On the Dynamics of a General Predator-Prey System

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Abstract: Problem statement: In this study a general two-dimensional predator-prey model is considered. The dynamic and existence of equilibrium points are studied. Conclusion/Recommendations: Hopf bifurcation is discussed. The existence and uniqueness of limit cycles is proved. Special cases are considered to justify our results.

Key words: Predator-prey system, functional response, limit cycles, hopf bifurcation, mathematics subject classification, analyzing models, interacting species, predator population, efficiency rate

INTRODUCTION

There has been a demanding need for developing and analyzing models of interacting species in ecosystems. Predator-prey models are one of the most important models of two-species interaction. In this study we are concerned with a general 2-dimensional predator-prey system of the form

\[ \frac{dx}{dt} = xg(x) - yp(x) - q_1E_1x, \]
\[ \frac{dy}{dt} = -Dy + yp(x) - q_2E_2y \]  

where, x and y are the prey and the predator population sizes, respectively. The parameters s, D and k are positive and represent the conversion efficiency rate of prey to predator, predator death rate and the carrying capacity of the prey population, respectively. The parameters E_1 \geq 0, E_2 \geq 0 denote the harvesting efforts for the predator and prey respectively. The expressions q_1E_1x and q_2E_2y represent the catch of the respective species, where q_1 and q_2 denote the catch ability coefficients of the prey and predator, respectively. The functions g(x) and p(x) denote the functional response of the prey and predator, respectively and satisfy the assumptions p(0) = 0 and g(0)>0, for all x>0. There have been considerable interests in the dynamics of the predator-prey models of some special cases of (1) by several authors (Attili, 2001; Attili and Mallak, 2006; Xiao and Ruan, 2001; Hesaaraki and Moghadas, 2001; Kar and Matsuda, 2007; Moghadas and Corbett, 2008; Moghadas and Alexander, 2005; Sugie et al., 1997; Ruan and Xiao, 2001). Hasik (2000); Sesay et al. (2010); Moghadas and Alexander (2006); Saha and Bandyopadhyay (2005) and Tao et al. (2011) the authors considered Eq. 1 in the special case g(x) = r(1 - \frac{x}{k}).

The aim of this study is to discuss the qualitative properties of the general predator-prey system (1). We discuss the existence and stability of equilibria and nonexistence criteria for limit cycles. We explore the uniqueness of limit cycles using Kuang and Freedman approach (Moghadas and Corbett, 2008) and some applications. It is natural due to biological considerations to expect that the solutions of (1) must to be positive and bounded. So we give the following result which is a partial extension of those of Freedman and So, 1985) and (Saha and Bandyopadhyay, 2005).The paper end with a brief discussion.

**Theorem 1.1:** All the solutions of Eq. 1 which start in \( \mathbb{R}_+^2 \) are positive and uniformly bounded.

**Proof:** According to the ecological consideration the positivity of the solutions of (1) is obtained. To show the roundedness', we set:

\[ g(x) = r(1 - \frac{x}{k}). \]
Then we easily obtain:
\[
\frac{dw}{dt} + \frac{dw}{dx} + \frac{1}{s} \frac{dy}{dt} = xg(x) - q_1E_1 x - \frac{D}{s} y - q_3E_3 y
\]
i.e. for any \( r \), we have:
\[
\frac{dw}{dt} + rw \leq x(\alpha + r - q_1E_1) \leq M^2 = M_1, \text{say}
\]
where, \( \alpha = \max g(x) \) and \( x \leq M = (\alpha + r - q_1E_1) \).

Thus applying the theory of differential inequality (Freedman and So, 1985), we obtain:
\[
0 < w(x, y) < \frac{M}{r}(1-e^{-n}) + w(x(0), y(0))e^{-n}.
\]

So the limits as tends to \( \infty \) yields
\[
0 < w(x, y) < \frac{M}{r},
\]

The nature of Equilibria: We discuss the stability properties of the equilibria \( \mathbf{p}_0(0,0), \mathbf{p}_1(x_1,0) \) and \( \mathbf{p}_2(x^*,y^*) \). The Jacobian matrix of (1) around \( \mathbf{p}_0(0,0) \) is:
\[
J_{\mathbf{p}_0(0,0)} = \begin{bmatrix}
g(0) = q_1E_1 & 0 \\
0 & -D - q_2E_2
\end{bmatrix}
\]
i.e., we have two eigenvalues \( \lambda_1 = -D - q_2E_2 \) and \( \lambda_2 = q_1E_1 \).

