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On Sandwich Theorems of Analytic Functions Involving Noor Integral Operator

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Abstract: In this research, we introduce sufficient conditions for subordination and superordination for subclass of analytic functions containing Noor integral operator.

Key words: Noor integral operator, subordination, superordination

INTRODUCTION

Let H be the class of functions analytic in U and H[a, n] be the subclass of H consisting of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...$$

Let A be the subclass of H consisting of functions of the form:

$$f(z) = z + a_2 z^2 + ...$$

Let F and G be analytic functions in the unit disk U. The function F is subordinate to G, written $F \prec G$, if G is univalent F(0) = G(0) and F(U)⊂G(U). In general, given two functions F(z) and G(z), which are analytic in U, the function F(z) is said to be subordination to G(z) in U if there exists a function h(z), analytic in U with h(0) and $|h(z)|<1$ for all $z\in U$ such that F(z) = G(h(z)) for all $z\in U$.

Let $\varphi : C^2 \to C$ and let h be univalent in U. If p is analytic in U and satisfies the differential subordination $\varphi(p(z)), zp'(z)) \prec h(z)$.

Then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, $p \prec q$. If p and $\varphi(p(z)), zp'(z))$ are univalent in U and satisfy the differential superordination $h(z) \prec \varphi(p(z)), zp'(z))$ then p is called a solution of the differential superordination.

An analytic function q is called subordinant of the solution of the differential superordination if $q \prec p$.

Denote by $D^\alpha : A \to A$ the operator defined by:

$$D^\alpha f(z) := \frac{z}{(1-z)^{\alpha+1}} * f(z), \ \alpha > -1$$

where, (*) refers to the Hadamard product or convolution. Then implies that:

$$D^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!}, \ n \in N_0 = N \cup \{0\}.$$

We note that $D^0 f(z) = f(z)$ and $D' f(z) = zf'(z)$.

The operator $D^n f$ is called Ruscheweyh derivative of n-th order of f.

Noor[1] defined and studied an integral operator $I_n : A \to A$ analogous to $D^n f$ as follows:

Let $f_n(z) = \frac{z}{(1-z)^{n+1}}$, $n \in N_0$ and let $f_n^{(-1)}$ be defined such that:

$$f_n(z) * f_n^{(-1)}(z) = \frac{z}{1-z} \quad (1)$$

Then:

$$I_n f(z) = f_n^{(-1)}(z) * f(z) = \left(\frac{z}{(1-z)^{n+1}}\right)^{-1} * f(z) \quad (2)$$

Note that $I_0 f(z) = zf'(z)$ and $I_1 f(z) = f(z)$. The operator $I_n f(z)$ is called the Noor Integral of n-th order of f. Using (1), (2) and a well- known identity for $D^n f$ we have:

$$(n+1) I_n f(z) - n I_{n+1} f(z) = z(I_{n+1} f(z))' \quad (3)$$

Using hypergeometric functions ${}_2F_1$, (2) becomes:

$$I_n f(z) = [z \ {}_2F_1(1,1;n+1,z)] * f(z)$$

where, ${}_2F_1(a,b;c,z)$ is defined by:

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$$ {}_2F_1(a,b;c,z) = 1+\frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!}+.... $$

The following definitions can be found in[2].

Definition 1: Let $f\in A$ Then $f \in S^*$ (the starlike subclass of A) if and only if for $z\in U$:

$$ \operatorname{Re}\left\{\frac{z[I_n f(z)]'}{I_n f(z)}\right\} > 0,\ n\in N_0 . $$

Definition 2: Let $f\in A$. Then $f\in N_{(n)}^*$, $n\in N_0$ if and only if $I_n f\in S^*$ (the starlike subclass of A) for $z\in U$.

Definition 3: Let $f\in A$. Then $M^*_{(n)}$ for N_0 if and only if there exists $g\in N_{(n)}^*$ such that, for $z\in U$:

$$ \operatorname{Re}\left\{\frac{z[I_n f(z)]'}{I_n g(z)}\right\} > 0 . $$

In the present study, we apply a method based on the differential subordination in order to obtain subordination results involving Noor Integral operator for a normalized analytic function f:

$$ q_1(z) \prec \frac{z[I_n f(z)]'}{I_n f(z)} \prec q_2(z) , $$

and

$$ q_1(z) \prec \frac{z[I_n f(z)]'}{[I_n g(z)]} \prec q_2(z) . $$

In order to prove our subordination and superordination results, we need the following definition and lemmas in the sequel.

