Homomorphisms on Lattices of Measures

1Norris Sookoo and 2Peter Chami
1University of Trinidad and Tobago, O’meara, Arima, Trinidad, West Indies
2Department of Computer Science, Mathematics and Physics, University of the West Indies, Cave Hill, Barbados, West Indies

Abstract: Problem statement: Homomorphisms on lattices of measures defined on the quotient spaces of the integers were considered. These measures were defined in terms of Sharma-Kaushik partitions. The homomorphisms were studied in terms of their relationship with the underlying Sharma-Kaushik partitions. Approach: We defined certain mappings between lattices of Sharma-Kaushik partitions and showed that they are homomorphisms. These homomorphisms were mirrored in homomorphisms between related lattices of measures. Results: We obtained the structure of certain homomorphisms of measures. Conclusion: Further information about homomorphisms between lattices of measures of the type considered here can be obtained by investigating the underlying lattices of Sharma-Kaushik partitions.

Key words: Measure, lattice, ideal, partition

INTRODUCTION

Systems of measures having different structures and properties have long been the subject of investigation. Maharam[5] studied a family of measures with orthogonality properties and Schmidt[8] proved that a particular ordered Banach space of vector measures is a Banach lattice. Systems of measures satisfying compatibility conditions were studied by Niederreiter and Sookoo[6,7], who obtained conditions under which a partial density can be extended to a density. Sookoo and Chami[9] investigated the lattice structure of certain sets of lattices of measures defined on the quotient spaces of the integers.

We consider mappings that preserve certain elements of the structure of lattices of such measures, namely homomorphisms. We investigate some of the forms that homomorphisms may take.

The measures that we consider are defined in terms of SK-partitions of the ring of integer’s module q. The studies of these partitions have been conducted by Kaushik[2-4].

We consider homomorphisms given in terms of a function defined on the class sizes of the underlying partitions. Later, we consider homomorphisms that change the number of classes or alter class sizes in a pre-determined manner.

Definitions and notations:

Notation: Let $F_q = \{0, 1,..., q-1\}$ be the ring of integers modulo $q$, $q \in \{2,3,...\}$.

Definition: Given $F_q$, $q \geq 2$, a partition $P = \{B_0, B_1,\ldots, B_{m-1}\}$ of $F_q$ is called an SK-partition if:

- $B_0 = \{0\}$ and $q-a \in B_i$ if $a \in B_i$, $i = 1, 2,..., m-1$
- If $a \in B_i$ and $b \in B_j$; $i, j = 0,1,\ldots,m-1$ and if $j$ precedes $i$ in the order of the partition $P$, (written as $i > j$), then $\min\{a,q-a\} > \min\{b,q-b\}$.
- If $i > j$, $(i, j \in \{0,1,...,m-1\})$ and $i \neq m - 1$, then:

$$|B_i| \geq |B_j| \text{ and } |B_{m-1}| \geq \frac{1}{2}|B_{m-2}|$$

where, $|B_i|$ stands for the size of the set $B_i$

Notation: A partition $B = \{B_0, B_1,\ldots, B_{m-1}\}$ is denoted by $B = (B_0, B_1,\ldots, B_{m-1})$ where $b_i = |B_i|, i = 1, 2,..., m - 1$.

Notation: $\mathcal{P}$ is the set of all SK-partitions.

The concept of an ideal is well known in lattice theory, Birkhoff[1].

Corresponding Author: Peter Chami, Department of Computer Science, Mathematics and Physics, University of the West Indies, Cave Hill, Barbados, West Indies

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Definition: Let \((L, \preceq)\) be a lattice. A subset \(A\) of \(L\) is called an ideal, if:
- \(a, b \in A \Rightarrow a \lor b \in A\)
- \(a \in A\) and \(c \in L \Rightarrow c \preceq a \Rightarrow c \in A\)

Notation: \(\mathbb{Z}/q\mathbb{Z}\) is the quotient group of integers modulo \(q\) with the discrete topology.

Definition: Given a partition \(P\) of \(F_q\), we define a measure \(\mu_P\) on \(\mathbb{Z}/q\mathbb{Z}\) as follows:
\[
\mu_P(i + q\mathbb{Z}) = j, \text{ if } i \in B_j, i = 0, 1, \ldots, q - 1
\]

Note: In this study, the SK-partitions that we consider must satisfy the condition that for each partition the class sizes never decrease as the subscript of the classes increases.

