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On a Univalent Class Involving Differential Subordinations with Applications

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Abstract: Problem statement: By means of the Hadamard product (or convolution), new class of analytic functions was formed. This class was motivated by many authors. **Approach:** By using the concept of the subordination and superordination, we define certain differential inequalities and first order differential subordinations. **Results:** As their applications, we obtain some sufficient conditions for univalence which generalize and refine some previous results. Sandwich theorem is also obtained. **Conclusion:** Therefore, we posed a new class of analytic functions which generalized some well known subclasses. This class involves the $E(\Phi,\Psi)-$ family of functions.

Key words: Convex functions, close-to-convex functions, differential subordination, superordination, unit disk, sandwich theorem, hadamard product, dziok-srivastava linear operator, differential sandwich

INTRODUCTION

Let H be the class of functions analytic in the unit disk $U = \{z : |z| < 1\}$ and for $a \in C$ (set of complex numbers) and $n \in N$ (set of natural numbers), let H [a,n] be the subclass of H consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. Let A be the class of functions F, analytic in U and normalized by the conditions $f(0) = f'(0) - 1 = 0$. Given two functions

$$f, g \in A,\ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

their convolution or Hadamard product $f(z) * g(z)$ is defined by:

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n,\ z \in U.$$

Let F be analytic in U, g analytic and univalent in U and f (0)= g (0) Then, by the symbol $f(z) \prec g(z)$ (f subordinate to g) in U, we shall mean $f(U) \subset g(U)$.

Let $\varphi : C^2 \to C$ and let h be univalent in U. If p is analytic in U and satisfies the differential subordination $\varphi(p(z)), zp'(z)) \prec h(z)$ then P is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, if $p \prec q$. If P and $\varphi(p(z)), zp'(z))$ are univalent in U and satisfy the differential superordination $h(z) \prec \varphi(p(z)), zp'(z))$ then P is called a solution of the differential superordination. An analytic function q is called subordinant of the solution of the differential superordination if $q \prec p$.

Juneja defined the family $E(\Phi,\Psi)$, so that:

$$\Re\{\frac{f(z) * \Phi(z)}{f(z) * \Psi(z)}\} > 0,\ z \in U$$

where:

$$\Phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n$$

$$\Psi(z) = z + \sum_{n=2}^{\infty} \psi_n z^n$$

are analytic in U with the conditions

$$\phi_n \geq 0, \psi_n \geq 0, \phi_n \geq \psi_n \text{ for } n \geq 2$$

$$f(z) * \Psi(z) \neq 0.$$

This type of class was motivated by many authors namely (Lewandowski *et al.*,1976; Kumar *et al.*, 1995; Kwon , 2007; Ravichandran *et al.*, 2002; Obradovi´c and Joshi, 1998; Joshi *et al.*, 1998; Singh and Gupta, 1996; Xu and Yang, 2005). Note that this family was then extended and studied in the work due to Ibrahim and Darus.

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MATERIALS AND METHODS

In the present study, we consider a new class $H(\alpha,\lambda,\delta,\Phi(z);\Psi(z))$ as follows Eq. 1:

$$\frac{z(f(z)*\Phi(z))'}{f(z)*\Psi(z)}\{(1-\alpha)\frac{z(f(z)*\Phi(z))'}{f(z)*\Psi(z)}$$

$$+\alpha(1+\frac{\lambda z(f(z)*\Phi(z))''}{(f(z)*\Phi(z))'}-\frac{\delta z(f(z)*\Psi(z))'}{(f(z)*\Psi(z))})\}\prec F(z),) \qquad (1)$$

where $\alpha\in[0,1],\lambda,\delta\in R$ and F is the conformal mapping of the unit disk U with F (0)= 1.

Remark 1: As special cases of the class $H(\alpha,\lambda,\delta,\Phi(z);\Psi(z))$ are the following well known classes: $H(1,\lambda,0,\frac{z}{1-z};\frac{z}{1-z})$ (Lewandowski *et al*., 1976) (Xu and Yang, 2005). Also this class reduces to the classes of starlike functions, convex functions and close-to-convex functions for various Φ and ψ.