Definition 4: Miller and Mocanu[3] . Denote by Q the set of all functions f(z) that are analytic and injective on $\{\overline{U}-E(f)$ where $E(f):=\{\xi \in\partial U: \lim_{z\to\xi} f(z)=\infty\}$ and are such that:

$$ f'(\xi) \neq 0 \ \text{for}\ \xi \in\partial U-E(f). $$

Lemma 1: Miller and Mocanu[4]. Let q(z) be univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing q(U) with $\phi(w) \neq 0$ when $w\in q(U)$.

Set:

$$ Q(z):=zq'(z)\phi(q(z)),\ h(z):=\theta(q(z))+Q(z) $$

Suppose that:

- Q(z) is starlike univalent in U and
- $\operatorname{Re}\left\{\frac{zh`(z)}{Q(z)}\right\}>0$ for $z\in U$.

If

$$ \theta(p(z))+ zp'(z)\phi(p(z)) \prec \theta(q(z))+ zq'(z)\phi(q(z)) $$

then $p(z)\prec q(z)$ and q(z) is the best dominant.

Lemma 2: Shanmugam, *et al.*[5]. Let q(z) be convex univalent in the unit disk U and Ψ and γ in *C* with:

$$ \operatorname{Re}\left\{1+\frac{zq''(z)}{q'(z)}+\frac{\psi}{\gamma}\right\} > 0 \tag{4} $$

If p(z) is analytic in U and $\psi p(z)+\gamma zp'(z) \prec \psi q(z)+\gamma zq'(z)$ then $p(z)\prec q(z)$ and q is the best dominant.

Lemma 3: Bulboaca[6]. Let q(z) be convex univalent in the unit disk U and ϑ and υ be analytic in a domain D containing q(U). Suppose that:

- $zq'(z)\varphi(q(z))$ is starlike univalent in U and
- $\operatorname{Re}\left\{\frac{\vartheta'(q(z))}{\varphi(q(z))}\right\}>0$ for $z\in U$.

If $p(z)\in H[q(0),1]\cap Q$, with $p(U)\subseteq D$ and $\vartheta(q(z))+zp'(z)\varphi(p(z))$ is univalent in U and

$$ \vartheta(q(z))+ zq'(z)\varphi(q(z)) \prec \vartheta(p(z))+ zp'(z)\varphi(p(z)) $$

then $q(z)\prec p(z)$ and q(z) is the best subordinant.

Lemma 4: Miller and Mocanu[3]. Let q(z) be convex univalent in the unit disk U and $\gamma\in C$.

Further, assume that $\operatorname{Re}\{\overline{\gamma}\}>0$. If $p(z)\in H[q(0),1]\cap Q$ with $p(z)+\gamma zp'(z)$ is univalent in U then $q(z)+\gamma zq'(z)\prec p(z)+\gamma zp'(z)$ implies $q(z)\prec p(z)$ and q(z) is the best subordinant.

SANDWICH RESULTS

By making use of Lemmas 1 and 2, we prove the following subordination results.

Theorem 1: Let $q(z) \neq 0$ be univalent in U such that $\frac{z q'(z)}{q(z)}$ is starlike univalent in U and:

$$\mathrm{Re}\left\{1+\frac{\alpha}{\gamma} q(z)+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right\} > 0 \qquad (5)$$
$$\alpha,\ \gamma \in C, \gamma \neq 0.$$

If $f \in A$ satisfies the subordination:

$$\alpha \frac{z[I_n f(z)]'}{I_n f(z)} + \gamma\left[1+\frac{z[I_n f(z)]''}{[I_n f(z)]'} - \frac{z[I_n f(z)]'}{I_n f(z)}\right]$$

$$\prec \alpha\, q(z) + \frac{\gamma z\, q'(z)}{q(z)}$$

then

$$\frac{z[I_n f(z)]'}{I_n f(z)} \prec q(z) \qquad (6)$$

q(z) is the best dominant.