Therefore if \( g(0) > q_1E_1 \) then \( \mathbf{p}_0(0,0) \) is a saddle point, while if \( g(0) < q_1E_1 \) then \( \mathbf{p}_0(0,0) \) is asymptotically stable. Further at the boundary equilibrium \( \mathbf{p}_1(x_1,0) \), the Jacobian matrix has the eigenvalues:
\[
\lambda_1 = x_1g'(x_1) + g(x_1) - q_1E_1 \text{ and } \lambda_2 = -(D + q_2E_2 - sp(x_1)).
\]
i.e., if we assume that \( D + q_2E_2 > sp(x_1) \), then \( \mathbf{p}_1(x_1,0) \) is a saddle point while if \( x_1g'(x_1) + g(x_1) > q_1E_1 \), then \( \mathbf{p}_1(x_1,0) \) is asymptotically stable.

Now to discuss the stability of the interior equilibrium \( \mathbf{p}_2(x^*,y^*) \), the Jacobean matrix around \( \mathbf{p}_2 \) is:
\[
J_{\mathbf{p}_2(x^*,y^*)} = \begin{bmatrix}
x^* g(x^*) + g(x^*) - y^* p(x^*) - q_1E_1 - p(x^*) \\
sp(x^*) y^* p(x^*)
\end{bmatrix}
\]
The eigenvalues of \( J_{\mathbf{p}_2} \) obey the equation:
\[
\lambda^2 - (\text{trace } J_{\mathbf{p}_2}) \lambda + \det J_{\mathbf{p}_2} = 0
\]

Since it is clear that \( \det J_{\mathbf{p}_2} = sp(x^*)p(x^*) > 0 \), so the sign of the characteristic roots depend only on trace \( J_{\mathbf{p}_2}(x^*,y^*) \). Following (Pimply, 1974), we introduce an auxiliary parameter \( \beta \) such that \( J_{\mathbf{p}_2} \) takes the form:
\[
J_{\mathbf{p}_2} = \begin{bmatrix}
x^*[g(x^*) + g(x^*) - y^* p(x^*) - q_1E_1 - p(x^*)] \\
sp(x^*) y^* p(x^*)
\end{bmatrix}
\]
Since by the Routh-Hurwitz criterion \( P_2(x^*,y^*) \) is stable or unstable if (trace \( J_{\mathbf{p}_2} \)) < 0 or > 0, respectively.

This leads to if \( \beta < \frac{y^* p(x^*) + q_1E_1}{x^* g(x^*) + g(x^*)} \), then critical point \( P_2(x^*,y^*) \) is locally asymptotically stable. On the other
hand if $\beta$ is chosen such that $\beta > \frac{y^* p(x^*) + q E_1}{x^* g(x^*) + q E_1}$, then $P_2(x^*, y^*)$ is unstable in the positive quadrant.

However in the case $\beta = \beta^*$, trace $J_{\beta^*} = 0$. In this case the characteristic roots are pure imaginary. Further since $\frac{d}{d\beta}(\text{trace} \, J) < 0$ for $\beta \neq \beta^*$, then by Hopf bifurcation Theorem (Hassard et al., 1981), we have small amplitude periodic solution at $\beta = \beta^*$. Therefore the two interacting populations oscillate around the unique nontrivial positive equilibrium $P_2(x^*, y^*)$.

Existence of limit cycles: Now we discuss the existence and nonexistence of limit cycles of the general system (1). (Attili, 2001; Attili and Mallak, 2006; Hesaaraki and Moghadas, 2001; Kar and Matsuda, 2007; Moghadas and Corbett, 2008; Sesay et al., 2010). We start with a criterion for nonexistence of limit cycles.

