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Some Subordination Results Associated With Certain Subclass of Analytic Meromorphic Functions

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Abstract: For functions belonging to each of the subclasses $S^*_w(\beta)$ and $C^*_w(\beta)$ of normalized analytic functions in the open unit disk D, which are investigated in this paper when $0\le\beta<1$, the authors derive several subordination results involving the Hadamard product (or convolution) of the associated functions. A number of interesting consequences of some of these subordination results are also discussed.

Key words: Univalent functions, convex functions, subordination principle, hadamard product (or convolution), subordinating factor sequence

INTRODUCTION

Let A be the class of functions f normalized by:

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^n \qquad (1)$$

which are analytic in the open unit disk $D = \{z\in C: |z| \le 1\}$.

As usual, we denote by S he subclass of A, consisting of functions which are also univalent in D. We recall here the definitions of the well-known classes of starlike function and convex functions:

$$S^* = \left\{ f \in A : \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in D \right\},$$

$$C^* = \left\{ f \in A : \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, z \in D \right\},$$

Let w be a fixed point in D and A (w) = {f∈H (D): f(w) = f'(w)-1=0}.
In[15], Kanas and Ronning introduced the following classes S_w = {f∈A (w): f is univalent in D} $C^*_W = \left\{ f \in A(w) : \mathrm{Re}\left(1 + \frac{(z-w)f''(z)}{f'(z)}\right) > 0, z \in D \right\}$. Later Acu and Owa[1] studied the classes extensively.

Let S_W denoted the subclass of $A(w)$ consisting of the function of the form:

$$f(z) = \frac{\alpha}{z-w} + \sum_{n=1}^{\infty} a_n (z-w)^n \qquad (2)$$

$(a_n \ge 0, z\in D)$. where α = Res (z,w), 0<α≤1 with $z \ne w$.

The class s^*_w is defined by geometric property that the image of any circular arc centered at w is starlike with respect to f (w) and the corresponding class C^*_w is defined by the property that the image of any circular arc centered at w is convex.

We observe that the definitions are somewhat similar to the ones introduced by Goodman in[13,14] for uniformly starlike and convex functions, except that in this case the point w is fixed.
The functions f (z) in S_w is said to be starlike functions of order β if and only if:

$$\mathrm{Re}\left\{\frac{(z-w)f'(z)}{f(z)}\right\} > \beta \quad (z\in D) \qquad (3)$$

for some $\beta(0 \le \beta < 1)$. We denote by $S^*_w(\beta)$ the class of all starlike functions of order β.

Similarly, a functions f (z) in S_w is said to be convex of order β if and only if:

$$\mathrm{Re}\left\{1 + \frac{(z-w)f''(z)}{f'(z)}\right\} > \beta \quad a(z\in D) \qquad (4)$$

for some $\beta(0 \le \beta < 1)$.

It follows from the definitions 3 and 4 that:

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$$f(z) \in S_W^*(\beta) \Leftrightarrow zf'(z) \in C_W^*(\beta) \quad (5)$$

We denote by C^*_w (β) the class of all convex functions of order β.

For the function f (z) in the class S_w, we define:

- $I^0 f(z) = f(z)$
- $I^1 f(z) = (z-w)f'(z) + \frac{2\alpha}{z-w}$
- $I^2 f(z) = (z-w)(I^1 f(z))' + \frac{2\alpha}{z-w}$

and for k = 1,2,3,... we can write:

$$\begin{aligned} I^k f(z) &= (z-w)(I^{k-1} f(z))' + \frac{2\alpha}{z-w} \\ &= \frac{\alpha}{z-w} + \sum_{n=1}^{\infty} n^k a_n (z-w)^n \end{aligned} \quad (6)$$

The differential operator I^* studied extensively by[10,11] and in the case w = 0 was given by[9].

We note that the class S^*_0 (β) and various other subclasses of S^*_0 (β) have been studied rather extensively by[1-8,10-12,16-25].

Next, we will recall each of the following coefficient inequalities associated with the function classes S^*_w (k, β) and C^*_w (k, β) as well as some significant definitions which will contribute to this study.

Definitions and preliminaries: Theorem A[11] if $f \in S_w$, given by 2, satisfies the coefficient inequality:

$$\sum_{n=1}^{\infty} n^k (n+\beta) a_n \le \alpha(1-\beta) \quad (7)$$

with $\beta (0 \le \beta < 1)$ and $0 < \alpha \le 1$, then $f \in S_W^*(k,\beta)$.