Proof: Our aim is to apply Lemma 1. Setting $p(z) := \frac{z[I_n f(z)]'}{I_n f(z)}$. By computation shows that:

$$\frac{zp'(z)}{p(z)} = 1+\frac{z[I_n f(z)]''}{[I_n f(z)]'} - \frac{z[I_n f(z)]'}{I_n f(z)}$$

which yields the following subordination:

$$\alpha\, p(z) + \frac{\gamma z\, p'(z)}{p(z)} \prec \alpha\, q(z) + \frac{\gamma z\, q'(z)}{q(z)}$$
$$\alpha, \gamma \in C.$$

By setting $\theta(\omega) := \alpha\omega$ and $\varphi(\omega) := \frac{\gamma}{\omega}, \gamma \neq 0$, it can be easily observed that $\theta(\omega)$ is analytic in C and $\varphi(\omega)$ is analytic in $C/\{0\}$ and that $\varphi(\omega) \neq 0$ when $C/\{0\}$. Also, by letting $Q(z) = zq'(z)\,\varphi(q(z)) = \gamma z \frac{q'(z)}{q(z)}$ and $h(z) = \theta(q(z)) + Q(z) = \alpha\, q(z) + \gamma z \frac{q'(z)}{q(z)}$.

We find that Q(z) is starlike univalent in U and that:

$$\mathrm{Re}\left\{\frac{z\, h'(z)}{Q(z)}\right\}$$
$$= \left\{1+\frac{\alpha}{\gamma} q(z)+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right\} > 0.$$

Then the relation (6) follows by an application of Lemma 1.

Corollary 1: If $f \in A$ and assume that (5) holds then:

$$1+\frac{z[I_n f(z)]''}{[I_n f(z)]'} \prec \frac{1+Az}{1+Bz} + \frac{(A-B)z}{(1+Az)(1+Bz)}$$

implies $\frac{z[I_n f(z)]'}{I_n f(z)} \prec \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$ and $\frac{1+Az}{1+Bz}$ is the best dominant.

Proof: By setting $\alpha = \gamma = 1$ and $q(z) := \frac{1+Az}{1+Bz}$ where $-1 \leq B < A \leq 1$.

Corollary 2: If $f \in A$ and assume that (5) holds then:

$$1+\frac{z[I_n f(z)]''}{[I_n f(z)]'} \prec \frac{1+z}{1-z} + \frac{2z}{1-z^2}$$

implies $\frac{z[I_n f(z)]'}{I_n f(z)} \prec \frac{1+z}{1-z}$ and $\frac{1+z}{1-z}$ is the best dominant.

Proof: By setting $\alpha = \gamma = 1$ and $q(z) := \frac{1+z}{1-z}$.

Corollary 3: If $f \in A$ and assume that (5) holds then:

$$1+\frac{z[I_n f(z)]''}{[I_n f(z)]'} \prec e^{Az} + Az$$

implies $\frac{z[I_n f(z)]'}{I_n f(z)} \prec e^{Az}$ and e^{Az} is the best dominant.

Proof: By setting $\alpha = \gamma = 1$ and $q(z) := e^{Az}$, $|A| < \pi$.

Theorem 2: Let q(z) be convex univalent in the unit disk U and γ in C satisfies $\mathrm{Re}\left\{1+\frac{zq''(z)}{q'(z)}+\frac{1}{\gamma}\right\} > 0,\ \gamma \in C$.

If f in $M_{(n)}^{*}$ for $n \in N_0$ and exists $g \in N_{(n)}^{*}$ such that

$\frac{z[I_n f(z)]'}{I_n g(z)}$ is analytic in U and the subordination

$$\frac{z[I_n f(z)]'}{I_n g(z)}\{1+[1+\frac{z(I_n f(z))''}{(I_n(z))'}-\frac{z(I_n g(z))'}{I_n g(z)}]\}$$
$$\prec q(z) + \gamma zq'(z),\ \gamma \in C$$

holds then:

$$\frac{z[I_n f(z)]'}{I_n g(z)} \prec q(z) \quad (7)$$

and q(z) is the best dominant.