In order to obtain our results, we need the following lemmas.

Lemma 1: Miler and Mocanu (2000). Let q (z) be univalent in the unit disk U and θ and φ be analytic in a domain D containing q (U) with $\varphi(w)\neq 0$ when $w\in q(U)$. Set:

$$Q(z):=zq'(z)\varphi(q(z)),h(z):=\theta(q(z))+Q(z).$$

Suppose that:
Q (z) is starlike univalent in U

$$\Re\{\frac{zh'(z)}{Q(z)}\}>0 \text{ for } z\in U.$$

If:

$$\theta(p(z))+zp'(z)\varphi(p(z))\prec\theta(q(z))+zq'(z)\varphi(q(z))$$

then $p(z)\prec q(z)$ and q(z) is the best dominant.

Definition 1: (Miller and Mocanu, 2003) Denote by Q the set of all functions f(z) that are analytic and injective on $\overline{U}-E(f)$ where $E(f):=\{\zeta\in\partial U:\lim_{z\to\zeta}f(z)=\infty\}$ and are such that $f'(\zeta)\neq 0$ for $\zeta\in\partial U-E(f)$.

Lemma 2: Bulboaca (2002). Let q(z) be convex univalent in the unit disk U and ϑ and φ be analytic in a domain D containing q(U) Suppose that :

$zq'(z)\phi(q(z))$ is starlike univalent in U

$$\Re\{\frac{\vartheta'(q(z))}{\phi(q(z))}\}>0 \text{ for } z\in U.$$

If $p(z)\in H[q(0),1]\cap Q$, with $p(U)\subseteq D$ and $\vartheta(p(z))+zp'(z)\phi(z)$ is univalent inU and

$$\vartheta(q(z))+zq'(z)\phi(q(z))$$

$$\prec\vartheta(p(z))+zp'(z)\phi(p(z))$$

then $q(z)\prec p(z)$ and q(z) is the best subordinant.

RESULTS AND DISCUSSION

In this section, we prove a subordination theorem by using Lemma 1 and as applications of this result, we find the sufficient conditions for f∈A to be univalent.

Theorem 1: Let q,q(z)≠ 0 be a univalent function in U and g(z)≠ 0 be analytic in C such that for nonnegative real numbers μ and ν Eq. 2:

$$\Re\{1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\}$$

$$>\max\{0,(\frac{\mu}{\nu})\Re(q(z)[1+\frac{g'(z)}{g(z)}(\frac{q(z)}{q'(z)}+\frac{\nu z}{\mu q(z)})])\} \qquad (2)$$

If $p(z)\neq 0, z\in U$ satisfies the differential subordination:

$$g(z)[\mu p(z)+\nu\frac{zp'(z)}{p(z)}] \qquad (3)$$

then $p\prec q$ and q is the best dominant.

Proof: Define the functions θ and ϕ as follows:

$$\theta(w(z)):=\mu w(z)g(z) \quad \text{and} \quad \varphi(w(z)):=\frac{\nu g(z)}{w(z)}.$$

Obviously, the functions θ and ϕ are analytic in domain $D=C\setminus\{0\}$ and $\varphi(w)\neq 0$ in D Now, define the functions Q and h as follows:

$$Q(z):=zq'(z)\varphi(q(z))=\nu g(z)\frac{zq'(z)}{q(z)},$$

$$h(z):=\theta(q(z))+Q(z)=\mu q(z)g(z)+\nu g(z)\frac{zq'(z)}{q(z)}.$$

Then in view of condition (2), we obtain Q is starlike in U and $\Re\{\frac{zh'(z)}{Q(z)}\} > 0$ for $z\in U$ Furthermore, in view of condition (3) we have;

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)).$$

Therefore, the proof follows from Lemma 1. By letting $\mu = 1, \nu = \alpha, g(z) := \frac{zf'(z)}{\Phi(z)}$ and $p = \frac{zf'(z)}{f(z)}$ in Theorem 1 we have

Corollary 1: Let q,q(z)≠ 0 be a univalent function in U and $\frac{zf'(z)}{\Phi(z)} \neq 0$ be analytic in U satisfy (2). If Eq. 4:

$$\frac{zf'(z)}{f(z)} \neq 0, z \in U \text{ and}$$

$$\frac{zf'(z)}{\Phi(z)}[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha(1+\frac{zf''(z)}{f'(z)})]$$

$$\prec \frac{zf'(z)}{\Phi(z)}[q(z) + \alpha\frac{zq'(z)}{q(z)}], \tag{4}$$

then $\frac{zf'(z)}{f(z)} \prec q$ and q is the best dominant.