Definition: We define the class-size ordering \(\leq_s\) on \(\mathbb{Z}/q\mathbb{Z}\) as follows. For elements \(P\) and \(Q\) of \(\mathbb{Z}/q\mathbb{Z}\):
\[
P = \{B_0, B_1, \ldots, B_{m-1}\}
\]
And:
\[
Q = \{C_0, C_1, \ldots, C_{m-1}\}; m, m' \in \{2, 3, 4, \ldots\}
\]
Where:
- \(P\) is an SK-partition of \(F_q\)
- \(Q\) is an SK-partition of \(F_{q'}\) with \(q, q' \in \{2, 3, \ldots\}\) and \(q, q' \in \{2, 3, \ldots\}\)

Note: Clearly:
\[
\mu_P \leq \mu_Q \Leftrightarrow P \leq Q
\]

Remark: Clearly, from the above definition \(P \leq Q \Leftrightarrow |B_i| \leq |C_i|, i = 0, 1, \ldots, m-1\).

Note: \(\leq_s\) is a partial ordering on \(\mathbb{Z}/q\mathbb{Z}\).

Note: Let \(m \leq m'\) and:
\[
A = \{(1, a_1, a_2, \ldots, a_{m-1})\}, B = \{(1, b_1, b_2, \ldots, b_{m'-1})\}.
\]
It is easy to show that:
\[
A \lor B = ((1, \max\{a_1, b_1\}, \max\{a_2, b_2\}, \ldots, \max\{a_{m-1}, b_{m-1}\}),
\max\{a_{m-1}, b_{m-1}\}, \max\{a_{m-1}, b_{m-2}\}, \ldots, \max\{a_{m-1}, b_{m-1}\})
\]
\[
A \land B = ((1, \min\{a_1, b_1\}, \min\{a_2, b_2\}, \ldots, \min\{a_{m-1}, b_{m-1}\}),
\]
and that \((\mathbb{Z}/q\mathbb{Z}, \leq_s)\) is a lattice.

MATERIALS AND METHODS

Function on the class sizes:

Theorem 1: Let \(\phi_i : \mathbb{Z}/q\mathbb{Z} \rightarrow \mathbb{Z}/q\mathbb{Z}\) be defined by:
\[
\phi_i[(1, g_1, g_2, \ldots, g_{m-1})] = ((1, f(g_1), f(g_2), \ldots, f(g_{m-1}))
\]
for any element \((1, g_1, g_2, \ldots, g_{m-1})\) of \(\mathbb{Z}/q\mathbb{Z}\), where \(m \in \{2, 3, \ldots\}\) and \(f\) is a function from \(\{2, 4, 6, \ldots\}\) to \(\{2, 4, 6, \ldots\}\).

\(\phi_i\) is a lattice homomorphism if and only if \(f\) is non-decreasing.

Proof: Let \(\phi_i\) be a lattice homomorphism on \(\mathbb{Z}/q\mathbb{Z}\) and let:
\[
A, B \in \mathbb{Z}/q\mathbb{Z} \Rightarrow A = \{(1, a_1, a_2, \ldots, a_{m-1})\}, B = \{(1, b_1, b_2, \ldots, b_{m'-1})\}
\]
where, \(a_1, a_2, \ldots, a_{m-1}, b_1, b_2, \ldots, b_{m'-1}\) are fixed, positive, even integers. We assume, without loss of generality, that \(m \leq m'\).
Corollary 2: Define \( \psi_i : \mu_p \to \mu_p \) by:

\[
\psi_i[[1, g_1, g_2, \ldots, g_{m-1}]] = ([1, f(g_1), f(g_2), \ldots, f(g_{m-1})])
\]

for any element \( ((1, g_1, g_2, \ldots, g_{m-1})) \) of \( M_p \). \( \psi_i \) is a lattice homomorphism iff \( f \) is non-decreasing.

