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Global Analysis of an Epidemic Model with Non-Linear Incidence Rate

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Abstract: Problem statement: In this study the equilibrium points and their local stability are found for SIQ and SIQR epidemic models with three forms of the incidence rates. Also, we study the global stability of the equilibrium by constructing the new forms of Liapounov functions. **Conclusion/Recommendations:** We explored the existence of Hopf bifurcation for some parameters in the given model.

Key words: Epidemic models, infectious disease, endemic model, local stability, global stability, hop bifurcation, quarantined individuals, infected individuals, periodic solutions

INTRODUCTION

One intervention procedure to control the spread of infectious diseases is to isolate some infective, in order to reduce transmissions of the infection to susceptible. Isolation may have been the first infection control method, since biblical passages refer to the ostracism of lepers and plague sufferers were often isolated. The word quarantine originally corresponded to a period of forty days, which is the length of time that arriving ships suspected of plague infection were constrained from intercourse with the shore inMediterranean ports in the 15-19th centuries (McNeill, 1998). The word quarantine has evolved to mean forced isolation or stoppage of interactions with others .Over the centuries quarantine has been used to reduce the transmission of human diseases such as leprosy, plague, cholera, typhus, yellow fever, smallpox, diphtheria, tuberculosis, measles, mumps, Ebola and Lassa fever, Rubella, Herpes simplex, Hepatitis B, Chagas and the most notorious AIDS (Hassard *et al*., 1981; Hethcote, 2000, Castillo-Chavez *et al*., 1989). For human and animal diseases, horizontal transmission typically occurs through direct or indirect physical contact with hosts, or through a disease vector such as mosquitoes, ticks, or other biting disease agents. An epidemic is an outbreak of disease over short time pe- 1 riod; disease is said to be endemic if it persists in a population over a long period of time. In order to study the effects of quarantine on endemic infectious diseases, the endemic models that include a new class Q of quarantined individuals, who have been removed and isolated either voluntarily or coercively from the infectious class. For some milder disease, quarantined people could be people who choose to stay home from school or work because they are sick. For other more severe disease, quarantined people could be those who are forced into isolation. It is assumed that these quarantined individuals do not mix with others, so that they do not infect susceptible. In these models susceptible in the S class become infected and move to the infectious class I In the three SIQ models for infectious that do not confer immunity, susceptible become infected and then some infected individuals remain in the I class for their entire infectious period before they return to the susceptible class, while other infected individuals are transferred into a quarantined class Q and remain there until they are no longer infectious, at which time they return to the susceptible class. In the SIQR models for infectious that confer permanent immunity, susceptible become infected and then some infected individuals stay in the I class while they are infectious and then move to the removed class R upon recovery. Other infected individuals are transferred into the quarantined class Q while they are infectious and then move into the removed class R. The models here have a variable total population size, because they have recruitment into the susceptible class by births or immigration and they have both natural and disease-related deaths (Dreismann, 1996). In these models we identify the basic reproduction numbers that are the thresholds, find the disease-free and endemic equilibrium and determine their stability. The incidence is the infection rate of susceptible individuals through their contacts with infective. Let S (t) be the number of

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susceptible at time t, I (t) be the number of infective, Q(t) be the number quarantined individuals, R(t) be the number of removed people and N(t) be the total population size. If β is the average number of adequate contacts of person per unit time, then I/N is the infectious fraction, $\beta I/N$ is the average number of contacts with infective per unit time of one susceptible and βIH (S, I,Q) is the number of new cases with per unit time due to the S susceptible. This form is called the general incidence (Hethcote, 1976; Dreismann, 1996; EL-Owaidy and EL-Sheikh, 1990). This study is organized as follows. We study the local and global stability of the reduced three dimension SIQ epidemic model by using the Routh-Hurwth criterion and we study the existence of periodic solutions. We also discuss the local and global stability of the four dimensional epidemic model SIQR with general nonlinear incidence rate. We introduce the sufficient condition for existence of Hopf bifurcation for some parameters. The study end with a brief discussion.