By setting $\mu = 1-\alpha, \nu = \alpha, g(z) := 1$ and $p = \frac{zf'(z)}{f(z)}$ in Theorem 2 we obtain the following result which can be found in (Singh *et al*., 2009), Theorem 3.2]:

Corollary 2: Let q,q(z) ≠ 0 be a univalent function in U. If $\frac{zf'(z)}{f(z)} \neq 0, z \in U$ and Eq. 5:

$$(1-2\alpha)\frac{zf'(z)}{f(z)} + \alpha(1+\frac{zf''(z)}{f'(z)})$$

$$\prec (1-\alpha)q(z) + \alpha\frac{zq'(z)}{q(z)}, \tag{5}$$

then $\frac{zf'(z)}{f(z)} \prec q$ and q is the best dominant.

By assuming $\mu = 1-\alpha, \nu = \alpha, g(z) := 1$ and $p(z) = \frac{zf'(z)}{\Phi(f(z))}$ in Theorem 2 we obtain the following result which can be found in (Singh *et al*., 2009) Theorem 3.3].

Corollary 3: Let q,q(z) ≠ 0 be a univalent function in U. If $\frac{zf'(z)}{\Phi(f(z))} \neq 0, z \in U$ and

$$(1-\alpha)\frac{zf'(z)}{\Phi(f(z))} + \alpha(1+\frac{zf''(z)}{f'(z)} - \frac{z\Phi(f(z))}{\Phi'(f(z))})$$

$$\prec (1-\alpha)q(z) + \alpha\frac{zq'(z)}{q(z)}, \tag{6}$$

then $\frac{zf'(z)}{\Phi(f(z))} \prec q$ and q is the best dominant Eq. 6.

Finally, by assuming:

$$\mu = 1-\alpha, \nu = \alpha, g(z) = p(z) = \frac{z(f(z)*\Phi(z))'}{f(z)*\Psi(z)}$$

in Theorem 1 we obtain the following result:

Corollary 4: Let q,q(z) ≠ 0 be a univalent function in U and $\frac{z(f(z)*\Phi(z))'}{f(z)*\Psi(z)} \neq 0$ be analytic in U satisfy (2). If the subordination Eq. 7:

$$\frac{z(f(z)*\Phi(z))'}{f(z)*\Psi(z)}\{(1-\alpha)\frac{z(f(z)*\Phi(z))'}{f(z)*\Psi(z)}$$

$$+\alpha(1+\frac{z(f(z)*\Phi(z))''}{(f(z)*\Phi(z))'} - \frac{z(f(z)*\Psi(z))'}{(f(z)*\Psi(z))})\}$$

$$\prec g(z)[(1-\alpha)q(z) + \alpha\frac{zq'(z)}{q(z)}], \tag{7}$$

holds then $\frac{z(f(z)*\Phi(z))'}{f(z)*\Psi(z)} \prec q$ and q is the best dominant.

Note that Corollary 4, gives sufficient conditions for functions f∈A to be in the class $H(\alpha,1,1,\Phi(z);\Psi(z))$.

An application of Theorem 1, next result shows the sufficient conditions for functions f∈A to be in the class $H(\alpha,\lambda,\delta,\Phi(z);\Psi(z))$. By assuming $\mu := \alpha$ and $\nu := 1-\alpha$, we have the following result:

Theorem 2: Let f∈A and q,q(z) ≠ 0 be a univalent function in U. Assume that $p(z) := \frac{z(f(z)*\Phi(z))'}{f(z)*\Psi(z)} \neq 0$ is analytic in U satisfies (2-3) for some g If $\lambda,\delta \in R$ then $f \in H(\alpha,\lambda,\delta,\Phi(z);\Psi(z))$.

