} \tag{7}$$

Then, the relation (7) shows that $P'(\varphi)$ is invertible for all $\varphi \in \Omega_1$ and it comes:

$$\left[P'(\varphi)\right]^{-1}-\left(I-\left[P'(\varphi_0)\right]^{-1}\left(P'(\varphi)-P'(\varphi_0)\right)^{-1}\left[P'(\varphi_0)\right]^{-1}\right.$$

Or still:

$$\left\|\left[P'(\varphi)\right]^{-1}\right\|\leq\frac{M}{(1-LM)\|\varphi-\varphi_0\|}$$

Given the Newton function as:

$$N(\varphi)=\varphi-\left[P'(\varphi)\right]^{-1}P(\varphi) \quad (8)$$

With φ and $N(\varphi)$ in Ω_1 and $\varphi_{n+1}=N(\varphi_n)$, we get:

$$\|N(N(\varphi))-N(\varphi)\|=\left[P'(N(\varphi))\right]^{-1}P(N(\varphi))$$

Hence:

$$\|P(N(\varphi))\|\leq\frac{L}{2}\|N(\varphi)-\varphi\|^2 \quad (9)$$

On the other hand

$$\|P'(N(\varphi))\|^{-1}\leq\frac{M}{(1-LM)\|N(\varphi)-\varphi_0\|} \quad (10)$$

From the relations (9) and (10) we obtain:

$$\|N(N(\varphi))-N(\varphi)\|\leq\frac{LM\|N(\varphi)-\varphi\|^2}{2(1-LM)\|N(\varphi)-\varphi_0\|}$$

or still:

$$\|\varphi_{n+1}-\varphi_n\|\leq\frac{LM\|\varphi_n-\varphi_{n-1}\|^2}{2(1-LM)\|\varphi_n-\varphi_0\|} \leq\frac{(LM)^n\|\varphi_1-\varphi_0\|^{2n}}{2^n(1-LM)^n\|\varphi_1-\varphi_0\|^n} \quad (11)$$

From the relation $\|\varphi_q-\varphi_p\|\leq\|\varphi_q-\varphi_{q-1}\|+\|\varphi_{q-1}-\varphi_{q-2}\|+\ldots+\|\varphi_{p+1}-\varphi_p\|$ it comes the sequence φ_n is Cauchy sequence in Banach space. Thus this sequence φ_n represents the Newton iterations are defined and converges to the solution φ in Ω_1 (Wouk, 1979):

Illustrating Examples

Example 1

Consider the nonlinear integral equation of Volterra:

$$\varphi(s)-\int_0^s\sin\varphi(t)\,dt=s+\cos s-1,\ 0\leq s,t\leq 1$$

where, the function $f(t_0)$ is chosen so that the exact solution is given by:

$$\varphi(t)=t$$

The approximate solution $\tilde{\varphi}(t)$ of $\varphi(t)$ is obtained by the adapted Newton-Kantorovich method.

Example 2

Consider the nonlinear integral equation of Volterra:

$$\varphi(s)-\int_0^s\varphi^2(t)\,dt=\exp(s)-\frac{1}{2}\left(\exp(2s)-1\right),\ 0\leq s,t\leq 1$$

where, the function $f(t_0)$ is chosen so that the exact solution is given by:

$$\varphi(t)=\exp(t)$$

The approximate solution $\tilde{\varphi}(t)$ of $\varphi(t)$ is obtained by the adapted Newton-Kantorovich method.

Example 3

Consider the nonlinear integral equation of Volterra:

$$\varphi(s)-\int_0^s\frac{1}{2}\varphi^2(t)\,dt=\sin s+\frac{1}{8}\sin 2s-\frac{1}{4}s,\ 0\leq s,t\leq 1$$

where, the function $f(t_0)$ is chosen so that the exact solution is given by:

$$\varphi(t)=\sin t$$

The approximate solution $\tilde{\varphi}(t)$ of $\varphi(t)$ is obtained by the adapted Newton-Kantorovich method.

