The Banach space $m_p(X)$, for $1 \leq p < 8$ has the Banach-Saks Property

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Abstract: Problem statement: In the theory of Banach spaces one of the problems which describes geometric property of Banach spaces is Banach-Saks Property. In this context we were known many Banach spaces which had this property such as $L^p[0, 1]$ for $1 < p \leq 2$. Approach: Following the sequential structure of the Banach sequence space $m_p(X)$, for $1 \leq p < 8$, defined in [1], we arrived to describe a geometric property of this Banach spaces. Results: In this note we showed that Banach spaces $m_p(X)$, for $1 \leq p < 8$ had the Banach-Saks Property. Conclusion/Recommendations: Based in present approach, we recommend using our method to study the weak Banach-Saks property in sequential Banach spaces.

Key words: Banach-saks property, scalar sequences

INTRODUCTION

The Banach-Saks property was studied in Banach spaces and several characterizations were given for it. In [5], was studied Banach-Saks property in the product of Banach spaces. Another characterizations was studied taking into consideration the Haar null sets property in sense of Christensen [4]. In this note we prove that the Banach space $m_p(X)$, for $1 \leq p < 8$ has the Banach-Saks property. The sequence space $m_p(X)$ was defined by [1]. In this section we briefly describe the notation and definitions which are used throughout the paper. Let $X$ be a Banach space with norm $\| \cdot \|$. Let $\Lambda$ denote the vector space of scalar sequences $(a_i)$, where $(a_i)$ are from $\mathbb{R}$, i.e.:

$$\Lambda = \{a = (a_i) : a_i \in \mathbb{R}\}$$

(Alternatively, we may also take $\Lambda$ to be the vector space of complex scalar sequences and what follows remains true in both cases, real and complex). The space $m_p(X)$ is defined as:

$$m_p(X) = \left\{ a = (a_i) \in \Lambda : \sum_{i=1}^{\infty} \|a_i x_i\| < \infty, \forall \chi \in \ell_p^\ast(X) \right\}$$

(1)

and is a Banach space under the norm:

$$\|a_i\|_p = \sup_{\chi \in \ell_p(X)} \left( \sum_{i=1}^{\infty} \|a_i \chi_i\| \right)^{\frac{1}{p}}$$

(2)

where $\ell_p\left((x_i)\right) = \sup\|a(x_i)\|_{\|\cdot\|}$, $a \in X'$ (see [1]). Here $\ell_p^\ast(X)$ stands for the Banach space:

$$\ell_p^\ast(X) = \left\{ x = (x_i) \in X : \left( \sum_{i=1}^{\infty} \|x^\ast(x_i)\| \right)^{\frac{1}{p'}} < \infty, x^\ast \in X^\ast \right\}$$

For the class of the scalar sequences $m_p(X)$, the following inclusion holds:

$$l_p \subseteq m_p(X) \subseteq L$$

(3)

for any $1 \leq p < 8$.

Definition [2]: A Banach space $X$ has the Banach-Saks property whenever every bounded sequence in $X$ has a subsequence, whose arithmetic mean converges in norm. All other notations are like as in [3].

MATERIALS AND METHODS

Theorem 1: The Banach space $m_p(X)$, for $1 \leq p < 8$ has the Banach-Saks property.

Proof: From the Definition 1 it is enough to prove that whenever bounded sequence in $m_p(X)$ has a subsequence whose arithmetic mean converges in norm, then that space has the Banach-Saks property. Let $(b_n)$ be any bounded sequence in $m_p(X)$. It mean
that there exists a positive constant \( K \in \mathbb{R} \), such that the following estimation:
\[
\left\| (b_n)_{n=p} \right\|_{p,p} \leq K
\]  
(4)
holds, for every \( n \in \mathbb{N} \). On the other side from relation (3) follows that \( (b_n) \in l_\infty \), so there exists a constant \( K_1 \) such that:
\[
|b_n| \leq K_1
\]  
(5)
for all \( n \in \mathbb{N} \). It is well-known that there exists a subsequence \( (b_{n_k}) \) of sequence \( (b_n) \), such that \( \lim_{k \to \infty} b_{n_k} = K_2 \). Taking into consideration relation (5) we get the following:
\[
\frac{|b_1 + b_2 + \ldots + b_{n_k}|}{k} \leq K_1
\]  
(6)
From relation (4), it follows that the following estimation:
\[
\left( \sum_{n=1}^\infty \left\| b_n \right\|_p \right)^{1/p} \leq K_3
\]  
(7)
holds, for some constant \( K_3 \). Now from relations (6) and (7) we get the following:
\[
\left\| \frac{b_1 + b_2 + \ldots + b_{n_k}}{k} \right\|_{p,p} \leq K_4
\]  
(8)
for some constant \( K_4 \). Let us denote by \( \left( y_{n_k} \right) = \left\| \frac{b_1 + b_2 + \ldots + b_{n_k}}{k} \right\|_{p,p} \). Then from (8) it follows that there exists a subsequence \( \left( y_{n_k} \right) \) of \( \left( y_n \right) \), such that:
\[
\lim_{s \to \infty} y_{n_k} = K_5
\]  
(9)
Now the scalar sequence \( \left( b_{n_k} \right) \) is the required one which satisfies the condition:
\[
\left\| \frac{b_1 + b_2 + \ldots + b_{n_k}}{s} \right\|_{p,p} \to K_4, \ s \to \infty
\]

**RESULTS AND DISCUSSION**

Here we discuss our results obtained in the previous section. Theorem 1, shows that Banach sequential space \( m_p(X) \), for \( 1 \leq p < 8 \), has a geometric property: The Banach-Saks Property. A helpful fact which is used to prove the Theorem 1 is relation: \( l_p \subseteq m_p(X) \subseteq l_\infty \).

**CONCLUSION**

In this note we give an approach which we recommend in order to study the weak Banach-Saks property in sequential Banach spaces.

**REFERENCES**