Sandwich Theorem: By employing the concept of the superordination (Lemma 2), we pose the sandwich theorem containing functions $f \in A$.

Theorem 3: Let q(z) be convex univalent in the unit disk U. Suppose that g ia an analytic in the unit disk such that

$$\nu g(z)\frac{zq'(z)}{q(z)}$$ is starlike univalent in U and

$$\frac{\mu}{\nu}\Re\{q(z)q'(z)\} > 0$$ for $Z \in U$.

If $p(z) \in H[q(0),1] \cap Q$, with $p(U) \subseteq D$ and $g(z)[\mu p(z) + \nu \frac{zp'(z)}{p(z)}]$ is univalent in U and

$$g(z)[\mu q(z) + \nu \frac{zq'(z)}{q(z)}] \prec g(z)[\mu p(z) + \nu \frac{zp'(z)}{p(z)}]$$

then $q(z) \prec p(z)$ and q(z) is the best subordinant.

Proof: Define the functions θ and ϕ as follows:

$$\vartheta(w(z)) := \mu w(z) g(z) \quad \text{and} \quad \phi(w(z)) := \frac{\nu g(z)}{w(z)}.$$

Obviously, the functions ϑ and φ are analytic in domain $D = C \setminus \{0\}$ and $\phi(w) \neq 0$ in D hence the assumptions of Lemma 2 are satisfied.

Combining Theorem 1 and Theorem 3 we get the following sandwich theorem:

Theorem 4: Let $q_1(z), q_2 \neq 0$ be convex and univalent in U respectively. Suppose that g ia an analytic in U such that Eq. 8:

$$\nu g(z)\frac{zq_1'(z)}{q_1(z)}$$ is starlike univalent in U and

$$\frac{\mu}{\nu}\Re\{q_1(z)q_1'(z)\} > 0$$ for $z \in U$ and

$$\Re\{1 + \frac{zq_2''(z)}{q_2'(z)} - \frac{zq_2'(z)}{q_2(z)}\}$$
$$> \max\{0, (\frac{\mu}{\nu})\Re(q_2(z)[1 + \frac{g'(z)}{g(z)}(\frac{q_2(z)}{q_2'(z)} + \frac{\nu z}{\mu q_2(z)})])\}. \qquad (8)$$

If $p(z) \neq 0 \in H[q(0),1] \cap Q$,

With:

$$p(U) \subseteq D$$

And:

$g(z)[\mu p(z) + \nu \frac{zp'(z)}{p(z)}]$ is univalent in U and

$$g(z)[\mu q_1(z) + \nu \frac{zq_1'(z)}{q_1(z)}] \prec g(z)[\mu p(z) + \nu \frac{zp'(z)}{p(z)}]$$
$$\prec g(z)[\mu q_2(z) + \nu \frac{zq_2'(z)}{q_2(z)}]$$

Then:

$$q_1(z) \prec p(z) \prec q_2(z), \quad (z \in U)$$

and $q_1(z), q_2(z)$ are the best subordinant and the best dominant respectively.

By letting $p(z) := \frac{zf'(z)}{f(z)}$ in Theorem 4, we have

Corollary 5: Let the conditions of Theorem 4 on the functions q_1 and q_2 hold. If for $f \in A$

$$\frac{zf'(z)}{f(z)} \neq 0 \in H[q(0),1] \cap Q,$$

With:

$$(\frac{zf'}{f})(U) \subseteq D$$

And:

$$g(z)[(\mu - \nu)\frac{zf'(z)}{f(z)} + \nu(1 + \frac{zf''(z)}{f'(z)})]$$
Is univalent in U and:

$$g(z)[\mu q_1(z) + \nu \frac{zq_1'(z)}{q_1(z)}]$$
$$\prec g(z)[(\mu - \nu)\frac{zf'(z)}{f(z)} + \nu(1 + \frac{zf''(z)}{f'(z)})]$$
$$\prec g(z)[\mu q_2(z) + \nu \frac{zq_2'(z)}{q_2(z)}]$$

Then Eq. 9:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z), \quad (z \in U) \qquad (9)$$

And $q_1(z), q_2(z)$ are the best subordinant and the best dominant respectively.

