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Transient Solution of the M/M/C_1 Queue with Additional C_2 Servers for Longer Queues and Balking

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Abstract: The goal of this research is to discuss the M/M/C_1 queue with additional C_2 servers for longer queues and balking. By using generating function technique the transient probabilities are derived in terms of the modified Bessel function.

Keywords: M/M/1queue, balking, transient solution, generating function, modified Bessel function

INTRODUCTION

Queueing theory is very useful in a wide range of application in our life, from computer networks and telecommunications, to chemical kinetics and epidemiology.

Some topics in queueing theory interested by discussion of single server queue as one the above applications. The M/M/1 queue is analyzed by several researchers. The transient solution of a single-server system with balking concept was considered in[2,5,6].

Baht[3] studied the queue M/M/1 with an additional server for longer queues. A complete theoretical solution to the single-server case was provided in the case steady state.

The system M/M/C_1 queue with additional C_2 servers for longer queues has been discussed in the case steady state[1].

In this research, the transient solution is obtained for transient solution of the M/M/C_1 queue with additional C_2 servers for longer queues and balking. We introduce the general case of the M/M/1 queue with an additional server when the basic server is C_1 with additional C_2 servers for longer queues and balking concept.

The important of this type of queues appears for instance, in a bank more windows are opened for the service when the queues in front of the already open windows get too long. Also, this type of queues is sometimes used even in the case of airlines, buses and so forth.

SYSTEM MODEL

In this research, the authors deal with the M/M/C_1 queue with additional C_2 servers for longer queues plus balking, and consideration the following assumptions:

(**a**) Customers arrive at the system one by one according to a Poisson process with rate λ. On arrival a customer either decides to join the queue with probability β or balk with probability $1-\beta$, where

$$\beta = \text{prob.}\{\text{a unit joins the queue}\},$$

where

$$0 \le \beta < 1 \ \text{ if } \ n = C_1(1)\infty \text{ and } \ \beta = 1 \text{ if } \ n = 0\,(1)\,\overline{C_1 - 1}.$$

(**b**) The customers are served on a first-come, first served (FCFS) discipline. The service times are assumed to be distributed according to an exponential distribution with the following density function:

$$s(t) = \mu e^{-\mu t}, \ t \ge o, \ \mu > o,$$

where μ is the service rate.

ANALYZING THE PROBLEM

The probability differential difference equations for M/M/C_1 queue with additional C_2 servers for longer queues plus balking are given as follows,

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$$P_0'(t) = -\lambda P_0(t) + \mu_1 P_1(t) \tag{1}$$

$$P_n'(t) = -(\lambda + n\mu_1)P_n(t) + \lambda P_{n-1}(t) + (n+1)\mu_1 P_{n+1}(t)\ ,\ 1 \le n < c_1 \tag{2}$$

$$P_n'(t) = -(\beta\lambda + c_1\mu_1)P_n(t) + \lambda P_{n-1}(t) + c_1\mu_1 P_{n+1}(t)\ ,\ n = c_1 \tag{3}$$

$$P_n'(t) = -(\beta\lambda + c_1\mu_1)P_n(t) + \beta\lambda P_{n-1}(t) + c_1\mu_1 P_{n+1}(t)\ ,\ c_1 + 1 \le n \le N - 1 \tag{4}$$

$$P_n'(t) = -(\beta\lambda + c_1\mu_1)P_n(t) + \beta\lambda P_{n-1}(t) + \mu P_{n+1}(t)\ ,\ n = N \tag{5}$$

$$P_n'(t) = -(\beta\lambda + \mu)P_n(t) + \beta\lambda P_{n-1}(t) + \mu P_{n+1}(t)\ ,\ n \ge N + 1 \tag{6}$$

where,

$$\mu = c_1\mu_1 + c_2\mu_2$$

Define

$$q_n(t) = \begin{cases} e^{(\lambda + n\mu_1)t}[(n\mu_1 P_n(t) - \lambda P_{n-1}(t))] & ,\quad 1 \le n \le c_1 \\ e^{(\beta\lambda + c_1\mu_1)t}\, c_1\mu_1 P_n(t) - \lambda e^{(\lambda + c_1\mu_1)t} P_{n-1}(t) & ,\quad n = c_1 \\ e^{(\beta\lambda + c_1\mu_1)t}[(c_1\mu_1 P_n(t) - \beta\lambda P_{n-1}(t))] & ,\quad c_1 + 1 \le n \le N \\ e^{(\beta\lambda + \mu)t}[(\mu P_n(t) - \beta\lambda P_{n-1}(t))] & ,\quad n \ge N + 1 \end{cases} \tag{7}$$

and consider

$$H(z,t) = \sum_{n=-\infty}^{\infty} q_n(t) z^n \tag{8}$$

Differentiating (7) and (8) with respect to t, and using (1)-(6) we get

$$\frac{\partial H(z,t)}{\partial t} = (\lambda z + \frac{\mu}{z})\ H(z,t) + G(t) \tag{9}$$