Theorem B: If $f \in S_W$, given by 2, satisfies the coefficient inequality:

$$\sum_{n=1}^{\infty} n^{k+1} (n+\beta) a_n \le \alpha(1-\beta) \quad (8)$$

with $\beta (0 \le \beta < 1)$ and $0 < \alpha \le 1$, then $f \in C_W^*(k,\beta)$.

Proof: It is easy to check that if:

$$f(z) \in S_W^*(\beta) \Leftrightarrow zf'(z) \in C_W^*(\beta)$$

Then we have $f \in C_W^*(k,\beta)$. Hence the theorem.

In view of Theorem A and Theorem B, we now introduce the subclasses $S_w^*(\beta) \subset ST_w(\beta)$ $C_w^*(\beta) \subset CV_w(\beta)$ which consist of functions $f \in S_w$ whose Taylor-Maclaurin coefficients a_n satisfy the inequalities 3 and 4, respectively.

In our proposed investigation of functions in the classes S^*_w (β) and C^*_w (β) we shall also make use of the following definitions and results.

Definition 1: (Hadamard Product or Convolution). Given two functions f, $g \in S_w$ where f is given by 5 and g (z) is defined by:

$$g(z) = \frac{\alpha}{z-w} + \sum_{n=1}^{\infty} b_n (z-w)^n \quad (9)$$

$(b_n \ge 0, z \in D)$. The Hadamard product (or convolution) f*g is defined (as usual) by:

$$\begin{aligned} (f*g)(z) &= \frac{\alpha}{z-w} + \sum_{n=1}^{\infty} a_n b_n \\ (z-w)^n &= (g*f)(z) \end{aligned} \quad (10)$$

Definition 2: (Subordination Principle). For two functions f and g, analytic in D, we say that the function F (z) is subordinate to g (z) in D and write f $f \prec g$ or $f(z) \prec g(z)$.

If there exists a Schwarz function w (z), analytic in D with w (0) = 0 and |w (z)|<1 such that $f(z) = g(w(z))$.

In particular, if the function g is univalent in D, the above subordination is equivalent to $f(0) = g(0)$ and $f(D) \subset g(D)$.

Definition 3: (Subordinating Factor Sequence). A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever f (z) of the form (2) is analytic, univalent and convex in D, we have the subordination given by:

$$\begin{aligned} &\sum_{n=1}^{\infty} a_n b_n (z-w)^n \prec f(z) \\ &(z \in D, a_1 = 1) \end{aligned} \quad (11)$$

Theorem C: (cf. Wilf [26]). The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if:

$$\Re\left(1 + 2\sum^{\infty} b_n z^n\right) > 0, \ (z \in D) \quad (12)$$

Subordination results for the classes: $S_w^*(\beta)$ AND ST_w (β) Our first main result (Theorem 1 below) provides a sharp subordination result involving the function class $S_w^*(\beta)$.

Theorem 1: Let the function f defined by 2 be in the class $S_w^*(\beta)$. Also let Ω denote the familiar class of functions $f \in S_w$ which are also univalent and convex in D, then:

$$\frac{1+\beta}{1+\beta+\alpha-\alpha\beta}(f*g)(z) \prec g(z) \tag{13}$$

$(z \in D, 0 \le \beta < 1, 0 < \alpha \le 1)$ and

$$\Re(f(z)) > \frac{1+\beta+\alpha-\alpha\beta}{2(1+\beta)} \tag{14}$$

The following constant factor in the subordination result (13):

$$\frac{1+\beta}{1+\beta+\alpha-\alpha\beta}$$

cannot be replaced by a larger one.

Proof: Let $f \in S_w^*(\beta)$ and suppose that:

$$g(z) = \frac{\alpha}{z-w} + \sum_{n=1}^{\infty} c_n (z-w)^n \in \Omega.$$

Then we readily have:

$$\frac{1+\beta}{1+\beta+\alpha-\alpha\beta}(f*g)(z) = \frac{1+\beta}{1+\beta+\alpha-\alpha\beta}\left(\frac{\alpha}{z-w} + \sum_{n=1}^{\infty} c_n a_n (z-w)^n\right) \tag{15}$$

Thus, by Definition 3, the subordination result 13 will hold true if:

$$\left\{\frac{1+\beta}{1+\beta+\alpha-\alpha\beta} a_n\right\}_{n=1}^{\infty} \tag{16}$$

is a subordinating factor sequence (with, of course, $a_1 = 1$).

