Measures on the Quotient Spaces of the Integers

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Abstract: SK-partitions were introduced by Sharma and Kaushik, who defined distances (metrics) between vectors in terms of partitions of the alphabet set $F_q$, the set of the ring of integers modulo $q$. These distances were applied in Coding Theory. The research examined algebraic and topological aspects of SK-partitions and related sets of measures. A lattice of SK-partitions was introduced and shown to have distributive sub lattices. Generator sets of the lattice were obtained and the structure of its ideals and filters examined. Corresponding results for lattices of measures were presented.

Keywords: Measure, Lattice, Metric, Ideal, Generator set

INTRODUCTION

Measures on the quotient spaces of the integers were studied by Niederreiter and Sookoo[9] in connection with the uniform distribution of sequences. Systems of measures were studied by Maharan[8], who considered a family of measures with orthogonality properties and also by Schmid[10], who proved that a certain ordered Banach space of vector measures is a Banach lattice. We examined the algebraic aspects of systems of measures defined on quotient spaces of the integers modulo $q$ for different natural numbers $q$. These measures were defined in terms of SK-partitions of the ring of integers modulo $q$. Studies in Coding Theory involving SK-Partitions were carried out by Kaushik[2,3,4,5,6,7], Sharma and Dial[11] and Sharma and Kaushik[12,13,14]. From results obtained we deduced comparable results for systems of measures.

DEFINITIONS AND NOTATIONS

Notation: Let $F_q = \{0, 1, \ldots, q-1\}$ be the ring of integers modulo $q$, $q \in \{2, 3, \ldots\}$.

Definition: Given $F_q$, $q \geq 2$, a partition $P = \{B_0, B_1, \ldots, B_{q-1}\}$ of $F_q$ is called an SK-partition if

1. $B_0 = \{0\}$, and $q-a \in B_i$ if $a \in B_j$, $i = 1, 2, \ldots, m-1$
2. If $a \in B_j$ and $b \in B_j$; $i, j=0,1,\ldots,m-1$, and if
   j precedes i in the order of the partition $P$, written as $i>j$, then $\min\{a, q-a\} > \min\{b, q-b\}$.
3. If $i>j \ (i, j \in \{0,1,\ldots,m-1\})$ and $i \neq m-1$, then
   
   $|B_i| = |B_j| + |B_{m-1}| - 1 - |B_{m-2}|$
   where $|B_i|$ stands for the size of the set $B_i$.

Notation: $\mathcal{F}_P$ is the set of all SK-partitions.

The concepts of a generator set, an ideal and a filter are well known in lattice theory, Birkhoff[1].

Definition: $G$ is said to be a generator set of a lattice $L$ if every element of $L$ is the upper bound of elements of $G$.

Definition: Let $(L, \leq)$ be a lattice. A subset $A$ of $L$ is called an ideal, if

1. $a, b \in A \Rightarrow a \lor b \in A$
2. $a \in A$ and $c \in L \Rightarrow c \leq a \Rightarrow c \in A$

Definition: Let $(L, \leq)$ be a lattice. A subset $H$ of $L$ is called a filter if

1. $a, b \in H \Rightarrow a \land b \in H$
2. $a \in H$ and $c \in L \Rightarrow c \geq a \Rightarrow a \in H$.

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Notation: For an SK-partition $P$ of $F_q$, $|P|$ denotes the number of classes of $P$, and $[P]_i$ denotes the $(i+1)$th class, for $i = 0, 1, \ldots, |P| - 1$. Also if $P = \{B_0, B_1, \ldots, B_{m-1}\}$, then $B_i^1$ denotes $\left\{x \in B_i \mid x < \frac{q}{2}\right\}$ and $B_i^2$ denotes $\left\{x \in B_i \mid x > \frac{q}{2}\right\}$, i.e., $i = 1, 2, \ldots, m-1$.

Note: Clearly $\mu_P \leq \mu_Q \iff P \leq_s Q$

Remark: Clearly, from the above definition $P \leq_s Q \iff |B_i| \leq |C_i|$, $i = 0, 1, \ldots, m-1$.

Theorem 3.1: $\leq_s$ is a partial ordering on $\mathfrak{S}_P$.

