On a Class of Nonhomogeneous Fields in Hilbert Space

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Abstract: Two-parametric semigroups of operators in Hilbert space with bounded infinitesimal doubly commuting operators are studied. The characteristics describing deviation of a semigroup from unitary one, when infinitesimal operators are unitary, in particular, nonunitary index, have been introduced. Necessary and sufficient conditions for nonunitary index finiteness have been obtained.

Keywords: Nonhomogeneous Fields, Multi-parametric Semigroup, Doubly Commuting Operators

INTRODUCTION

One-parametric semigroups of operators were studied adequately, both from theoretical and applied pointviews [1]. A few works in harmonic analysis are devoted to study multi-parametric semigroups [2, 3]. We study the nonhomogeneous field $u(x_1, x_2)$ in Hilbert space $H$ which is presented in the form

$$u(x_1, x_2) = e^{ix_1 T_1 + ix_2 T_2} u_0,$$

where $u_0 \in H$, $T_1$ and $T_2$ are bounded doubly commuting operators [4]. Consider a scalar product

$$\langle u(x_1, x_2), u(y_1, y_2) \rangle_H = K(x_1, y_1; x_2, y_2).$$

Then if $T_j = T_j^*$ ($j = 1, 2$), the function $K(x_1, y_1; x_2, y_2)$ depends only on corresponding differences $K(x_1 - y_1; x_2 - y_2)$ and the field is homogenous.

If $T_1 \neq T_1^*$ or $T_2 \neq T_2^*$ or both operators $T_j$ ($j = 1, 2$) are non self-adjoint operators, then the field $u(x_1, x_2)$ is nonhomogeneous. In addition, if $T_j$ ($j = 1, 2$) belongs to a certain class of non self-adjoint operators, one may invoke spectral theory of doubly commuting non self-adjoint operators to study the field $u(x_1, x_2)$.

Functional Characteristic of the Nonhomogeneous Field: Consider the case when $T_j$ ($j = 1, 2$) are doubly commuting unitary or quasi-unitary operators and introduce some numerical and functional characteristics, describing deviation of the field in the form

$$u(x_1, x_2) = e^{ix_1 T_1 + ix_2 T_2} u_0,$$

where $T_j$ are unitary operators. Note that for unitary doubly commuting operators (we call the corresponding field to be unitary) function $K(x_1, y_1; x_2, y_2)$ may be presented in the form

$$K(x_1 - y_1; x_2 - y_2, x_1 + y_1; x_2 + y_2) = \int_0^{2\pi} e^{i(x_1 - y_1) \cos f_1(\lambda) + i(x_2 - y_2) \cos f_2(\lambda)} \times e^{-(x_1 + y_1) \sin f_1(\lambda) - (x_2 + y_2) \sin f_2(\lambda)} dF_\lambda,$$

where, $f_k(\lambda)$ real-value functions, $\Delta F_\lambda = \langle \Delta E_\lambda u_0, u_0 \rangle$, and $E_\lambda$ is the spectral function of unitary operator $T_0 = \int_0^{2\pi} e^{i\lambda} dE_\lambda$.

The above form of $K$ follows from the Neuman theorem for generating operator $T_0$ of a set of mutually commuting selfadjoint (unitary) operators [5].

Taking into the account the well-known fact for commuting operators $T_1$ and $T_2$ one of them is a function of another [5]. It is not difficult to verify that if $T_1$ and $T_2$ are the unitary commutative operators then the function $K(x_1, y_1; x_2, y_2)$ satisfies the following equation

$$L_{s_j} K(x_1, y_1; x_2, y_2) = 0, \quad (j = 1, 2) \quad (2)$$
where
\[ L_{xy} = I - \frac{\partial^2}{\partial x \partial y} . \]

From the applied point of view \( K(x_1, y_1; x_2, y_2) \) is the correlation function for some random field, because \( K(x_1, y_1; x_2, y_2) \) is Hermitian nonnegative function. Hence there exists Gaussian normal field for which \( K(x_1, y_1; x_2, y_2) \) is the correlation function and the results obtained may be interpreted as a correlation theory for nonhomogeneous random field. Hereafter we will consider that \( H = H_0 = \sum_{x_1, y_2 \geq 0} T_{x_1}^* T_{y_2} u_0, \quad (x_j \text{ are integers}). \)

Let us consider the field
\[ u^*(x_1, x_2) = e^{i\tau_1 T_{x_1}^* + i\tau_2 T_{y_2}^*} u_0, \]
which, henceforth, we will call it the adjoint field.