In view of Theorem C, this is equivalent to the following inequality:

$$\Re\left(1 + 2\sum_{n=1}^{\infty} \frac{1+\beta}{1+\beta+\alpha-\alpha\beta} a_n (z-w)^n\right) > 0 \tag{17}$$

$(z \in D)$

Now, since $(n+\beta), (0 \le \beta < 1)$ is an increasing function of n, we have:

$$\begin{aligned}
&\Re\left(1 + 2\sum_{n=1}^{\infty} \frac{1+\beta}{1+\beta+\alpha-\alpha\beta} a_n (z-w)^n\right) \\
&= \Re\left(1 + \frac{2}{1+\beta+\alpha-\alpha\beta}\sum_{n=1}^{\infty} (1+\beta) a_n (z-w)^n\right) \\
&\ge 1 - \frac{2}{1+\beta+\alpha-\alpha\beta}\sum_{n=1}^{\infty} (n+\beta)|a_n| r^n \\
&> 1 - \frac{2\alpha(1-\beta)}{1+\beta+\alpha-\alpha\beta} r > 0 \\
&(|z-w| = r < 1)
\end{aligned} \tag{18}$$

where we have also made use of the assertion 7 of Theorem A. This evidently proves the inequality 17 and hence also the subordination result 13 asserted by Theorem 1.

The inequality 14 follows from 7 upon setting:

$$\begin{aligned}
g(z) &= \frac{1}{z-w}\left(\frac{\alpha}{1-(z-w)}\right) \\
&= \frac{\alpha}{z-w} + \sum_{n=1}^{\infty} (z-w)^n \in \Omega
\end{aligned} \tag{19}$$

Next we consider the function:

$$q(z) = \frac{\alpha}{z-w} - \frac{2\alpha(1-\beta)}{1+\beta+\alpha-\alpha\beta}(z-w) \quad (0 \le \beta < 1) \tag{20}$$

which is a member of the class $S_w^*(\beta)$. Then, by using 13, we have:

$$\frac{1+\beta}{1+\beta+\alpha-\alpha\beta} q(z) \prec \frac{1}{z-w}\left(\frac{\alpha}{1-(z-w)}\right), \quad (z \in D) \tag{21}$$

It is also easily verified for the function q (z) defined by 20 that:

$$\min\left\{\Re\left(\frac{1+\beta}{1+\beta+\alpha-\alpha\beta} q(z)\right)\right\} = \frac{-\alpha}{2} \tag{22}$$

which completes the proof of Theorem 1.

Corollary: Let the function f defined by 2 be in the class $ST_W(\beta)$. Then the assertions 13 and 14 of Theorem 1 hold true. Furthermore, the following constant factor:

$$\frac{1+\beta}{1+\beta+\alpha-\alpha\beta}$$

cannot be replaced by a larger one.

By taking α = 1 in the above corollary, we obtain.

Corollary: Let the function f defined by 2 be in the class $ST_W(\beta)$. Then

$$\left(\frac{1}{2}(1+\beta)\right)(f*g)(z)\prec g(z) \qquad (23)$$

and

$$\Re(f(z))>-\frac{1}{1+\beta} \qquad (24)$$

The constant factor $\left(\frac{1}{2}(1+\beta)\right)$ in the subordination result 25 cannot be replaced by a larger one.

Subordination results for the classes: $C^*_W(\beta)$ and $CV_W(\beta)$ Our proof of Theorem 2 below is much akin to that of Theorem1. Here we make use of Theorem B in place of Theorem A.

Theorem 2: Let the function f defined by 2 be in the class $C^*_W(\beta)$. Then:

$$\frac{1+\beta}{1+\beta+\alpha-\alpha\beta}(f*g)(z)\prec g(z) \qquad (25)$$

$(z\in D, 0\le\beta<1, 0<\alpha\le 1)$ and

$$\Re(f(z))>\frac{1+\beta+\alpha-\alpha\beta}{2(1+\beta)} \qquad (26)$$

The following constant factor in the subordination result 25:

$$\frac{1+\beta}{1+\beta+\alpha-\alpha\beta}$$

cannot be replaced by a larger one.

Corollary: Let the function f defined by 2 be in the class $CV_W(\beta)$. Then the assertions 25 and 26 of Theorem 2 hold true. Furthermore, the following constant factor:

$$\frac{1+\beta}{1+\beta+\alpha-\alpha\beta}$$

cannot be replaced by a larger one.

By taking α = 1 in the above corollary, we obtain.

Corollary: Let the function f defined by 2 be in the class $CV_W(\beta)$. Then

$$\left(\frac{1}{2}(1+\beta)\right)(f*g)(z)\prec g(z) \qquad (27)$$

and

$$\Re(f(z))>-\frac{1}{1+\beta} \qquad (28)$$

The constant factor $\left(\frac{1}{2}(1+\beta)\right)$ in the subordination result 27 cannot be replaced by a larger one.

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