Proposition 3.1: If $\frac{x^* g'(x^*) + g(x^*) - y^* p'(x^*) - q E_1}{p(x^*)} < 0$, then there is no periodic solutions of (1).

Proof: Since for any real function $\alpha(x, y)$ we have:

$$\text{div}\{F(x, y)\} = \frac{\alpha(x, y)[xg(x) - yp(x)]}{\alpha(x, y)[-Dy + sp(x)]} = \frac{x^* g'(x^*) + g(x^*) - y^* p'(x^*) - q E_1}{-Dy + sp(x)}.$$

choosing Dulac function $\alpha = 1$, then:

$$\text{div}\{F(x, y)\} = x^* g'(x^*) + g(x^*) - y^* p'(x^*) - q E_1 < 0$$

By Bendixon-Dulac Theorem (Kelley and Peterson, 2003), this is sufficient for nonexistence of periodic solutions of (1). The following result discusses the existence of limit cycles.

Theorem 3.1: The system (1) has at least one limit cycle in $\Omega = \{(x, y) : x > 0, y > 0\}$ if and only if Eq. 2:

$$x^* g'(x^*) + g(x^*) - y^* p'(x^*) - q E_1 < 0 \quad (2)$$

In order to prove sufficient condition of Theorem 3.1, we first note that one can show ((Freedman and So, 1985; Moghadas and Corbett, 2008) that, if:

$$\vartheta = \left\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq M, 0 \leq y \leq \frac{M}{r} \right\},$$

where $\frac{M}{r} = \max x$ then:

- The set $\vartheta$ is positively invariant
- For $(x_0, y_0) \in \mathbb{R}^2$, $(x(t), y(t)) \rightarrow \vartheta$ as $t \rightarrow \infty$

Now since by above, the characteristic polynomial at the nontrivial positive equilibrium $P_2(x^*, y^*)$ is:

$$p(\lambda) = \lambda^2 + \frac{y^* p(x^*) - x^* g(x^*)}{g(x^*) + q E_1} \lambda + sy^* p'(x^*) p(x^*)$$

Since $sy^* p'(x^*) > 0$, the roots of $p(\lambda)$ have positive real parts if and only if (2) holds. Consequently $P_2(x^*, y^*)$ is unstable if (2) holds. It is easy to check that the stable and unstable manifolds at $(0, 0)$ are on the x-axis and y-axis, respectively. If $P_2(x^*, y^*)$ is unstable, then by above discussion and Poincaré Bendixon Theorem, it follows that the $\omega$-limit set of every solution initiating at a point in the first quadrant is a limit cycle. Therefore, we have established that if (2) holds, then the system (1) has at least one limit cycle. This proves the sufficient condition.

Remark 3.1: We note that it can be shown that this condition is not only sufficient but also necessary (Attili, 2001; Attili and Mallak, 2006; Hesaaraki and Moghadas, 2001; Moghadas and Corbett, 2008).

Uniqueness of limit cycles: We discuss the uniqueness of limit cycles of the general system (1). Our criterion improves and partially generalizes those of (Hasik, 2000; Kuang and Freedman, 1988; Sugie et al., 1997). We first start with the most famous uniqueness result which was the motivation for several authors and criteria. As in (Kuang and Freedman, 1988) we consider the system Eq. 3:

$$\frac{dx}{dt} = xp(x) - y \varphi(x),$$
$$\frac{dy}{dt} = y(-\gamma + \psi(x)) \quad (3)$$

where, $\gamma > 0$, all the functions are sufficiently smooth on $[0, \infty)$ and satisfy Eq. 4:

$$\varphi(0) = \psi(0) = 0 \quad \text{and} \quad \varphi'(x) > 0, \psi(x) > 0 \quad \text{for} \quad x > 0 \quad (4)$$

The authors in (Kuang and Freedman, 1988) gave the following result on the uniqueness of limit cycles.