Inserting classes:

**Theorem 3:** Let:

\[
f_r : \mathcal{Z}_r \to \mathcal{Z}_r \ni f_r((1,a_1, a_2, \ldots, a_{m-1}))
\]

\[
= \{(1,2,2, \ldots, 2, a_1, a_2, \ldots, a_{m-1})\}
\]

\[
\leftarrow \text{rtwos} \to
\]

for any element \( ((1,a_1, a_2, \ldots, a_{m-1})) \) of \( \mathcal{Z}_r \), where \( r \) is a fixed, arbitrary element of \( \{1,2,\ldots\} \) \( \forall \ m \leq m-1 \).

\( f_r \) is a homomorphism on \( \mathcal{Z}_r \).

**Proof:** \( f_r \) is clearly a function.

Let:

\[
A = ((1,a_1, a_2, \ldots, a_{m-1}))
\]

\[
B = ((1,b_1, b_2, \ldots, b_{m-1}))
\]

For fixed, arbitrary numbers \( m, m' \in \{2,3,\ldots\} \).

We show that:

\[
f_r(A \lor B) = \left[ f_r(A) \right] \lor \left[ f_r(B) \right]
\]

\[
f_r(A \land B) = \left[ f_r(A) \right] \land \left[ f_r(B) \right]
\]

\[
\leftarrow \text{rtwos} \to
\]

\[
= \{(1,2,2, \ldots, 2, a_1, a_2, \ldots, a_{m-1})\}
\]

\[
\leftarrow \text{rtwos} \to
\]

In almost the same way, we can show that:

\[
\phi_i(A \land B) = \left[ \phi_i(A) \right] \land \left[ \phi_i(B) \right]
\]

Hence \( \phi_i \) is a lattice homomorphism from \( \mathcal{Z}_r \) to \( \mathcal{Z}_r \).

**Corollary 2:** Define \( \psi_i : \mu_p \to \mu_p \) by:

\[
\psi_i[[1, g_1, g_2, \ldots, g_{m-1}]] = ([1, f(g_1), f(g_2), \ldots, f(g_{m-1})])
\]

for any element \( ((1, g_1, g_2, \ldots, g_{m-1})) \) of \( M_p \). \( \psi_i \) is a lattice homomorphism iff \( f \) is non-decreasing.
\[ f([(1, \min \{a_1, b_1\}, \min \{a_2, b_2\}, \ldots, \min \{a_m, b_m\})] \]

\[ = ((1, 2, 2, \ldots, 2, \min \{a_1, b_1\}, \min \{a_2, b_2\}, \ldots), \min \{a_{m-1}, b_{m-1}\}) \]

\[ \leftarrow r \text{twos} \rightarrow \]

\[ \min \{a_1, b_1\} \]

\[ = ((1, 2, 2, \ldots, 2, a_1, a_2, \ldots, a_{m-1}) \land \]

\[ ((1, 2, 2, \ldots, 2, b_1, b_2, \ldots, b_{m-1})) \]

\[ \leftarrow \rightarrow \]

\[ \left[ f(A) \right] \land \left[ f(B) \right] \]

(2)

From (1) and (2), we conclude that \( f \) is a lattice homomorphism.

Remark: \( f \) maps an element of \( \mathcal{S}_{p,m} \) to an element of \( \mathcal{S}_{r,m} \) for each \( m \in \{2,3,\ldots\} \).

Corollary 4: Define \( \varphi_r : \mathcal{M}_p \rightarrow \mathcal{M}_p \) by:

\[ \varphi_r \left[ \left( (1, a_1, a_2, \ldots, a_{m-1}) \right)_p \right] = \left( (1, 2, 2, \ldots, 2, a_1, a_2, \ldots, a_{m-1}) \right)_p \]

for any \( \left( (1, a_1, a_2, \ldots, a_{m-1}) \right)_p \in \mathcal{M}_p \), where \( r \) is a fixed arbitrary natural number \( \geq r \leq m - 1 \).

\( \varphi_r \) is a homomorphism on \( \mathcal{M}_p \).

Increasing class sizes:

Theorem 5: Let \( h_1, h_2, \ldots, h_{m-1} \) be elements of \( \{0, \pm 2, \pm 4, \ldots\} \) and let:

\[ \mathcal{S}_{r,m} = \left\{ \left( (1, a_1, a_2, \ldots, a_{m-1}) \right)_p \mid 2 \leq a_1 + h_1 \leq a_2 + h_2 \leq \ldots \leq a_{m-1} + h_{m-1} \right\} \]

Then \( \mathcal{S}_{r,m} \leq \mathcal{S}_{p,m} \) is a sublattice of \( \left( \mathcal{S}_{p,m}, \leq \right) \).