Note that (Ali *et al*., 2004) have used the results of (Bulboac, 2002) and obtained sufficient conditions for certain normalized analytic functions f(z) to satisfy (9).

By assuming $p(z) := \frac{f(z)}{zf'(z)}$ in Theorem 4, we obtain

Corollary 6: Let the conditions of Theorem 4 on the functions q_1 and q_2 hold. If for $f \in A$

$$\frac{f(z)}{zf'(z)} \neq 0 \in H q(0),1] \cap Q,$$

With:

$$(\frac{f}{zf'})(U) \subseteq D$$

And:

$$g(z)[\mu \frac{f(z)}{zf'(z)} + \nu(\frac{zf'(z)}{f(z)} - 1 - \frac{zf''(z)}{f'(z)})]$$

is univalent in U and

$$g(z)[\mu q_1(z) + \nu \frac{zq_1'(z)}{q_1(z)}]$$

$$\prec g(z)[\mu \frac{f(z)}{zf'(z)} + \nu(\frac{zf'(z)}{f(z)} - 1 - \frac{zf''(z)}{f'(z)})]$$

$$\prec g(z)[\mu q_2(z) + \nu \frac{zq_2'(z)}{q_2(z)}]$$

Then Eq. 10:

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z), \quad (z \in U) \tag{10}$$

and $q_1(z), q_2(z)$ are the best subordinant and the best dominant respectively.

Note that (Shanmugam *et al*., 2006) posed sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy (10).

Again by considering $p(z) := \frac{z^2 f'(z)}{f^2(z)}$ in Theorem 4, we find

Corollary 7: Let the conditions of Theorem 4 on the functions q_1 and q_2 hold. If for $f \in A$.

$$\frac{z^2 f'(z)}{f^2(z)} \neq 0 \in H[q(0),1] \cap Q,$$

With:

$$\frac{z^2 f'}{f^2}(U) \subseteq D$$

And:

$$g(z)[\mu \frac{z^2 f'(z)}{f^2(z)} + \nu(\frac{zf''(z)}{f'(z)} + 2 - 2\frac{zf'(z)}{f(z)})]$$

is univalent in U and

$$g(z)[\mu q_1(z) + \nu \frac{zq_1'(z)}{q_1(z)}]$$

$$\prec g(z))[\mu \frac{z^2 f'(z)}{f^2(z)} + \nu(\frac{zf''(z)}{f'(z)} + 2 - 2\frac{zf'(z)}{f(z)})]$$

$$\prec g(z)[\mu q_2(z) + \nu \frac{zq_2'(z)}{q_2(z)}]$$

Then Eq. 11:

$$q_1(z) \prec \frac{z^2 f'(z)}{f^2(z)} \prec q_2(z), \quad (z \in U) \tag{11}$$

And $q_1(z), q_2(z)$ are the best subordinant and the best dominant respectively.

Note that (Shanmugam *et al*., 2006) estimated sufficient conditions for certain normalized analytic functions f (z) to satisfy (11).

Furthermore, by letting $p(z) := \frac{z(f*g)'(z)}{\Phi(f*g)(z)}$ in Theorem 4, we pose

Corollary 8: Let the conditions of Theorem 4 on the functions q_1 and q_2 hold. If for $f \in A$

$$\frac{z(f*g)'(z)}{\Phi(f*g)(z)} \neq 0 \in H[q(0),1] \cap Q,$$

With:

$$(\frac{z(f*g)'}{\Phi(f*g)})(U) \subseteq D$$

And:

$$g(z)[\mu\frac{z(f*g)'(z)}{\Phi(f*g)(z)}-\nu(\frac{zf''(z)}{f'(z)}+1-\frac{z\Phi'(f*g)(z)}{\Phi(f*g)(z)})]$$

is univalent in U and:

$$g(z)[\mu q_1(z)+\nu\frac{zq_1'(z)}{q_1(z)}]$$

$$\prec g(z)[\mu\frac{z(f*g)'(z)}{\Phi(f*g)(z)}-\nu(\frac{zf''(z)}{f'(z)}+1-\frac{z\Phi'(f*g)(z)}{\Phi(f*g)(z)})]$$

$$\prec g(z)[\mu q_2(z)+\nu\frac{zq_2'(z)}{q_2(z)}]$$

Then Eq. 12:

$$q_1(z)\prec\frac{z(f*g)'(z)}{\Phi(f*g)(z)}\prec q_2(z),\quad (z\in U) \tag{12}$$

and $q_1(z), q_2(z)$ are the best subordinant and the best dominant respectively.