Three-dimensional reduced epidemic model: The total population N(t) is divided into three compartments with N(t) = S(t)+ I(t) + Q(t), where S is the number of individuals in the susceptible class, I is the number of individuals who are infectious but not quarantined and Q is the number of individuals who are quarantined. The latten period, in which the person is infected but not yet infectious, is neglected and it is assumed that an infection does not confer immunity (Li *et al*., 1999; 2001; Li and Wang, 2002). The following Eq. 1a three-dimensional system:

$$S'(t) = A-\beta IH(S, I,Q)-dS + \gamma I + \varepsilon Q,$$
$$I'(t) = \beta IH(S, I,Q)-(\gamma + \delta + d + \alpha)I$$
$$Q'(t) = \delta I-(\varepsilon + d + \alpha)Q, \tag{1a}$$

Where:
A, d and β = Positive constants
γ, δ, ε and α = Non-negative constants

The constant A is the recruitment rate of susceptibles corresponding to births and immigration, d is the per capita natural mortality rate, βIH (S, I, Q) is general nonlinear incidence, δ is the rate constant for individuals leaving the infective compartment I for the quarantine compartment Q, α is the disease-related death rate constant in compartments I and Q and γ and ε are the rates at which individuals recover and return to susceptible compartment S for compartments I and Q, respectively. The total population size N(t) is variable with $N'(t) = A- dN-\alpha(I + Q)$. In the absence of disease, the population size N approaches carrying capacity A/d. the differential equation for N implies that solutions of (1a) starting in the positive outhunt R^3+ defined by Eq. 2a:

$$\Gamma = \{(S, I, Q) \in R^3_+ : S + I + Q \le A/d\} \tag{1b}$$

The linear zed problem corresponding to (1a) is Eq. 1c and 1d:

$$X' = MX, \text{ where } X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, (x_1, x_2, x_3) \in R^3_+ \tag{1c}$$

$$M = \begin{pmatrix} -\beta I\frac{\partial H}{\partial S} - d & -\beta I\frac{\partial H}{\partial I} - \beta H + \gamma & -\beta I\frac{\partial H}{\partial Q} + \varepsilon \\ \beta I\frac{\partial H}{\partial S} & \beta I\frac{\partial H}{\partial I} + \beta H - (\gamma + \delta + d + \alpha) & \beta I\frac{\partial H}{\partial Q} \\ 0 & \delta & -(\varepsilon + d + \alpha) \end{pmatrix} \tag{1d}$$

It is clear that (2a) has $\bar{P}_0 =$ (A/d, 0, 0) as a trivial equilibrium (a disease-free equilibrium). The Jacobean matrix of (1d) at $\bar{P}_0$ is Eq. 1e:

$$M_{\bar{P}_0} = \begin{pmatrix} -d & -\beta H + \gamma & \varepsilon \\ 0 & \beta H - (\gamma + \delta + D + \alpha) & 0 \\ 0 & \delta & -(\varepsilon + d + \alpha) \end{pmatrix} \tag{1e}$$

The eigenvalues are $\lambda_1 = -d < 0$, $\lambda_2 = -(\varepsilon + d + \alpha) < 0$ and $\lambda_3 = \beta H-(\gamma + \delta + d + \alpha) \le 0$. Set $R_q = \frac{\beta H}{(\gamma + \delta + d + \alpha)}$. If $R_q \le 1$, then the disease-free equilibrium $\bar{P}_0$ = (A / d, 0, 0) is locally asymptotically stable. If $R_q > 1$, then the equilibrium $\bar{P}_0$ is unstable.

Now we consider the nontrivial equilibrium P_0 = ($S_\circ$, $I_\circ$, $Q_\circ$) of system (1a) Eq. 1f,

Where:

$$S_\circ = \frac{I_0}{d}\left[A\text{-}(\delta + d + \alpha) + \frac{\varepsilon\delta}{\varepsilon + d + \alpha}\right], Q_\circ = \frac{\delta}{\varepsilon + d + \alpha} I_\circ \tag{1f}$$

The Jacobean matrix of at P_0 is Eq. 1g:

$$MP_\circ = \begin{pmatrix} -\beta I_0 H_{S_0} - d & -\beta I_0 H_{I_0} - \beta H_\circ + \gamma & -\beta I_\circ H_{Q_0} + \varepsilon \\ \beta I_0 H_{S_0} & \beta I_0 H_{I_0} + \beta H_\circ - (\gamma + \delta + d + \alpha) & \beta I_\circ H_{Q_0} \\ 0 & \delta & -(\varepsilon + d + \alpha) \end{pmatrix} \tag{1g}$$