Proof: We prove that the g.l.b. and the l.u.b. of two arbitrary elements of \( \mathcal{S}_{p,m} \) are also in \( \mathcal{S}_{r,m} \).

Let \( A \) and \( B \) be two arbitrary elements of \( \mathcal{S}_{p,m} \) as:

\[ A = \left( (1, a_1, a_2, \ldots, a_{m-1}) \right)_p \]

And:

\[ B = \left( (1, b_1, b_2, \ldots, b_{m-1}) \right)_p \]

\[ A \lor B = \left( (1, \max \{a_1, b_1\}, \max \{a_2, b_2\}, \ldots, \max \{a_{m-1}, b_{m-1}\}) \right)_p \]

Now, since \( A, B \in \mathcal{S}_{p,m} \):

\[ 2 \leq a_1 + h_1 \leq a_2 + h_2 \leq \ldots \leq a_{m-1} + h_{m-1} \]

And:

\[ 2 \leq b_1 + h_1 \leq b_2 + h_2 \leq \ldots \leq b_{m-1} + h_{m-1} \]

\[ : 2 \leq \max \{a_1, b_1\} + h_1 \leq \max \{a_2, b_2\} + h_2 \leq \]

\[ \cdots \leq \max \{a_{m-1}, b_{m-1}\} + h_{m-1} \]

\[ A \lor B \in \mathcal{S}_{r,m} \]

Similarly:

\[ A \land B \in \mathcal{S}_{r,m} \]

From (3) and (4), we see that \( \left( \mathcal{S}_{p,m}, \leq \right) \) is a sublattice of \( \left( \mathcal{S}_{r,m}, \leq \right) \).

Corollary 6: Let \( h_1, h_2, \ldots, h_{m-1} \) be elements \( \{0, \pm 2, \pm 4, \ldots\} \):

\[ M_{p,m} = \left\{ \mu \mid \mu \in \mathcal{S}_{p,m} \right\} \]

\[ \left( M_{p,m}, \leq \right) \]

is a sublattice of \( \left( M_{p,m}, \leq \mu \right) \).

Theorem 7: Let \( \mathcal{S}_{p,m}, h_1, h_2, \ldots, h_{m-1} \) be as in the previous theorem.

Also, let \( g : \mathcal{S}_{p,m} \rightarrow \mathcal{S}_{p,m} \) be defined by:

\[ g \left[ \left( (1, a_1, a_2, \ldots, a_{m-1}) \right)_p \right] = \left( (1, a_1 + h_1, a_2 + h_2, \ldots, a_{m-1} + h_{m-1}) \right)_p \]

for any element \( \left( (1, a_1, a_2, \ldots, a_{m-1}) \right)_p \) of \( \mathcal{S}_{p,m} \).

\( g \) is a lattice homomorphism.

Proof: Clearly \( g \) is a function that maps elements of \( \mathcal{S}_{p,m} \) to elements of \( \mathcal{S}_{p,m} \).

We show that \( g \) is a homomorphism.

Let:
A, B ∈ ℑ ∋

A = ((1, a_1, a_2, ..., a_{m-1}))

And:

B = ((1, b_1, b_2, ..., b_{m-1}))

g(A ∨ B) = g(((1, a_1, a_2, ..., a_{m-1})) ∨ ((1, b_1, b_2, ..., b_{m-1})))

= ((1, max(a_1, b_1), max(a_2, b_2), ..., max(a_{m-1}, b_{m-1})))

= (((1, a_1 + h_1, a_2 + h_2, ..., a_{m-1} + h_{m-1})) ∨
   (1, b_1 + h_1, b_2 + h_2, ..., b_{m-1} + h_{m-1})))

= [g(A)] ∨ [g(B)]

Similarly, it can be shown that:

\[ g(A ∧ B) = [g(A)] ∧ [g(B)] \]

Hence g is a lattice homomorphism on ℑ_{P,m}.

**Remark:** Clearly ℑ_{P,2e} is a sublattice of ℑ_P; for if A, B ∈ ℑ_{P,2e} then A ∨ B and A ∧ B would each have at least e classes and so A ∨ B, A ∧ B ∈ ℑ_{P,2e}.