Proof: Our aim is to apply Lemma 2. Setting $p(z) := \frac{z[I_n f(z)]'}{I_n g(z)}$. By computation shows that:

$$zp'(z) = \frac{z[I_n f(z)]'}{I_n g(z)}\left[1 + \frac{z(I_n f(z))''}{(I_n f(z))'} - \frac{z(I_n g(z))'}{I_n g(z)}\right]$$

which yields the following subordination $p(z) + \gamma z p'(z) \prec q(z) + \gamma z q'(z), \quad \gamma \in C$. Thus in view of Lemma 2, (7) holds.

Theorem 3: Let q(z) ≠ 0 be convex univalent in the unit disk U. Suppose that:

$$\mathrm{Re}\left\{\frac{\alpha}{\gamma} q(z)\right\} > 0, \ \alpha, \gamma \in C, \gamma \neq 0 \ \text{for} \ z \in U \quad (8)$$

and $\frac{z q'(z)}{q(z)}$ is starlike univalent in U. If $\frac{z[I_n f(z)]'}{I_n f(z)} \in H[q(0),1] \cap Q$ where $f \in A$,

$\alpha\left\{\frac{z[I_n f(z)]'}{I_n f(z)}\right\} + \gamma\left\{1 + \frac{z[I_n f(z)]''}{[I_n f(z)]'} - \frac{z[I_n f(z)]'}{I_n f(z)}\right\}$ is univalent in U and the subordination

$$q(z) + \frac{\gamma z q'(z)}{q(z)} \prec \alpha\left\{\frac{z[I_n f(z)]'}{I_n f(z)}\right\} + \gamma\left\{1 + \frac{z[I_n f(z)]''}{[I_n f(z)]'} - \frac{z[I_n f(z)]'}{I_n f(z)}\right\}$$

holds, then:

$$q(z) \prec \frac{z[I_n f(z)]'}{I_n f(z)} \quad (9)$$

and q is the best subordinant.

Proof: Our aim is to apply Lemma 3.

Setting $p(z) := \frac{z[I_n f(z)]'}{I_n f(z)}$. By computation shows that:

$$\frac{zp'(z)}{p(z)} = 1 + \frac{z[I_n f(z)]''}{[I_n f(z)]'} - \frac{z[I_n f(z)]'}{I_n f(z)}$$

which yields the following subordination $\alpha\, q(z) + \frac{\gamma z\, q'(z)}{q(z)} \prec \alpha\, p(z) + \frac{\gamma z\, p'(z)}{p(z)}$ for $\alpha, \gamma \in C$.

By setting $\theta(\omega) := \alpha\omega$ and $\varphi(\omega) := \frac{\gamma}{\omega}, \gamma \neq 0$ it can be easily observed that $\theta(\omega)$ is analytic in C and $\varphi(\omega)$ is analytic in $C \setminus \{0\}$ and that $\varphi(\omega) \neq 0$ when $\omega \in C \setminus \{0\}$. Also, we obtain

$$\mathrm{Re}\left\{\frac{\theta'(q(z))}{\varphi(q(z))}\right\} = \mathrm{Re}\left\{\frac{\alpha}{\gamma} q(z)\right\} > 0 .$$

Then (9) follows by an application of Lemma 3.

Theorem 4: Let q(z) be convex univalent in the unit disk U and $\gamma \in C$. Further, assume that $\mathrm{Re}\{\bar{\gamma}\} > 0$. If $\frac{z[I_n f(z)]'}{I_n g(z)} \in H[q(0),1] \cap Q$, with $\frac{z[I_n f(z)]'}{I_n g(z)}\{1 + [1 + \frac{z(I_n f(z))''}{(I_n f(z))'} - \frac{z[I_n g(z)]'}{I_n g(z)}]\}$ is univalent in U then

$$q(z) + \gamma z q'(z) \prec \frac{z[I_n f(z)]'}{I_n g(z)}\{1 + [1 + \frac{z(I_n f(z))''}{(I_n f(z))'}\} - \frac{z(I_n g(z))'}{I_n g(z)}]\}$$

implies:

$$q(z) \prec \frac{z[I_n f(z)]'}{I_n g(z)} \quad (10)$$

and q(z) is the best subordinant.