Where:

$$H_{S_\circ} = \frac{\partial H}{\partial S}\Big|_{S=S_\circ}, H_{I_\circ} = \frac{\partial H}{\partial S}\Big|_{I=I_\circ}, H_{Q_\circ} = \frac{\partial H}{\partial Q}\Big|_{Q=Q_\circ}, H_\circ = H(S_\circ, I_\circ, Q_\circ)$$

We assume that H_{S_0}, H_{I_0}, H_{Q_0} and $H_\circ$ are positive. Set $k_1 = \gamma + \delta + d + \alpha$, $k_2 = \varepsilon + d + \alpha$, $k_3 = \beta I_\circ H_{S\circ}$, $k_4 = I_\circ H_{I\circ}$ and $k_5 = \beta I_\circ H_{Q\circ}$. When $\beta H_\circ = k_1$ we have Eq. 1h:

$$M_{P_0} = \begin{pmatrix} -k_3-d & -k_4-k_1+\gamma & -k_5+\varepsilon \\ k_3 & k_4 & k_5 \\ 0 & \delta & -k_2 \end{pmatrix} \tag{1h}$$

The characteristic equation of M_{P_0} at $P_\circ$ is Eq. 1i and 1j:

$$\lambda^3 + \alpha_1\lambda^2 + \alpha_2\lambda + \alpha_3 = 0 \tag{1i}$$

Where:

$$\begin{aligned} \alpha_1 &= d + k_2 + k_3 - k_4, \\ \alpha_2 &= (k_1 + k_2 - \gamma - 2k_4)k_3 - (d + k_2)k_4 + dk_2 - \delta k_5, \\ \alpha_3 &= ((k_1 - \gamma - 2k_4)k_2 - \delta\varepsilon)k_3 - d(k_2k_4 + \delta k_5). \end{aligned} \tag{1j}$$

Since the Routh-Hurwitz criterion (Lizana and Rivero, 1996; Nani and Freedman, 2000; Thieme, 1992; 1993, Anderson and May, 1991) and says that $P_\circ = (S_\circ, I_\circ, Q_\circ)$ is locally asymptotically stable if $\alpha_1 > 0$, $\alpha_3 > 0$ and $\alpha_1\alpha_2 - \alpha_3 > 0$, then we have the following theorem.

Theorem 2.1: Assume that the following conditions are satisfied:

$$(A_1)\ L_1 + H_{S_0} > H_{I_0}, \text{ where } L_1 = \frac{2d+\varepsilon}{\beta I_0},$$

$$(A_2)\ L_2\frac{H_{Q_0}}{H_{S_0}} + L_3\frac{H_{I_0}}{H_{S_0}} + L_4H_{I_0} + L_5 < 1,$$

Where:

$$L_2 = \frac{d\delta}{k_1k_2}, L_3 = \frac{d}{k_1}, L_4 = \frac{2\beta I_0}{k_1}, L_5 = \frac{\delta\varepsilon + \gamma k_2}{k_1k_2}$$

$$(A_3)\ H_{S_0}^2 + \sigma_1H_{S_0} + \sigma_2 + \sigma_3H_{S_0}H_{I_0}^2 > H_{I_0} + \sigma_4H_{S_0}H_{Q_0} + \sigma_5H_{Q_0} + \sigma_6H_{S_0}H_{I_0}$$

Where:

$$\sigma_1 = \frac{(d\delta + d\alpha + \delta\varepsilon + \varepsilon + 2d + \alpha)(\varepsilon + 2d + \alpha)}{(\beta I_\circ)^2(2\alpha + \delta + \varepsilon + 2d)},$$

$$\sigma_2 = \frac{d^2(\varepsilon + d + \alpha) + d(\varepsilon + d + \alpha)^2}{(\beta I_\circ)^2(2\alpha + \delta + \varepsilon + 2d)}$$

$$\sigma_3 = \sigma_1(\varepsilon + 2d + \alpha) + \frac{2\beta I_\circ(\varepsilon + 2d + \alpha)}{(\varepsilon + 2d + \alpha)(2\alpha + \delta + \varepsilon + 2d)},$$

$$\sigma_4 = \frac{\delta\beta I_\circ}{(\varepsilon + 2d + \alpha)^2}, \quad \sigma_5 = \frac{\delta(\varepsilon + d + \alpha)}{(\varepsilon + 2d + \alpha)^2}$$

$$\text{and } \sigma_6 = \frac{\beta I_\circ(2\varepsilon + 6d + 3\alpha + 2)}{(\varepsilon + 2d + \alpha)^2}$$

Then the equilibrium point $P_\circ = (S_\circ, I_\circ, Q_\circ)$ is locally asymptotically stable.