**Theorem 9:** Let i_1, i_2, ..., i_{r-1} be fixed, arbitrary, natural numbers ≥ i_1 ≤ i_2 ≤ ... ≤ i_{r-1} and let U be a mapping on:

ℑ_{P,2i_{r-1}} ⊃ U[((1, a_1, a_2, ..., a_{m-1}))] = ((1, a_{i_1}, a_{i_2}, ..., a_{i_{r-1}}))

∀ ∈ ℑ_{P,2i_{r-1}}

U is a homomorphism from ℑ_{P,2i_{r-1}} to ℑ_{P,r}.

**Proof:** U is clearly a function.

Also, for any element:

\[ U[((1, a_1, a_2, ..., a_{m-1}))] = ((1, a_{i_1}, a_{i_2}, ..., a_{i_{r-1}})) ∈ ℑ_{P,r} \]

Since:

\[ a_{i_1} ≤ a_{i_2} ≤ ... ≤ a_{i_{r-1}} \]

∴ U is a function from ℑ_{P,2i_{r-1}} to ℑ_{P,r}.

Now, let A, B ∈ ℑ_{P,2i_{r-1}} ⊃ :

A = ((1, a_1, a_2, ..., a_{m-1}))

B = ((1, b_1, b_2, ..., b_{m-1}))

For numbers:

\[ m', m'' ∈ \{i_1, i_2, ..., i_{r-1} + 1, i_{r-1} + 2, ...\} \]

⇒ m' ≤ m''

\[ U(A ∨ B) = U(((1, a_1, a_2, ..., a_{m-1})) ∨ ((1, b_1, b_2, ..., b_{m-1}))) \]

= U(((1, max(a_1, b_1), max(a_2, b_2), ..., max(a_{m-1}, b_{m-1})),
    max{a_{m-1}, b_{m-1}}, max{a_{m-1}, b_{m-1}}),
    max{a_{m-1}, b_{m-1}}, max{a_{m-1}, b_{m-1}}))

= ((1, max(a_{i_1}, b_{i_1}),
    max{a_{i_1}, b_{i_1}}, max{a_{i_1}, b_{i_1}}))

\[ U(A) ∨ U(B) = [U(A)] ∨ [U(B)] \]
Similarly $U(A \land B) = [U(A)] \land [U(B)]$.

Hence $U$ is a lattice homomorphism on $\mathfrak{A}_{p,2^s}$.

**Notation:** Let $M_{p,2^s} = \{ \mu \in \mathfrak{A}_{p,2^s} \}$ where $e \in \{2,3,\ldots\}$.

**Remark:** $M_{p,2^s}$ is a sublattice of $M_p$.

**Corollary 10:** Let $i_1, i_2, \ldots, i_{s-1}$ be as in Theorem 9 and let $\psi_\mu$ be a mapping on:

$1,a_1,a_2,a_s \leq \mu \in \mathfrak{I}$

$$f((1,a_1,a_2,a_s)) = f((1,b_1,b_2,b_{s-1})) = \max \{a_{s-1},b_{s-1}\}$$

Where:

$$k_s = \begin{cases} \max\{a_s, b_s\} & \text{if } \max\{a_s, b_s\} \leq t_s \\ t_s & \text{if } \max\{a_s, b_s\} > t_s \end{cases}$$

$$\forall i = 1,2,s-1$$

$$u_i = \begin{cases} a_i & \text{if } a_i \leq t_i \\ t_i & \text{if } a_i > t_i \end{cases}$$

$$\forall i = 1,2,s-1$$

$$v_i = \begin{cases} b_i & \text{if } b_i \leq t_i \\ t_i & \text{if } b_i > t_i \end{cases}$$

**Reducing some class sizes:**

**Theorem 11:** Let $t_1, t_2, \ldots, t_{s-1}$ be positive, even numbers $\geq t_1 \leq t_2 \leq \ldots \leq t_{s-1}$ and let $f$ be a function on:

$$\mathfrak{A}_{p,2^s} \ni f((1,a_1,a_2,a_s)) = (1,b_1,b_2,b_{s-1})$$

$$\forall (1,a_1,a_2,a_s) \in \mathfrak{A}_{p,2^s}$$

where:

$$h_i = \begin{cases} a_i & \text{if } a_i \leq t_i \\ t_i & \text{if } a_i > t_i \end{cases}$$

for $i = 1,2,\ldots,s-1$.