It is obvious that for the field \( e^{-i\tau_1 T_{x_1}^* + i\tau_2 T_{y_2}^*} u \) (\( T_1 \) and \( T_2 \) double commuting operators) to be unitary it is necessary and sufficient that \( K \) should be in accordance with
\[ L_{x_1, y_1} K(x_1, y_1; x_2, y_2) = 0, \quad (j = 1, 2) \]

**Lemma 1:** Let \( H_u = H_0^* = H \), and \( u(x_1, x_2) = e^{i\tau_1 T_{x_1}^* + i\tau_2 T_{y_2}^*} u_0 \). Then the necessary and sufficient for \( T_1 \) and \( T_2 \) to be commutative is that
\[ \frac{\partial^2}{\partial x_1 \partial y_2} \widetilde{K}(x_1, y_1; x_2, y_2) = \frac{\partial^2}{\partial x_2 \partial y_1} \widetilde{K}(x_1, y_1; x_2, y_2), \]
where
\[ \widetilde{K}(x_1, y_1; x_2, y_2) = \{u(x_1, x_2), u^*(y_1, y_2)\}. \]

The lemma proof follows from the definition of the function, \( \widetilde{K}(x_1, y_1; x_2, y_2) \) and a relationship
\[ \frac{\partial^2 \widetilde{K}}{\partial x_1 \partial y_2} = -\{T_{x_1} T_{y_2} u(x_1, x_2), u^*(y_1, y_2)\}. \]

If \( L_{x_1, y_1} L_{x_2, y_2} K(x_1, y_1; x_2, y_2) \neq 0 \), then the function
\[ W(x_1, y_1; x_2, y_2) = L_{x_1, y_1} L_{x_2, y_2} K(x_1, y_1; x_2, y_2) \]
may be considered as a functional characteristic of deviation infinitesimal commutative operators \( T_1 \) and \( T_2 \) from unitary operators.

If \( T_1 \) and \( T_2 \) are doubly commuting operators (\( [T_1, T_2] = 0, \ [T_1, T_2]^* = 0 \)), then from (3) we may obtained the following presentations for \( W \):
\[ W(x_1, y_1; x_2, y_2) = \left\{ (I - T_1 T_2^*) (I - T_2 T_1^*) u(x_1, x_2), u(y_1, y_2) \right\}. \]

The presentation (4) is significant for further studies.

**Remark 1:** To reconstruct \( K(x_1, y_1; x_2, y_2) \) by \( W(x_1, y_1; x_2, y_2) \) one may solve Darboux-Goursat problem for equation
\[ L_{x_1, y_1} L_{x_2, y_2} K(x_1, y_1; x_2, y_2) = W(x_1, y_1; x_2, y_2) \]
twice, and defining appropriate conditions additionally.

**Remark 2:** If the operators \( T_1 \) and \( T_2 \) are commuting operators, but are not doubly commuting, then \( W(x_1, y_1; x_2, y_2) = \left\{ (I - T_1 T_2^*) (I - T_2 T_1^*) u(x_1, x_2), u(y_1, y_2) \right\} \) and further analysis is based on assumption of commutant \([T_1, T_2^*]\) properties, for example \( T_1, T_2^* \) and \([T_1, T_2^*]\) form Lie algebra.

**Theorem 1:** If \( \dim H_0 = r < \infty \), where
\[ H_0 = (I - T_1 T_1^*) H \cap (I - T_2 T_2^*) H, \]
then
\[ W(x_1, y_1; x_2, y_2) = \sum_{a=1}^{r} \lambda_a \Phi_a(x_1, x_2) \Phi_a(y_1, y_2), \]
where \( \Phi_a(x_1, x_2) = \{u(x_1, x_2), h_a\}, h_a \in H_0, \) and \( \lambda_a \) are real numbers.