Now we choose α as a bifurcation parameter for system (1a). Let α_c be the value of α at which the characteristic Eq. 1i has two pure imaginary roots $\lambda_{1,2}$. Thus we have the following result.

Theorem 2.2: If the assumption (1q) holds, then at $\alpha = \alpha_c$,there exists one parameter family of periodic solutions bifurcating from the critical point $P_\circ = (S_\circ, I_\circ, Q_\circ)$ with period T, where $T \to T_\circ$ as $\alpha \to \alpha_c$ and where $T_\circ = 2\pi/\omega_0 = 2\pi/\sqrt{\alpha_2}$ and α_2 is given in (1j).

Proof: Since there exists at least one real root of the cubic Eq. 1i λ_3 say, we have the following Eq. 1k and 1l factorization:

$$(\lambda - \lambda_3)[\lambda^2 + (\lambda_3 + \alpha_1)\lambda + (\lambda_3^2 + \alpha_1\lambda_3 + \alpha_2)] = 0 \tag{1k}$$

Since, by (1j):

$$\lambda_1 + \lambda_2 + \lambda_3 = -\alpha_1 \tag{1l}$$

Also at $\alpha = \alpha_c$, we obtain Eq. 1m:

$$\begin{aligned} &\lambda_3 = -\alpha_1, \quad \lambda_1 = -\bar{\lambda}_2, \\ &\lambda_{1,2} = -\frac{1}{2}\left\{(\lambda_3 + a_1) \pm \sqrt{(\lambda_3 + \alpha_1)^2 - 4(\lambda_3^2 + \alpha_1\lambda_3 + \alpha_2)}\right\} \end{aligned} \tag{1m}$$

Thus, at $\alpha = \alpha_c$, (1j) can be written in the following form Eq. 1n:

$$D_\alpha(\alpha_1) = \alpha_1\alpha_2 - \alpha_3 \tag{1n}$$

Hence, since $\alpha_2 > 0$ and $\alpha_3 > 0$, at $\alpha = \alpha_c$, we should have $\lambda_3 = -\alpha_1 < 0$. Also, the critical value $\alpha = \alpha_c > 0$ is the solution of (2n) which can be seen by (2k) to be the quadratic equation in α as follows Eq. 1o and 1p:

$$-c_1\alpha^2 - c_2\alpha + c_3 = 0 \tag{2o}$$

Where:

$$c_1 = d + \beta I_\circ (H_{S_\circ} - H_{I_\circ})$$
$$c_2 = 2\beta I_\circ (\varepsilon + 2d)(H_{S_0} - H_{I_0}) + 2d\left(\varepsilon + 2d + \frac{1}{2}\right)$$
$$+\beta^2 I_0^2 (H_{S_0}^2 - 3H_{S_\circ} H_{I_\circ} + H_{S_0}^2) - \delta\beta I_\circ H_{Q_0}$$
$$c_3 = (\varepsilon + d)(\beta I_\circ \delta(H_{S_\circ} + H_{Q_\circ}) + 2d^2 + d\varepsilon) \quad (1p)$$
$$+\beta I_\circ (\varepsilon + 2d)^2 (H_{S_0} - H_{I_0}) + (\varepsilon + 2d)$$
$$(\beta^2 I_0^2 (H_{S_0} + H_{I_0})^2) + \beta^2 I_0^2 (\delta H_{S_0}^2 + 2H_{S_\circ} H_{I_\circ}^2$$
$$+\delta H_{I_0} H_{Q_0} + (2d + 2) H_{S_0} H_{Q_0} + \delta H_{S_0} H_{Q_0})$$