Example 4

Consider the nonlinear integral equation of Fredholm:

$$\varphi(t)-\int_0^1\frac{t^2x^2}{1+\varphi^2(x)}dx-\left(\frac{1}{2}-\ln(2)\right)t^2+\sqrt{t},\ 0\leq x,t\leq 1$$

where, the function $f(t_0)$ is chosen so that the exact solution is given by:

$$\varphi(t)=\sqrt{t}$$

The approximate solution $\tilde{\varphi}(t)$ of $\varphi(t)$ is obtained by the adapted Newton-Kantorovich method.

Example 5

Consider the nonlinear integral equation of Fredholm:

$$\varphi(t)-\frac{1}{5}\int_0^1\cos(\pi t)\sin(\pi x)\varphi^3(x)dx=\sin(\pi t),\ 0\le x,t\le 1$$

where, the function $f(t_0)$ is chosen so that the exact solution is given by:

$$\varphi(t)=\sin\pi t+\frac{1}{3}\left(20-\sqrt{391}\right)\cos\pi t$$

The approximate solution e'(t) of '(t) is obtained by the adapted Newton-Kantorovich method.

Conclusion

A numerical method for solving nonlinear Volterra and Fredholm integral equations, based on an adapted Newton-Kontorovich methods is presented. The efficiency of this method is tested by solving some examples for which the exact solution is known. This allows us to estimate the exactness with our numerical results and compare those with another results. For nonlinear volterra integral equations our method is compared with the ones, the Haar wavelets and collocation, the fixed point technique with cubic B-spline scaling function, Adomian decomposition method and block pulse functions by collocation method. treated by (Babolian and Shahsavaran, 2007) Table 1, (Maleknejad *et al.*, 2013) Table 2, (Awawdeh *et al.*, 2009) Table 3 and (Shahsavaran, 2011) Table 4 respectively. On the other hand for nonlinear Fredholm integral equations our method is compared with the ones, the Haar wavelet method, the Urysohn form by Newton-Kantorovich-quadrature method and A numerical method treated by ((Lepik and Tamme, 2007)) Table 5, (Saberi-Nadja and Heidari, 2010) Table 6 and (Awawdeh *et al.*, 2009) Table 7 respect1vely.

Table 1. We present the exact and the approximate solutions of the equation in the example 1 in some arbitrary points, the error for N = 10 is compared with the ones treated in (Babolian and Shahsavaran, 2007)

Values of t	Exact solution φ	Approx solution $\tilde{\varphi}$	Error	Error (Babolian and Shahsavaran, 2007)
0.000000	0.00000e+00	0.00000e+00	0.000e+00	0e+00
0.200000	2.000000e-01	2.000000e-01	7.4733e-10	4e-04
0.400000	4.000000e-01	4.000000e-01	3.1445e-09	6e-04
0.600000	6.000000e-01	6.000000e-01	7.3628e-09	7e-04
0.800000	8.000000e-01	8.000000e-01	1.3385e-08	9e-04
1.000000	1.00000e+00	1.00000e+00	2.0917e-08	1e-03

Table 2. We present the exact and the approximate solutions of the equation in the example 1 in some arbitrary points, the error for N = 10 is compared with the ones treated in (Maleknejad *et al.*, 2013)

Values of t	Exact solution φ	Approx solution $\tilde{\varphi}$	Error	Error (Maleknejad *et al.*, 2013)
0.000000	0.00000e+00	0.00000e+00	0.00e+00	0.00e+00
0.200000	2.000000e-01	2.000000e-01	7.47e-10	4.22e-08
0.400000	4.000000e-01	4.000000e-01	3.14e-09	1.09e-08
0.600000	6.000000e-01	6.000000e-01	7.36e-09	2.35e-08
0.800000	8.000000e-01	8.000000e-01	1.33e-08	1.42e-08
1.000000	1.00000e+00	1.00000e+00	2.09e-08	2.63e-08

Table 3. We present the exact and the approximate solutions of the equation in the example 2 in some arbitrary points, the error for N = 10 is compared with the ones treated in (Abdelwahid, 2010)