**Proof:** Consider the orthonormal basis \( \{h_{a\alpha}\}^r_{\alpha=1} \) in \( H_0 \), consisting of eigenvector contraction of self-adjoint operator \((I - T_1 T_1^*)(I - T_2 T_2^*)\) onto its invariant subspace \( H_0 \). Since
\[ B_H = (I - T_1 T_1^*)(I - T_2 T_2^*)u(x_1, x_2) \]
\[ = \sum_{a=1}^r \langle Bu(x_1, x_2), h_a \rangle h_a \]
\[ = \sum_{a=1}^r \langle u(x_1, x_2), Bh_a \rangle h_a \]
\[ = \sum_{a=1}^r \lambda_a \langle u(x_1, x_2), h_a \rangle h_a, \]
where \( Bh_a = \lambda_a h_a \) and \( \lambda_a \) are eigenvalues of the operator \( B \).

As a result, we obtain
\[ W(x_1, y_1; x_2, y_2) = \sum_{a=1}^r \lambda_a \Phi_a(x_1, x_2) \Phi_a(y_1, y_2). \]

Remark: that the function \( K(x_1, y_1; x_2, y_2) \) defines the Hilbert-valued function \( u(x_1, x_2) \) quite completely. The next assertion is valid.

**Lemma 2:** Consider the two functions \( u_1(x_1, x_2) \) and \( u_2(x_1, x_2) \) with values belonging to the Hilbert spaces \( H_{u_1} = \bigvee_{x_1, x_2 \geq 0} u_j(x_1, x_2) \) respectively, where the scalar product is generated by the respective function \( K(x_1, y_1; x_2, y_2) = \langle u_j(x_1, x_2), u_j(y_1, y_2) \rangle_{H_j} \) \( = K_j(x_1, y_1; x_2, y_2). \)

If \( K_1(x_1, y_1; x_2, y_2) = K_2(x_1, y_1; x_2, y_2) \), then there exists a unitary transformation \( U \in [H_1, H_2] \) such that \( u_2(x_1, x_2) = U u_1(x_1, x_2). \) Moreover if \( u(x_1, x_2) = e^{i x_1 T_1 + i x_2 T_2} u_0 \), then \( u_2(x_1, x_2) \) is also generated by two-parametric semigroup of operators \( u_2(x_1, x_2) = e^{i x_1 T_1 + i x_2 T_2} u_0. \)

**Proof:** Consider lineals
\[ L_j = \left\{ \sum_{a, \beta=1}^{n_1, n_2} C_{a, \beta} u_j(x_\alpha, x_\beta) \right\}, \quad n_1, n_2 < \infty, \]
where, \( C_{a, \beta} \) are complex numbers. For \( h_1^{(j)}, h_2^{(j)} \in L_j \) define binary form
\[ \{ h_1^{(j)}, h_2^{(j)} \}_{L_j} = \sum_{a, \beta=1}^{n_1, n_2} \sum_{p, q=1}^{m_1, m_2} C_{a, \beta} \phi_{p, q} K_j(x_\alpha, y_\beta; x_\beta, y_\alpha), \]
where,
\[ h_1^{(j)} = \sum_{a, \beta=1}^{n_1, n_2} C_{a, \beta} u_j(x_\alpha, x_\beta), \]
\[ h_2^{(j)} = \sum_{p, q=1}^{m_1, m_2} \phi_{p, q} u_j(x_p, x_q). \]

Then \( L_j \) become pre-Hilbert spaces. Define isometric (by virtue of equality \( K_1(x_1, y_1; x_2, y_2) = K_2(x_1, y_1; x_2, y_2) \)), transformation of \( L_1 \) into \( L_2 \):
\[ U \left\{ \sum_{a, \beta=1}^{n_1, n_2} C_{a, \beta} u_1(x_\alpha, x_\beta) \right\} = \left\{ \sum_{a, \beta=1}^{n_1, n_2} C_{a, \beta} u_2(x_\alpha, x_\beta) \right\}. \]

Extending \( U \) for closures \( L_1 \) and \( L_2 \) we get the first assertion of the Lemma. The second part of the Lemma follows immediately from the evident relationships:
\[ u_2(x_1, x_2) = U u_1(x_1, x_2) = U e^{i x_1 T_1 + i x_2 T_2} u_0 = e^{i x_1 T_1 + i x_2 T_2} u_0, \]
where \( B_j = U T_j U^{-1}, u_0 = U u_0. \)