Conversely, knowing that $\alpha_1 > 0$, $\alpha_3 > 0$ and $\alpha > 0$, then we can solve (2o) for $\alpha_c > 0$ and we then know that $\alpha_2 > 0$, $\lambda_3 = -\alpha_1 < 0$ and $\lambda_{1,2}$ are conjugate imaginary. Now, choosing both d and H_{S_0} to be sufficiently small and H_{I_0} sufficiently large, then we get Eq. 1q:

$$d + \beta I_0 H_{S_0} > \beta I_\circ H_{I_0} \quad (1q)$$

But since by (1o) and (1q) $c_1 > 0$, $c_2 > 0$ and $c_3 > 0$ Eq. 1r:

$$D_\circ(\alpha_1) = c_3 > 0, \lim D_\alpha(\alpha_1) = \infty \quad (1r)$$

Thus α_c is uniquely determined. Now, since by (1i), $\lambda_3 = -\alpha_1 < 0$ and Eq. 1s:

$$\begin{aligned} D_\alpha(\alpha_1) &= \alpha_1\alpha_2 - \alpha_3 = (\alpha_1 + \lambda_3)(\lambda_1\lambda_2 - \alpha_1\lambda_1) \\ \operatorname{sgn} D_\alpha(\alpha_1) &= \operatorname{sgn}(\alpha_1 + \lambda_3) \end{aligned} \quad (1s)$$

Consequently we have Eq. 1t and 1u:

$$\operatorname{Re}\lambda_{1,2} = \frac{1}{2}(\alpha_1 + \lambda_3) < 0 \text{ for } \alpha > \alpha_c \quad (1t)$$

$$\operatorname{Re}\lambda_{1,2} > 0 \text{ for } \alpha < \alpha_c \quad (1u)$$

By the above discussion, we see that as α is increased though α_c, there exists a pair of complex conjugate imaginary eigenvalues $\lambda_{1,2}$ of the Jacobian matrix M_{P_0}. Since at $\alpha = \alpha_c$, $\lambda_3 = -\alpha_1$, $\lambda_{1,2} = \pm i\sqrt{\alpha_2} = \pm i\omega_\circ$, where it is clear that $\omega_\circ > 0$. Now, since for $\lambda_1 = -\lambda_2$ Eq. 1v:

$$\operatorname{Re}\lambda_2 = \frac{1}{2}(\lambda_2 + -\bar{\lambda}_2) = 0 \text{ at } \alpha = \alpha_c \quad (1v)$$

and by above discussion we see that Re $\lambda_2 > 0$ for $\alpha < \alpha_c$ and Re $\lambda_2 = < 0$ for $\alpha > \alpha_c$ thus Eq. 1w:

$$\begin{aligned} \frac{d}{d\alpha}(\operatorname{Re}\lambda_2)\big|_{\alpha=\alpha_c} &= \frac{-1}{2}\frac{d}{d\alpha}(\alpha_1 + \lambda_3)\big|_{\alpha=\alpha_c} \\ &= R_e\left(\frac{d}{d\alpha}\lambda_2\right)\big|_{\alpha=\alpha_c} < 0 \end{aligned} \quad (1w)$$

This completes the proof.

Fourth-dimensional reduced epidemic model: Assume that infection confers permanent immunity, so that individuals can move from theI and Q classes to the R class, where R(t) is the number of individuals with permanent immunity and N(t) = S(t) + I(t) + Q(t)+R(t). In this model, the flow is from the S class to the I class and then either directly to the R class or to the Q class and then to the R class. This model is given by Eq. 1x:

$$\begin{aligned} S'(t) &= A - \beta IH(S, I, R) - dS, \\ I'(t) &= \beta IH(S, I, R) - (\gamma + \delta + d + \alpha_1)I, \\ Q'(t) &= \delta I - (\varepsilon + d + \alpha_2)Q, \\ R'(t) &= \gamma I + \varepsilon Q - dR \end{aligned} \quad (1x)$$

where δ and ε are the removal rate constants from group I and Q respectively. Also α_1 and α_2 represent the extra disease-related death rate constants in classes I and Q, respectively. The general nonlinear incidence rate is $\beta IH(S, I,R)$ and the other parameters are the same as in the first model. The total population size N (t) satisfies $N'(t) = A-dN-\alpha_1 I-\alpha_2 Q$, so that the population size N approaches the carrying capacity A/d when there is no disease. The differential equation for N implies that solutions of (1a) starting in the positive orthant R_4^+ defined by Eq. 1y:

$$\begin{aligned} \Gamma &= \{(S, I, Q, R) \in \mathbb{R}_+^4 \\ &+ : S + I + Q + R \le A/d\} \end{aligned} \quad (1y)$$