Theorem 4.1: Kuang and Freedman (1988) if there exist constants $x^*$ and $m$ with $0 < x^* < m$ such that:
The system (3) has exactly one limit cycle which is globally asymptotically stable. In view of Theorem 4.1 and those of (Sugie et al., 1997; Hasik, 2000), we give the following uniqueness theorem for the limit cycles of our general system (1).

**Theorem 4.2:** Assume that \( H(x) = x^g(x^*) + g(x^*) \cdot y^p(x^*) \cdot q(x^*) \geq 0 \). Then the system (1) has a unique limit cycle.

**Proof:** Rewriting the system (1) as the system (3) of Kuang and Freedman (1988) with:

\[
\gamma = D + q, E_2, p(x) = g(x) - q, E_1, q(x) = p(x) \text{ and } \psi(x) = sp(x)
\]

Then it is clear that our functions satisfy the conditions (4) of (Kuang and Freedman 1988). Moreover:

\[
\frac{d}{dx} \left( \frac{x(p(x))}{\psi(x)} \right) = \frac{d}{dx} \left( \frac{x^* g(x^*) - q_1 E_1}{P(x^*)} \right) = \frac{x^* [g(x^*) - q_1 E_1]}{P(x^*)} - \frac{x^* g(x^*) - q_1 E_1}{P(x^*)} \cdot \frac{p'(x^*)}{E_1} = \frac{x^* g(x^*) - q_1 E_1}{p(x^*)} \cdot \frac{y^p(x^*)}{p(x^*)} - \frac{H(x)}{p(x^*)} > 0
\]

Setting \( W(x) = \frac{H(x)}{sp(x) - D - q_1 E_2} \) then:

\[
\frac{d}{dx} \left( \frac{x^* g(x^*) - q_1 E_1}{P(x^*)} \right) = \frac{d}{dx} \left( \frac{x^* [g(x^*) - q_1 E_1]}{P(x^*)} \right) = \frac{D + q_1 E_2}{sp(x) - D - q_1 E_2}
\]

Hence since \( H(x) > 0, sp(x) > 0, D + q_1 E_2 > sp(x^*) \), then \( W'(x^*) < 0 \).

Therefore the condition of Theorem 4.1 (Freedman and So, 1985) is satisfied and this completes the uniqueness of limit cycles.

**Applications:** Now we give some examples from the ecological literature. The numerical simulations may justify the results.

**Example 5.1:** Consider the special case:

\[
g(x) = \left( \frac{1 - x}{k} \right) p(x) = \frac{mx}{\alpha + bx}
\]

Then we have the system:

\[
\begin{align*}
\frac{dx}{dt} &= rx \left( 1 - \frac{x}{k} \right) - y \frac{mx}{\alpha + bx} - q, E_1 x, \\
\frac{dy}{dt} &= - Dy + sy \frac{mx}{\alpha + bx} - q, E_1 y
\end{align*}
\]

Clearly all the assumptions hold. Thus the critical point \((x^*, y^*) = \left( \frac{\alpha D + q, E_1}{s m - D q, E_1 b}, \left( \frac{1 - x^*}{k} \right) q, E_1 \left( \frac{\alpha + bx^*}{x^*} \right) \right) \right) \) and the Jacobian is:

\[
A = \begin{bmatrix}
-2r x^* & -s \alpha m x^* & -s \alpha q, E_1 & & -s \alpha m x^* \\
\frac{k}{\alpha + bx^*} & s \alpha m x^* & -s \alpha q, E_1 & & -s \alpha m x^* \\
\frac{k}{\alpha + bx^*} & s \alpha m x^* & -s \alpha q, E_1 & & -s \alpha m x^* \\
\end{bmatrix}
\]

choosing \( r = 1, k = 0.5, m = 2, \alpha = 0.51, b = 1, q_1 = 0.001, E_1 = 1, D = 1, s = 10, q_2 = 0.02, E_2 = 0.5, x(0) = 0.01, y(0) = 0.2, T_1 = 200. \)

It is clear that \( \frac{r - 2r x^*}{k} \frac{\alpha m x^*}{(\alpha + bx^*)} - q, E_1 < 0 \). Thus the condition of Theorem 3.1. holds. Then there exists at least one limit cycle (Fig. 1).