**Nonunitary index:** Let us now define a numerical characteristic for the field deviation from the unitary field. Let us call the nonunitary index the maximal rank of quadratic forms
\[ \sum_{j, m=1}^n W(x_1^{(j)}, y_1^{(j)}; x_2^{(m)}, y_2^{(m)}) Z_j Z_m, \quad n < \infty. \]
For the unitary field a nonunitary property coefficient is equal to 0, since \( W(x_1, y_1; x_2, y_2) = 0 \).

**Theorem 2:** In order that the field \( u(x_1, x_2) = e^{i x_1 T_1 + i x_2 T_2} u_0 \) has a finite nonunitary index it is necessary and sufficiently that \( \dim H_0 = r < \infty \), where \( T_1 \) and \( T_2 \) are doubly commuting operators and...
Proof:
Sufficiency: When \( \dim H_0 = r < \infty \), there exists representation (5) for \( W(x_1, y_1; x_2, y_2) \) and
\[
\sum_{\ell, m=1}^{n} W(x_1^{(\ell)}, y_1^{(\ell)}; x_2^{(m)}, y_2^{(m)}) Z_{\ell} \overline{Z}_{m} = \sum_{v=1}^{2} \lambda_{v} | \zeta_{v} |^{2},
\]
where \( \zeta_{v} = \sum_{\ell=1}^{n} \Phi_{v}(x_1^{(\ell)}, x_2^{(\ell)}) Z_{\ell} \). It follows that the rank of quadratic form does not exceed \( r \).

Necessity: Let us consider the sequence of pairs of real numbers
\[
x_{\ell} = (x_1^{(\ell)}, x_2^{(\ell)}) , \quad (\ell = 1, n).
\]
Then
\[
\sum_{\ell, m=1}^{n} W(x_{\ell}, x_m) Z_{\ell} \overline{Z}_{m} = \{(I - T_1 T_1)(I - T_2 T_2) h, h\}
\]
where \( h = \sum_{\ell=1}^{n} Z_{\ell} \mu(x_1^{(\ell)}, x_2^{(\ell)}) \).
Let
\[
H_n = \left\{ h : h = \sum_{\ell=1}^{n} Z_{\ell} \mu(x_1^{(\ell)}, x_2^{(\ell)}) \right\}, \quad H_n \subseteq H_u.
\]
Consider the subspace
\[
G_n = P_n (I - T_1 T_1)(I - T_2 T_2) P_n H_u, \quad \text{where} \quad P_n \quad \text{is the projection operator onto subspace} \quad H_n.
\]
It is obvious that \( G_n \subseteq P_n H_0 \) and the rank of form
\[
\sum_{\ell, m=1}^{n} W(x_{\ell}, x_m) Z_{\ell} \overline{Z}_{m} \]
is equal to \( \dim G_n \). It is evident that \( H_1 \subseteq H_2 \subseteq \ldots \subseteq H_n \subseteq \ldots \) and
\[
\lim_{n \to \infty} P_n = I , \quad \text{hence rank} \quad W > \dim G_n \quad \text{and} \quad \lim_{n \to \infty} G_n = \dim H_0 \quad \text{This implies that rank} \quad \dim H_0 \leq r .
\]
Similarly one may prove the next theorem.

Theorem 3: In order that the field
\[
u(x_1, x_2) = e^{ix_1 + ix_2} u_0,
\]
has a finite nonunitary index it is necessary and sufficient that the subspaces
\[
H_0^{(j)} = (I - T_j T_j) H \quad (j = 1, 2)
\]
be finite-dimensional where, \( u_0 \in H \), \( T_j \) are doubly commuting operators.

Further development of suggested approach is related to the spectral theory for the doubly commuting contraction systems and their triangular and universal models\(^6\). Thus, one may derive canonical representation for \( W(x_1, y_1; x_2, y_2) \) and perform harmonic analysis of two-parametric semigroups \( e^{ix_1 T_1 + ix_2 T_2} \) when \( T_1 \) and \( T_2 \) are doubly commuting contractions.

REFERENCES