Note that (Shanmugam *et al*., 2007) posed sufficient conditions for certain normalized analytic functions f(z) to satisfy (12).

Finally, by setting $p(z):=(\frac{H_m^l[\alpha_1]f(z)}{z})^\delta$, where $f\in A$ and $H_m^l[\alpha_1]$ is the Dziok-Srivastava linear operator (Dziok and Srivastava, 2003), in Theorem 4, we have.

Corollary 9: Let the conditions of Theorem 4 on the functions q_1 and q_2 hold. If for $f\in A$,

$$(\frac{H_m^l[\alpha_1]f(z)}{z})^\delta\neq 0\in H[q(0),1]\cap Q,$$

With:

$$((\frac{H_m^l[\alpha_1]f}{z})^\delta)(U)\subseteq D$$

And:

$$g(z)[\mu(\frac{H_m^l[\alpha_1]f(z)}{z})^\delta-\nu\delta z(\frac{z}{H_m^l[\alpha_1]f(z)}-1)]$$

is univalent in U and:

$$g(z)[\mu q_1(z)+\nu\frac{zq_1'(z)}{q_1(z)}]$$

$$\prec g(z)[\mu(\frac{H_m^l[\alpha_1]f(z)}{z})^\delta-\nu\delta z(\frac{z}{H_m^l[\alpha_1]f(z)}-1)]$$

$$\prec g(z)[\mu q_2(z)+\nu\frac{zq_2'(z)}{q_2(z)}]$$

Then Eq. 13.

$$q_1(z)\prec(\frac{H_m^l[\alpha_1]f(z)}{z})^\delta\prec q_2(z),\quad (z\in U) \tag{13}$$

and $q_1(z), q_2(z)$ are the best subordinant and the best dominant respectively.

Note that (Murugusundaramoorthy and Magesh, 2006) introduced sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy (13).

Corollary 10: Let the assumptions of Theorem 4 on the function:

$$p(z):=\frac{z(f(z)*\Phi(z))'}{f(z)*\Psi(z)}\{(1-\alpha)\frac{z(f(z)*\Phi(z))'}{f(z)*\Psi(z)}$$
$$+\alpha(1+\frac{\lambda z(f(z)*\Phi(z))''}{(f(z)*\Phi(z))'}-\frac{\delta z(f(z)*\Psi(z))'}{(f(z)*\Psi(z))})\}$$

holds. Then:

$$q_1(z)\prec\frac{z(f(z)*\Phi(z))'}{f(z)*\Psi(z)}\{(1-\alpha)\frac{z(f(z)*\Phi(z))'}{f(z)*\Psi(z)}$$

$$+\alpha(1+\frac{\lambda z(f(z)*\Phi(z))''}{(f(z)*\Phi(z))'}-\frac{\delta z(f(z)*\Psi(z))'}{(f(z)*\Psi(z))})\}\prec q_2(z),$$

and $q_1(z), q_2(z)$ are the best subordinant and the best dominant respectively.

CONCLUSION

We conclude that a new class of analytic functions has been introduced and this class generalizes some well-known subclasses. This class also involves the $E(\Phi,\Psi)-$ family of functions and consequently, we can replace these functions by any well- known linear operators, differential or integral operators such as

Carlson - Shaffer linear operator, Ruscheweyh differential operator, Sălăgean differential operator, Noor integral operator, Breaz and Guney integral operator and other generalizations. Some other work can also be found in the articles written by (Shaqsi and Darus, 2008) (Ibrahim and Darus, 2008; 2010; Al-Refai and Darus, 2009a; Al-Refai and Darus, 2009b; Al-Shaqsi *et al*., 2010; Darus *et al*., 2009).

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