**Example 5.2:** Consider the special case:

\[
g(x) = r(1 - \frac{x}{k}) p(x) = \frac{x^p}{\alpha + x^7}
\]

Then we have the Holling system:

\[
\begin{align*}
\frac{dx}{dt} &= rx \left( 1 - \frac{x}{k} \right) - y \frac{x^p}{\alpha + bx} - q, E_1 x, \\
\frac{dy}{dt} &= - Dy + sy \frac{x^p}{\alpha + bx} - q, E_1 y
\end{align*}
\]

Clearly the conditions \((H_1)-(H_3)\) and \((H_4)-(H_6)\) hold. Thus the critical point
Then there exists at least one limit cycle (Fig. 3).

Example 5.4: Consider the special case:

\[ g(x, k) = r\left(1 - \frac{x}{k}\right), \quad p(x) = 1 - e^{-x} \]

Then we have the system:

\[ \frac{dx}{dt} = rx \left(1 - \frac{x}{k}\right) - y(1 - e^{-x}) - q_1 x, \]
\[ \frac{dy}{dt} = -Dy + sy(1 - e^{-x}) - q_2 y \]

Clearly the conditions (H4)-(H6) hold. Thus the critical point:

\[ (x^*, y^*) = \left( -\frac{1}{\alpha} \ln \frac{s}{D + q_1 E_1}, \frac{rx \left(1 - \frac{x^*}{k}\right) - q_1 E_1 x^*}{1 - e^{-x^*}} \right) \]

Then there exists at least one limit cycle (Fig. 4).

Fig. 1: Existence of limit cycles for choosing \( r = 1, k = 0.5, \alpha = 0.51, q_1 = 0.001, E_1 = 1, D = 1, s = 10, q_2 = 0.02, E_2 = 0.5, T = 150, x(0) = 0.5, y(0) = 0.01, T_1 = 100. \)

It is clear that \( r - \frac{2rx^*}{k} - \frac{\alpha \alpha y^* x^{k-1}}{(\alpha + x^{k-1})^2} - q_1 E_1 x^* < 0 \). Thus the condition of Theorem 3.1 holds.

Then there exists at least one limit cycle (Fig. 4).
Fig. 2: Existence of limit cycles for choosing \( r = 0.1, k = 20, \alpha = 0.51, q_1 = 0.001, E_1 = 1, D = 1, s = 10, q_2 = 0.02, E_2 = 0.5, T = 150, x(0) = 0.5, y(0) = 0.001, p = 3\)

Fig. 3: Existence of limit cycles for \( r = 1, k = 0.5, m = 2, \alpha = 0.51, b = 1, q_1 = 0.001, E_1 = 1, D = 1, s = 10, q_2 = 0.02, E_2 = 0.5, T = 150, x(0) = 0.01, y(0) = 1, T_1 = 100, p = 2\)

Remark 5.2: we may note that our examples and graphs are different from those exist in (Kar and Matsuda, 2007; Moghadas and Corbett, 2008).

CONCLUSION

In this article we discuss the existence and stability of equilibrium points using Routh-Hurwitz approach. Theorem1 shows that all solutions of the model are positive and bounded. We give sufficient condition guarantees that the model has at least one limit cycle in the first quadrant of the xy plane. We give sufficient condition guarantees that the model has at least one limit cycle in the first quadrant of the xy plane. We give sufficient condition guarantees that the model has at least one limit cycle in the first quadrant of the xy plane if:

\[ x^*g(x^*) + g(x^*) - y^*p(x^*) - q_1E_i < 0 \]

Using Bendixon Dulac Theorem we discuss the case for which the predator-prey model (1) has no periodic solution. Using Kuang and Freedman technique we give sufficient condition for the uniqueness of limit cycles of (1). Numerical examples for some special cases of \( g(x) \) are given to justify the results with graphs showing the existence of limit cycles.

REFERENCES