where

$$\begin{aligned} G(t) = &-(\lambda z + \frac{\mu}{z})\ H(z,t) + \lambda z e^{\mu_1 t} F_1(z,t) - \frac{\mu_1 e^{-\mu_1 t}}{z} F_1(z,t) \\ &+ \mu_1 e^{-\mu_1 t} \frac{\partial F_1(z,t)}{\partial z} + (c_1\mu_1)^2 e^{(\beta\lambda + c_1\mu_1)t} P_{c_1+1}(t) z^{c_1} \\ &- \lambda c_1\mu_1 e^{(\lambda + c_1\mu_1)t} P_{c_1}(t) z^{c_1} + \lambda c_1\mu_1 e^{(\beta\lambda + c_1\mu_1)t} P_{c_1-1}(t) z^{c_1} \\ &- \lambda^2 e^{(\lambda + c_1\mu_1)t} P_{c_1-2}(t) z^{c_1} - \mu_1 e^{(\lambda + c_1\mu_1)t} P_{c_1-1}(t) z^{c_1} \\ &+ \frac{c_1\mu_1}{z} F_2(z,t) + \beta\lambda z F_2(z,t) + \frac{\mu}{z} F_3(z,t) \\ &+ \beta\lambda z F_3(z,t) + c_1\mu_1(c_2\mu_2 - \beta\lambda z) e^{(\beta\lambda + c_1\mu_1)t} P_{N+1}(t) z^N \end{aligned}$$

with

$$F_1(z,t) = \sum_{n=1}^{c_1-1} q_n(t) z^n\ ,\quad F_2(z,t) = \sum_{n=c_1+1}^{N} q_n(t) z^n \quad \text{and} \quad F_3(z,t) = \sum_{n=N+1}^{\infty} q_n(t) z^n\ .$$

The resulting in (9) is a linear differential equation in $H(z,t)$ and its solution is given by

$$H(z,t).\exp\left[-\left(\lambda z+\frac{\mu}{z}\right)t\right]=\int_0^t G(u)\exp\left[\left(\lambda z+\frac{n\mu}{z}\right)(t-u)\right].du+C \tag{10}$$

Put t = 0 in (10), then

$$C=H(z,0) \quad \text{and} \quad H(z,0)=z^a\left[(a\mu_1+c_1\mu_1+\mu-\lambda z(1+2\beta))(1-\delta_{0a})-\lambda z\delta_{0a}\right]$$

where δ_{0a} is Kronecker delta. So (10) can be written in the following form

$$\begin{aligned} H(z,t)&=\exp\left[\left(\lambda z+\frac{n\mu}{z}\right)t\right].z^a\left[(a\mu_1+c_1\mu_1+\mu-\lambda z\,(1+2\beta))\,(1-\delta_{0a})-\lambda z\delta_{0a}\right] \\ &+\int_0^t G(u)\exp\left[\left(\lambda z+\frac{n\mu}{z}\right)(t-u)\right].du \end{aligned} \tag{11}$$

Since

$$\exp\left\{\left(\lambda z+\frac{\mu}{z}\right)t\right\}=\sum_{n=-\infty}^{\infty}(vz)^n I_n(rt)$$

where $I_n(.)$ is the modified Bessel function with $r=2\sqrt{\lambda\mu}$ and $v=\sqrt{\lambda/\mu}$, then (8) is equivalent to

$$\begin{aligned} H(z,t)&=\sum_{n=-\infty}^{\infty}(vz)^n I_n(rt).z^a\left[(a\mu_1+c_1\mu_1+\mu-\lambda z(1+2\beta))(1-\delta_{0a})-\lambda z\delta_{0a}\right] \\ &+\int_0^t G(u)\sum_{n=-\infty}^{\infty}(vz)^n I_n(r(t-u)).du \end{aligned} \tag{12}$$

Comparing the coefficients of z^n in both sides in (9) for $n\ge 1$, one gets

$$\begin{aligned} q_n(t)&=(a\mu_1+c_1\mu_1+\mu)(1-\delta_{0a})v^{n-a}\,I_{n-a}(rt)-\lambda(1+2\beta)(1-\delta_{0a})v^{n-a-1}\,I_{n-a-1}(rt) \\ &-\lambda\delta_{0a}\,v^{n-a-1}\,I_{n-a-1}(rt)+\int_1^t G(u)\,v^n\,I_n(r(t-u)).du \end{aligned} \tag{13}$$

Using $q_n(t)=0$ for n<0 and $I_n(u)=I_{-n}(u)$ in (12) we find

$$\begin{aligned} \int_0^t G(u)\,v^n\,I_n(r(t-u)).du&=-(a\mu_1+c_1\mu_1+\mu)(1-\delta_{0a})v^{n-a}\,I_{n+a}(rt)+\lambda(1+2\beta)(1-\delta_{0a})v^{n-a-1}\,I_{n+a+1}(rt) \\ &+\lambda\delta_{0a}\,v^{n-a-1}\,I_{n+a+1}(rt) \end{aligned} \tag{14}$$

From (13) and (14) we obtain

$$\begin{aligned} q_n(t)&=(a\mu_1+c_1\mu_1+\mu)(1-\delta_{0a})v^{n-a}\left[I_{n-a}(rt)-I_{n+a}(rt)\right] \\ &-\lambda(1+2\beta)(1-\delta_{0a})v^{n-a-1}\left[I_{n-a-1}(rt)-I_{n+a+1}(rt)\right] \\ &-\lambda\delta_{0a}\,v^{n-a-1}\left[I_{n-a-1}(rt)-I_{n+a+1}(rt)\right] \end{aligned} \tag{15}$$

for n = 1,2,….