We discuss the existence and global stability of the equilibrium of (1a). The equilibrium points of Eq. 2b is obtained by the system of isoclines Eq. 1z:

$$\begin{aligned} &A - \beta IH(S, I, R) - dS = 0, \\ &\beta IH(S, I, R) - (\gamma + \delta + d + \alpha_1)I = 0, \\ &\delta I - (\varepsilon + d + \alpha_2)Q = 0, \\ &\gamma I + \varepsilon Q - dR = 0 \end{aligned} \quad (1z)$$

The possible equilibrium of (1x) are $P_\circ$ = (A/d, 0, 0, 0) and P* = (S*, I*, Q*, R*). The Jacobean matrix due to linearization of (1x) at the equilibrium point $P_\circ$ = (A/d, 0, 0, 0) is Eq. 2a:

$$J_{P_\circ} = \begin{pmatrix} -d & -\beta H_\circ & 0 & 0 \\ 0 & \beta H_\circ - (\gamma+\delta+d+\alpha_1) & 0 & 0 \\ 0 & \delta & -(\varepsilon+d+\alpha_2) & 0 \\ 0 & \gamma & \varepsilon & -d \end{pmatrix} \quad (2a)$$

The eigenvalues of $J_{P_\circ}$ at $P_\circ = (A/d, 0, 0, 0)$ are given by $\lambda_1 = \lambda_4 = -d < 0$, $\lambda_3 = -(\varepsilon+d+\alpha_2) < o$ and $\lambda_2 = \beta H_\circ-(\gamma+\delta+d+\alpha_1) \leq 0$. Set $R_q = \frac{\beta H_\circ}{(\gamma+\delta+d+\alpha_1)}$ the above discussion leads to the following results.

Theorem 3.1: If $R_q \leq 1$, then $P_\circ = (A/d, 0, 0, 0)$ is locally asymptotically stable. If $R_q > 1$, then the equilibrium $P_\circ$ is unstable.

Now we consider the endemic equilibrium point $P^* = (S^*, I^*, Q^*, R^*)$ of system (1x).

Where:

$$S^* = (A-(\gamma+\delta+d+\alpha_1))\frac{I^*}{d},$$
$$Q^* = \frac{\delta}{\varepsilon+d+\alpha_2} I^*, \quad (2b)$$
$$R^* = (\gamma+\varepsilon\delta(\varepsilon+d+\alpha_2))\frac{I^*}{d}$$

The Jacobean matrix at $P^* = (S^*, I^*, Q^*, R^*)$ is Eq. 2c:

$$J_{P^*} = \begin{pmatrix} -\beta I^* H_{S^*} - d & -\beta I^* H_{I^*} - \beta H^* & 0 & -\beta I^* H_{R^*} \\ 0 & \beta I^* H_{I^*} + \beta H^* - L_1 & 0 & \beta I^* H_{R^*} \\ 0 & \delta & -L_2 & 0 \\ 0 & \gamma & \varepsilon & -d \end{pmatrix} \quad (2c)$$

Where:

$$HS^* = \frac{\partial H}{\partial S}\Big|_{S=S^*}, H_{I^*} = \frac{\partial H}{\partial S}\Big|_{I=I^*}, H_{Q^*} = \frac{\partial H}{\partial Q}\Big|_{Q=Q^*}, H_{R^*} = \frac{\partial H}{\partial Q}\Big|_{R=R^*},$$
$$H^* = H(S^*, I^*, R^*), L_1 = (\gamma+\delta+d+\alpha_1) \text{ and } L_2 = (\varepsilon+d+\alpha_2)$$

The characteristic equation at the endemic equilibrium point is Eq. 2d and 2e:

$$\lambda^4 + \sigma_1\lambda^3 + \sigma_2\lambda^2 + \sigma_3\lambda + \sigma_4 = 0 \quad (2d)$$