$f$ is a lattice homomorphism from $\mathfrak{A}_{p,2^s}$ to $\mathfrak{I}$, where $I$ is the ideal:

$$\{(1,a_1,a_2,a_s) \in \mathfrak{A}_{p,2^s} | a_i \leq t_i; i = 1,2,3,\ldots,s-1\}$$

Of:

$$(\mathfrak{A}_{p,2^s}, \leq_s)$$

**Proof:** $f$ is clearly a function. Also for any two elements $A$ and $B$ of:

$$\mathfrak{A}_{p,2^s} \ni A = ((1,a_1,a_2,a_s))$$

$$B = ((1,b_1,b_2,b_{s-1}))$$

$$(m' \geq m)$$

$$f(A \lor B) = f((1,a_1,a_2,a_s)) \lor (1,b_1,b_2,b_{s-1}))$$

$$= f((1, \max\{a_1,b_1\}, \max\{a_2,b_2\}, \ldots, \max\{a_{s-1},b_{s-1}\}))$$

Now:

$$f(A \land B) = f\left((1,a_1,a_2,a_s)) \land (1,b_1,b_2,b_{s-1})\right)$$

$$= f\left(1, \min\{a_1,b_1\}, \min\{a_2,b_2\}, \ldots, \min\{a_{s-1},b_{s-1}\}\right)$$

$$= ((1,1,1,\ldots,1, \min\{a_{s-1},b_{s-1}\}))$$

$$\therefore f(A \lor B) = \left[f(A) \lor f(B)\right]$$

$$\therefore f(A \land B) = f\left((1,1,1,\ldots,1, \min\{a_{s-1},b_{s-1}\})\right)$$
Where:

\[
l_i = \begin{cases} 
\max\{a, b\} & \text{if } \min\{a, b\} \leq t \\
t_i & \text{if } \min\{a, b\} > t_i
\end{cases}
\]

\[
= (l_1, w_2, \ldots, w_{s-1}, a_1, a_2(\ldots, a_m)) \wedge
(l_1, x_2, \ldots, x_{s-1}, b_1, b_2(\ldots, b_m))
\]

\[=[f(A)] \wedge [f(B)]\]

Where:

\[
w_i = \begin{cases} 
a_i & \text{if } a_i \leq t_i \\
t_i & \text{if } a_i > t_i
\end{cases}
\]

(i = 1, 2, ..., s − 1)

And:

\[
x_i = \begin{cases} 
b_i & \text{if } b_i \leq t_i \\
t_i & \text{if } b_i > t_i
\end{cases}
\]

Where:

(i = 1, 2, ..., s − 1)

**Corollary 12:** Let \( t_1, t_2, \ldots, t_{s−1}, \) and \( I \) be as in Theorem 11. Also let:

\[
I_\mu = \{\mu_P | P \in I\}
\]

Define: \( \Psi_\mu \) on \( \mu_{P,2^s} \) by:

\[
\Psi_\mu\left(\left((1, a_1, a_2, \ldots, a_m)\right)_{\mu}\right) = \left((1, h_1, h_2, \ldots, h_{s−1}, a_1, a_2(\ldots, a_m))\right)_{\mu}
\]

\[
\forall\left((1, a_1, a_2, \ldots, a_m)\right)_{\mu} \in M_{P,2^s}
\]

Where:

\[
h_i = \begin{cases} 
a_i & \text{if } a_i \leq t_i \\
t_i & \text{if } a_i > t_i
\end{cases}
\]

(i = 1, 2, ..., s − 1); \( m \geq s \)

\( \Psi \) is a lattice homomorphism from \( M_{P,2^s} \) to \( I_\mu \).

**RESULTS AND DISCUSSION**

We have shown that there exists various lattice homomorphisms from lattice of SK-partitions to lattices of SK-partitions and that similar relationships exist between lattices of measures defined in term of SK-partitions.

This study furthers the study of systems of measures and relationships between such systems.

**CONCLUSION**

Using the approach used in this study, it is possible to do further study of lattices of measures defined in terms of SK-partitions by investigating lattices of these partitions.

**REFERENCES**


