Journal of Mathematics and Statistics 5 (1): 63-64, 2009
ISSN 1549-3644

The Banach space $m_p(X)$, for 1 £ p < 8 has the Banach-Saks Property

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Abstract: Problem statement: In the theory of Banach spaces one of the problems which describes geometric property of Banach spaces is Banach-Saks Property. In this context we were known many Banach spaces which had this property such as $L_p[0, 1]$ for $1 < p \le 2$. **Approach**: Following the sequential structure of the Banach sequence space $m_p(X)$, for $1 \le p < 8$, defined in[1], we arrived to describe a geometric property of this Banach spaces. **Results:** In this note we showed that Banach spaces $m_p(X)$, for $1 \le p < 8$ had the Banach-Saks Property. **Conclusion/Recommendations:** Based in present approach, we recommend using our method to study the weak Banach-Saks property in sequential Banach spaces.

Key words: Banach-saks property, scalar sequences

INTRODUCTION

The Banach-Saks property was studied in Banach spaces and several characterizations were given for it. In[5], was studied Banach-Saks property in the product of Banach spaces. Another characterizations was studied taking into consideration the Haar null sets property in sense of Christensen[4]. In this note we prove that the Banach space $m_p(X)$, for $1 \le p < 8$ has the Banach-Saks property. The sequence space $m_p(X)$ was defined by[1]. In this section we briefly describe the notation and definitions which are used throughout the paper. Let X be a Banach space with norm $\|.\|$. Let Λ denote the vector space of scalar sequences (a_i), where (a_i) are from $\mathbb{R}$, i.e.:

$$\Lambda = \{a = (a_i) : a_i \in \mathbb{R}\}$$

(Alternatively, we may also take Λ to be the vector space of complex scalar sequences and what follows remains true in both cases, real and complex). The space $m_p(X)$ is defined as:

$$m_p(X) = \left\{a = (a_i) \in \Lambda : \sum_i \|a_i x_i\|^p < \infty, \forall (x_i) \in l_w^p(X)\right\} \qquad (1)$$

and is a Banach space under the norm:

$$\|(a_i)\|_{p,p} = \sup_{\varepsilon_p((x_i)) \le 1} \left(\sum_{n \in \mathbb{N}} |a_n|^p \|x_n\|^p\right)^{\frac{1}{p}} \qquad (2)$$

where $\varepsilon_p((x_i)) = \sup_{\|a\| \le 1} \|a(x_i)\|_p$, $a \in X^*$ (see [1]). Here $l_w^p(X)$ stands for the Banach space:

$$l_w^p(X) = \left\{x = (x_i) \in X : \left(\sum_i |x^*(x_i)|^p\right)^{1/p} < \infty, x^* \in X^*\right\}$$

For the class of the scalar sequences $m_p(X)$, the following inclusion holds:

$$l_p \subseteq m_p(X) \subset l_\infty \qquad (3)$$

for any $1 \le p < 8$.

Definition 1[2]: A Banach space X has the Banach-Saks property whenever every bounded sequence in X has a subsequence, whose arithmetic mean converges in norm. All other notations are like as in[3].

MATERIALS AND METHODS

Theorem 1: The Banach space $m_p(X)$, for $1 \le p < 8$ has the Banach-Saks property.

Proof: From the Definition 1 it is enough to prove that whenever bounded sequence in $m_p(X)$ has a subsequence whose arithmetic mean converges in norm, then that space has the Banach-Saks property. Let (b_n) be any bounded sequence in $m_p(X)$. It mean that there exists a positive constant $K \in \mathbb{R}$, such that the following estimation:

$$\left\|(b_n)\right\|_{p,p} \le K \quad (4)$$

holds, for every $n \in \mathbb{N}$. On the other side from relation (3) follows that $(b_n) \in l_\infty$, so there exists a constant K_1 such that:

$$|b_n| \le K_1 \quad (5)$$

for all $n \in \mathbb{N}$. It is well-known that there exists a subsequence (b_{n_k}) of sequence (b_n), such that $\lim_{k\to\infty} b_{n_k} = K_2$. Taking into consideration relation (5) we get the following:

$$\left|\frac{b_1 + b_2 + ... + b_{n_k}}{k}\right| \le K_1 \quad (6)$$

From relation (4), it follows that the following estimation:

$$\left(\sum_{n\in\mathbb{N}} \|x_n\|^p\right)^{1/p} < K_3 \quad (7)$$

holds, for some constant K_3. Now from relations (6) and (7) we get the following:

$$\left\|\frac{b_1 + b_2 + ... + b_{n_k}}{k}\right\|_{p,p} \le K_4 \quad (8)$$

for some constant K_4. Let us denote by $(y_{n_k}) = \left\|\frac{b_1 + b_2 + ... + b_{n_k}}{k}\right\|_{p,p}$. Then from (8) it follows that there exists a subsequence (y_{n_s}) of (y_{n_k}), such that:

$$\lim_{s\to\infty} y_{n_s} = K_5 \quad (9)$$

Now the scalar sequence (b_{n_s}) is the required one which satisfies the condition:

$$\left\|\frac{b_1 + b_2 + ... + b_{n_s}}{s}\right\|_{p,p} \to K_5,\ s \to \infty$$

RESULTS AND DISCUSSION

Here we discus our results obtained in the previous section. Theorem 1, shows that Banach sequential space $m_p(X)$, for $1 \le p < 8$, has a geometric property: The Banach-Saks Property. A helpful fact which is used to prove the Theorem 1 is relation: $l_p \subseteq m_p(X) \subset l_\infty$.

CONCLUSION

In this note we give an approach which we recommend in order to study the weak Banach-Saks property in sequential Banach spaces.

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