Where:

$$\begin{aligned} &\sigma_1 = \varepsilon + 3d + \alpha_2 + \beta I^*(H_{S^*} - H_{I^*}), \\ &\sigma_2 = \beta I^* H_{S^*}(d+L_1+L_2) \\ &-\beta I^* H_{I^*}(2d+L_2) - \gamma\beta I^* H_{R^*} + 2dL_2 + d^2, \\ &\sigma_3 = \beta I^* H_{S^*}(dL_1 + dL_2 + L_1L_2) - d\beta I^* H_{I^*} \\ &(2L_2+d) - \beta I^* H_{R^*}(\delta\varepsilon + d\gamma + \gamma L_2) + d^2 L_2, \\ &\sigma_4 = d\beta I^*((L_1 H_{S^*} - dH_{I^*})L_2 - (\gamma L_2 + \delta\varepsilon)H_{R^*}) \end{aligned} \quad (3e)$$

The Routh-Hurwitz criterion (Lancaster, 1969; Nani and Freedman, 2000) that are necessary and sufficient for the local asymptotic stability of the endemic equilibrium point is that the coefficients are positive Eq. 2f:

$$(i)\sigma_i > 0,\ i = 1,2,3,4, (ii)\sigma_1\sigma_2\sigma_3 > \sigma^2{}_1\sigma_4 + \sigma^2{}_3 \quad (2f)$$

Lemma 3.2: Assume that the following conditions are satisfied Eq. 2g:

$$(i)\sigma_i > 0,\ i = 1,2,3,4, (ii)\sigma_2\sigma_3 - \sigma_4 > 0, (iii)\sigma_1\sigma_2\sigma_3 \leq \sigma^2{}_1\sigma_4 + \sigma^2{}_3 \quad (2g)$$

Then the characteristic Eq. 2d can be factorized into the form Eq. 2h:

$$(\lambda^2 + n_1)(\lambda + n_2)(\lambda + n_3) = 0,\ n_i > 0,\ i = 1,2,3, \quad (2h)$$

where, $\sigma_1 = n_2 + n_3$, $\sigma_2 = n_1 + n_2n_3$, $\sigma_3 = n_1(n_2 + n_3)$ and $\sigma_4 = n_1n_2n_3$, which implies that $n_1 = \frac{\sigma_3}{\sigma_1}$ and n_2, n_3 are satisfied by the quadratic Eq. 2i:

$$x^2 - \sigma_1 x + (\sigma_1 - \frac{\sigma_3}{\sigma_1}) = 0 \quad (2i)$$

The eigenvalues of (2c) are given by $\{i\sqrt{n_1}, -i\sqrt{n_1}, n_2, -n_3\}$. Thus, under the conditions of the above Lemma, the eigenvalues of the Jacobian matrix J_{P^*} have two pure imaginary roots for some value of α, say $\alpha = \alpha^*$. For $\alpha \in (\alpha^* - \in, \alpha^* + \in)$, the characteristic Eq. 2c cannot have real positive roots. But for $\alpha \in (\alpha^* - \in, \alpha^* + \in)$, the roots are in the general form Eq. 2j:

$$\begin{aligned} &\lambda_1(\alpha) = \theta(\alpha) + i\phi(\alpha), \\ &\lambda_2(\alpha) = \theta(\alpha) - i\phi(\alpha), \\ &\lambda_3(\alpha) = -n_2 \neq 0, \\ &\lambda_4(\alpha) = -n_3 \neq 0 \end{aligned} \quad (2j)$$

We now apply Hopf transversality criterion to (2d) in order to obtain the required condition for Hopfbifurcation to occur for this system. Hoofs transversality criterion is given by:

$$\mathrm{Re}\left[\frac{d\lambda_j}{d\alpha}\right]_{\alpha=\alpha^*} \neq 0,\ j = 1,2$$

Substituting $\lambda_j(\alpha) = \theta(\alpha) + i\phi(\alpha)$ into (2d),we obtain:

$$\begin{aligned}&4(\theta(\alpha) + i\phi(\alpha))^3(\theta'(\alpha) + i\phi'(\alpha)) + \sigma'_1(\theta(\alpha) + i\phi(\alpha))^3 +\\ &3\sigma_1(\theta(\alpha) + i\phi(\alpha))^2(\theta'(\alpha) + i\phi'(\alpha)) + \sigma'_2(\theta(\alpha) + i\phi(\alpha))^2\\ &+ 2\sigma_2(\theta(\alpha) + i\phi(\alpha))(\theta'(\alpha) + i\phi'(\alpha)) + 8\sigma'_3(\theta(\alpha) +\\ &i\phi(\alpha)) + \sigma_3(\theta'(\alpha) + i\phi'(\alpha)) + \sigma'_4 = 0\end{aligned}$$