Proof: Our aim is to apply Lemma 4. Setting $p(z) := \frac{z[I_n f(z)]'}{I_n g(z)}$. By computation shows that $zp'(z) = \frac{z[I_n f(z)]'}{I_n g(z)}\left\{1 + \frac{z(I_n f(z))''}{(I_n f(z))'} - \frac{z(I_n g(z))'}{I_n g(z)}\right\}$ which yields the following subordination $q(z) + \gamma z q'(z) \prec p(z) + \gamma z p'(z), \quad \gamma \in C$. Thus in view of Lemma 4, we obtain (10). By combining Theorems 1 and 3 and Theorems 2 and 4 to get the following Sandwich theorems.

Theorem 5: Let $q_1(z) \neq 0, q_2(z) \neq 0$ be convex univalent in the unit disk U satisfy (8) and (5) respectively. Suppose that and $\frac{z\, q_i'(z)}{q_i(z)}, i = 1,2$ is starlike univalent in U.

If $\frac{z[I_n f(z)]'}{I_n f(z)} \in H[q(0),1] \cap Q$ where $f \in A$, $\alpha\left\{\frac{z[I_n f(z)]'}{I_n f(z)}\right\} + \gamma\left\{1+\frac{z[I_n f(z)]''}{[I_n f(z)]'} - \frac{z[I_n f(z)]'}{I_n f(z)}\right\}$ is univalent in U and the subordination

$$q_1(z) + \frac{\gamma z\, q_1'(z)}{q_1(z)} \prec \alpha\left\{\frac{z[I_n f(z)]'}{I_n f(z)} + \gamma\left[1+\frac{z[I_n f(z)]''}{[I_n f(z)]'} - \frac{z[I_n f(z)]'}{I_n f(z)}\right]\right\} \prec \alpha\, q_2(z) + \frac{\gamma z\, q_2'(z)}{q_2(z)}$$

holds, then:

$$q_1(z) \prec \frac{z[I_n f(z)]'}{I_n f(z)} \prec q_2(z) \qquad (11)$$

and $q_1(z)$ is the best subordinant and $q_2(z)$ is the best dominant.

Theorem 5 reduces to the following known result obtained by Ali *et al*.[7]

Corollary 4: Let the assumption of Theorem 5 holds with $q_1(0) = q_2(0) = 1$. Then $q_1(z) \prec \frac{z[f(z)]'}{f(z)} \prec q_2(z)$ and $q_1(z)$ is the best subordinant and $q_2(z)$ is the best dominant.

Proof: By setting $\alpha = \gamma = 1$ and $n = 1$.

Corollary 5: Let the assumption of Theorem 5 holds. Then $q_1(z) \prec 1+\frac{z[f(z)]''}{[f(z)]'} \prec q_2(z)$ and $q_1(z)$ is the best subordinant and $q_2(z)$ is the best dominant.

Proof: By setting $\alpha = \gamma = 1$ and $n = 0$.

Theorem 6: Let $q_1(z)$, $q_2(z)$ be convex univalent in the unit disk U such that

$$\operatorname{Re}\left\{1+\frac{zq_2''(z)}{q_2'(z)}+\frac{1}{\gamma}\right\} > 0,\ \gamma \in C, \operatorname{Re}\{\bar{\gamma}\} > 0.$$

If $f \in M_{(n)}^*$ for $n \in N_0$ and exists $g \in N_{(n)}^*$ such that $\frac{z[I_n f(z)]'}{I_n g(z)} \in H[q_1(0),1] \cap Q$, with

$$\frac{z[I_n f(z)]'}{I_n g(z)}\{1+ [1+\frac{z(I_n f(z))''}{(I_n f(z))'} - \frac{z(I_n g(z))'}{I_n g(z)}]\}$$

is univalent in U, then

$$q_1(z) + \gamma z q_1'(z) \prec \frac{z[I_n f(z)]'}{I_n g(z)}\{1+ [1+\frac{z(I_n f(z))''}{(I_n f(z))'} - \frac{z(I_n g(z))'}{I_n g(z)}]\} \prec q_2(z) + \gamma z q_2'(z)$$

implies:

$$q_1(z) \prec \frac{z[I_n f(z)]'}{I_n g(z)} \prec q_2(z) \qquad (12)$$

and $q_1(z)$ is the best subordinant and $q_2(z)$ is the best dominant.

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