From (7) and by iteration method one can get

$$P_n(t)=\frac{\rho^n}{n!}P_0(t)+\frac{1}{\mu_1}\sum_{k=1}^{n}q_k(t)e^{-(\lambda+k\mu_1)t}\,\frac{(k-1)!\rho^{n-k}}{n!}\,,\ 1\le n\le c_1-1\,, \tag{16}$$

$$P_n(t) = \frac{\rho^{n-c_1+1}\beta^{n-c_1}}{c_1^{\,n-c_1+1}} e^{(1-\beta)\lambda t} P_{c_1-1}(t) + \frac{1}{\mu_1}\sum_{k=0}^{n-c_1} q_{c_1+k}(t) e^{-(\beta\lambda+c_1\mu_1)t} \frac{(\beta\rho)^{n-c_1-k}}{c_1^{\,n-c_1-k+1}}, \quad c_1 \le n \le N, \tag{17}$$

and

$$P_n(t) = (\beta\rho_1)^{n-N+1} P_{N-1}(t) + \frac{1}{\mu}\sum_{k=0}^{n-N} q_{N+k}(t) e^{-(\beta\lambda+\mu)t} (\beta\rho_1)^{n-N-k}, \quad n \ge N+1 \tag{18}$$

where $\rho = \frac{\lambda}{\mu_1}$ and $\rho_1 = \frac{\lambda}{\mu}$.

Substitute the value of $q_n(t)$ from (15) in (16), then the value of $P_0(t)$ is

$$P_0(t) = \int_0^t q_1(u) e^{-(\lambda+\mu_1)u}.du + \delta_{0a}.$$

SOME SPECIAL CASES

Case 1: The queue $M/M/1$

Let $C_1 = 1, C_2 = 0$, $\beta = 1$ and $N \to \infty$, then the results of Parthasarathy[1] are got as

$$P_n(t) = \rho^n P_0(t) + \frac{1}{\mu}\sum_{k=1}^{n} q_k(t) e^{-(\lambda+\mu)t} \rho^{n-k}$$

and

$$\begin{aligned} q_n(t) &= \mu(1-\delta_{0a}) v^{n-a} [I_{n-a}(rt) - I_{n+a}(rt)] \\ &\quad - \lambda\delta_{0a} v^{n-a-1} [I_{n-a-1}(rt) - I_{n+a+1}(rt)] \end{aligned}$$

Case 2: The queue $M/M/1$ **with balking**

Let $C_1 = 1, C_2 = 0$ and $N \to \infty$, then the results of R.O. Al-Seedy and K.A.M. Kotb[3] are got as

$$P_n(t) = \beta^{n-1}\rho^n e^{(1-\beta)\lambda t} P_0(t) + \frac{1}{\mu}\sum_{k=1}^{n} q_k(t) e^{-(\beta\lambda+\mu)t} (\beta\rho)^{n-k}$$

and

$$\begin{aligned} q_n(t) &= \mu(1-\delta_{0a}) v^{n-a} [I_{n-a}(rt) - I_{n+a}(rt)] \\ &\quad - \lambda v^{n-a-1} [\beta + \delta_{0a}(1-\beta)][I_{n-a-1}(rt) - I_{n+a+1}(rt)] \end{aligned}$$

Case 3: The queue $M/M/1$ **with an additional server**

Let $C_1 = 1, C_2 = 0$ and $t \to \infty$ then the results of Bhat[4] can be obtained as

$$P_n = \rho^n P_0, \qquad \text{for} \qquad 1 \le n \le N$$

$$P_n = \frac{1}{2^{n-N}} \rho^n P_0, \qquad \text{for} \qquad n \ge N+1$$

and

$$P_0 = \frac{(1-\rho)(2-\rho)}{2-\rho-\rho^{N+1}}$$

Case 4: The queue $M/M/1$ **with steady state**

Let $C_1 = 1, C_2 = 0, \beta = 1$, $N \to \infty$ and $t \to \infty$, then we find the results in Harris[6]

$$P_n = \rho^n P_0 \quad \text{for } n \ge 1, \text{ where } P_0 = 1-\rho$$

CONCLUSIONS

In this research, we obtained the transient probabilities of the M/M/C_1 Queue with additional C_2 servers for longer queues and balking. The Method is based on generating function technique, the obtained expression are in form terms of modified Bessel function.

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