Values of t	Exact solution φ	Approx solution $\tilde{\varphi}$	Error	Error (Abdelwahid, 2010)
0.000000	1.000000e+00	1.000000e+00	0.00e+00	0.00e+00
0.200000	1.221403e+00	1.221919e+00	5.16e-04	9.40e-04
0.400000	1.491825e+00	1.493531e+00	1.70e-03	3.06e-03
0.600000	1.822119e+00	1.826756e+00	4.63e-03	8.16e-03
0.800000	2.225541e+00	2.238233e+00	1.26e-02	2.16e-02
1.000000	2.718282e+00	2.756934e+00	3.86e-02	6.27e-02

Table 4. We present the exact and the approximate solutions of the equation in the example 3 in some arbitrary points, the error for $N = 10$ is compared with the ones treated in (Shahsavaran, 2011)

Values of t	Exact solution φ	Approx solution $\tilde{\varphi}$	Error	Error (Shahsavaran, 2011)
0.000000	0.00000e+00	0.00000e+00	0.00e+00	0.0e+00
0.200000	1.986693e-01	1.986672e-01	2.11e-06	8.4e-03
0.400000	3.894183e-01	3.894008e-01	1.75e-05	5.8e-03
0.600000	5.646425e-01	5.645828e-01	5.97e-05	5.0e-03
0.800000	7.173561e-01	7.172144e-01	1.41e-04	7.0e-03
1.000000	8.414710e-01	8.411949e-01	2.76e-04	4.1e-03

Table 5. We present the exact and the approximate solutions of the equation in the example 4 in some arbitrary points, the error for $N = 10$ is compared with the ones treated in (Lepik and Tamme, 2007)

Values of t	Exact solution φ	Appro solution $\tilde{\varphi}$	Error	Error (Lepik and Tamme, 2007)
0.000000	0.00000e+00	0.00000e+00	0.00e+00	2.7e-04
0.200000	4.472136e-01	4.472137e-01	1.10e-07	2.7e-04
0.400000	6.324555e-01	6.324560e-01	4.40e-07	2.7e-04
0.600000	7.745967e-01	7.745977e-01	9.90e-07	2.7e-04
0.800000	8.944272e-01	8.944290e-01	1.76e-06	2.7e-04
1.000000	1.00000e+00	1.00000e+00	2.75e-06	2.7e-04

Table 6. We present the exact and the approximate solutions of the equation in the example 5 in some arbitrary points, the error for $N = 10$ is compared with the ones treated in (Saberi-Nadja and Heidari, 2010)

Values of t	Exact solution φ	Approx solution $\tilde{\varphi}$	Error	Error (Saberi-Nadja and Heidari, 2010)
0.000000	7.542669e-02	7.542663e-02	5.44e-08	4.98e-02
0.200000	6.488067e-01	6.488067e-01	4.40e-08	4.03e-02
0.400000	9.743646e-01	9.743646e-01	1.68e-08	1.53e-02
0.600000	9.277484e-01	9.277484e-01	1.68e-08	1.53e-02
0.800000	5.267638e-01	5.267638e-01	4.40e-08	4.03e-02
1.000000	-7.542669e-02	-7.542663e-02	5.44e-08	1.53e-02

Table 7. We present the exact and the approximate solutions of the equation in the example 5 in some arbitrary points, the error for $N = 20$ is compared with the ones treated in (Awawdeh *et al*., 2009)

Values of t	Exact solution φ	Approx solution $\tilde{\varphi}$	Error	Error (Awawdeh *et al*., 2009)
0.000000	7.542669e-002	7.542669e-002	3.19e-016	5.53e-015
0.200000	6.488067e-001	6.488067e-001	2.22e-016	4.55e-015
0.400000	9.743646e-001	9.743646e-001	1.11e-016	1.77e-015
0.600000	9.277484e-001	9.277484e-001	1.11e-016	1.77e-015
0.800000	5.267638e-001	5.267638e-001	2.22e-016	4.55e-015
1.000000	-7.542669e-002	-7.542669e-002	3.19e-016	5.53e-015

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Author's Contributions

The authors contributed equally to the writing of this paper and they read and approved the final manuscript.

Ethics

Authors address any ethical issues that may arise after the publication of this manuscript.

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