By comparing the real and imaginary parts in both sides of the above equation, we get Eq. 2k and 2l:

$$\begin{aligned}&A(\alpha)\theta'(\alpha) - B(\alpha)\phi'(\alpha) + C(\alpha) = 0,\\ &B(\alpha)\theta'(\alpha) + A(\alpha)\phi'(\alpha) + D(\alpha) = 0\end{aligned} \qquad (2k)$$

Where:

$$\begin{aligned}&A(\alpha) = 4\theta(\theta^2 - 3\phi^2) + 3\sigma_1(\theta^2 - \varphi^2) + 2\sigma_2\theta + \sigma_3,\\ &B(\alpha) = 4\phi(3\theta^2 - \phi^2) + 2\phi(3\sigma_1 + \sigma_2),\\ &C(\alpha) = \sigma'_1\theta(\theta^2 - 3\phi^2) + \sigma'_2(\theta^2 - \varphi^2) + \sigma'_2\theta + \sigma'_4,\\ &D(\alpha) = \sigma'_1\phi(3\theta - \phi^2) + 2\sigma'_2\theta\phi + \sigma'_3\phi\end{aligned} \qquad (2l)$$

Thus, from (3j), we have Eq. 2m:

$$\mathrm{Re}\left[\frac{d\lambda_j}{d\alpha}\right]_{\alpha=\alpha^*} = \frac{\det\begin{pmatrix}-C(\alpha) & -B(\alpha)\\ -D(\alpha) & A(\alpha)\end{pmatrix}}{\det\begin{pmatrix}A(\alpha) & -B(\alpha)\\ B(\alpha) & A(\alpha)\end{pmatrix}} = -\frac{(AC + BD)}{A^2 + B^2} \qquad (2m)$$

Since (AC + BD) = 0, then $\mathrm{Re}\frac{d\lambda_j}{d\alpha}_{\alpha=\alpha^*} = 0$. The above discussion proves the following result.

Theorem 3.3: Suppose the equilibrium point P* = (S*, I*,Q*,R*) exists, $\sigma_i > 0$, i = 1, 2, 3, 4 and $\sigma_1\sigma_2\sigma_3 \le \sigma_1^2\sigma_4 + \sigma_3^2$; then the system (2b) exhibits an Hopf- Andronov-Poincare bifurcation in the first orthant, leading to a family of periodic solutions that bifurcate from P* for suitable values of α in the neighborhood of α = α*.

CONCLUSION

In this study, we studied a general SIQR model for the dynamics of an infectious disease. The incidence rate βIH (S, I, R) is of nonlinear form. We established the local asymptotic stability of the disease-free equilibrium points $\bar{P}_O = (A/d, 0, 0)$, $P_\circ$ = (A/d, 0, 0, 0) for systems (1a) and (2.1), respectively. Our results are consistent with those obtained by Li *et al*. (2001); Jing and Lin (1993); Wu and Feng (2000) and Greenhalgh (1992). The disease-free equilibrium point $\bar{P}_\circ = (A/d, 0, 0)$ is locally asymptotically stable in the interior of the feasible region and the disease always dies out. Also we showed that the endemic equilibrium point P° = ($S_\circ$, $I_\circ$, $Q_\circ$) exists and is locally asymptotically stable in the interior of the feasible region. The global stability of $\bar{P}_\circ = (A/d, 0, 0)$ and P° = ($S_\circ$, $I_\circ$, $Q_\circ$) was established using Lyapunov functions similar to those discussed by Li and Wang (2002) and Nani and Freedman (2000), respectively. We employed the mathematical tools of differential analysis, persistence theory Hopf-Andronnov-poincare bifurcation and 9 linear system theory to deduce the existence of a family of periodic solutions that bifurcate from P* = (S*, I*, Q*, R*).Our results obtained here improve and partially generalize those obtained in (Li *et al*., 1999; 2001; Li and Wang, 2002; Guckenheimer and Holmes, 1983; Anderson *et al*., 1982).

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