Proof: Let $P, Q \in \mathfrak{S}_P \Rightarrow P \leq_s Q$ and $Q \leq_s P$.

Then $m \leq m'$ and $m' \leq m$.

$\therefore m = m'$

(I) Also, $q \leq q'$ and $q' \leq q$

$q = q'$

(II) We also have $|B_i| \leq |C_i|$ and $|C_i| \leq |B_i|$, $i = 0, 1, \ldots, m-1$

$\therefore |B_i| = |C_i|$, $i = 0, 1, \ldots, m-1$

(III) From (I), (II) and (III), it follows that $P=Q$. Hence $\leq_s$ is antisymmetric. Also, $\leq_s$ is reflexive and transitive.

Corollary 3.2: $\leq_\mu$ is a partial ordering on $M_p$.

The following example showed that $\leq_s$ is not linear.

Example. Let $P = \{B_0, B_1, B_2, B_3\}$

where $B_0 = \{0\}$

$B_1 = \{1, 2, 21, 22\}$

$B_2 = \{3, 4, 5, 6, 17, 18, 19, 20\}$

$B_3 = \{7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$

and let $Q = \{C_0, C_1, C_2, C_3, C_4\}$

where $C_0 = \{0\}$

$C_1 = \{1, 2, 3, 26, 27, 28\}$

$C_2 = \{4, 5, 6, 23, 24, 25\}$

$C_3 = \{7, 8, 9, 10, 19, 20, 21, 22\}$

$C_4 = \{11, 12, 13, 14, 15, 16, 17, 18\}$

$|B_1| = 4 < 6 = |C_1|$

$\therefore Q \not\leq_s P$

also $|B_3| = 10 > 8 = |C_3|$

$\therefore P \not\leq_s Q$.

We devised a more convenient notation for an SK-partition which was expressed in terms of its class-
sizes, and which would also uniquely determine the SK-partition.

**Theorem 3.3:** $a_1, a_2, a_3, \ldots, a_{m-1}$ are positive, even integers satisfying $a_1 \leq a_2 \leq \ldots \leq a_{m-1}$ \iff there exists a unique SK-partition $P = \{B_0, B_1, \ldots, B_{m-1}\}$ in $\mathfrak{S}_p$ such that $|B_i| = a_i; i = 1, 2, \ldots, m-1$.

**Proof:** Suppose that $a_1, a_2, \ldots, a_{m-1}$ are positive, even integers such that $a_1 \leq a_2 \leq \ldots \leq a_{m-1}$ and that $q = 1 + \sum_{i=1}^{m-1} a_i$.

Any class of an SK-partition of $F_q$ is of the form $$\{x, x+1, \ldots, x+a, q-(x+a), q-(x+a-1), \ldots, q-x\}.$$ Also, from condition (2) of an SK-partition, $B_i$ and $B_{i+1}$ can only be classes of an SK-partition $P = \{B_0, B_1, \ldots, B_{m-1}\}$ of $F_q$ if the elements of $B_i$ less than $q/2$ are all less than the elements of $B_{i+1}$ less than $q/2, (i = 1, 2, \ldots, m-2)$. The only partition $P = \{B_0, B_1, \ldots, B_{m-1}\}$ of $F_q$ satisfying conditions (1) and (2) of an SK-partition must satisfy and following:

- $B_0 = \{0\}$
- $B_1 = \left\{ x \in F_q \mid 1 \leq x \leq \frac{a_1}{2} \right\}$
- $B_i = \left\{ x \in F_q \mid 1 + \sum_{j=1}^{i-1} \frac{a_j}{2} \leq x \leq \sum_{j=1}^{i} \frac{a_j}{2} \right\}$, $i = 2, 3, \ldots, m-1$.

$P$ also satisfies condition (3) of an SK-partition since $|B_i| = a_i$ (i = 1, 2, \ldots, m-1) and so $|B_i| \leq |B_2| \leq \ldots \leq |B_{m-1}|$.

Hence $P$ satisfies the three conditions of an SK-partition. Since $|B_i|$ is even for $i \in \{1, 2, \ldots, m-1\}$ and $|B_1| \leq |B_2| \leq \ldots \leq |B_{m-1}|$, $P \in \mathfrak{S}_p$.

Conversely, suppose that $P \in \mathfrak{S}_p \Rightarrow P = \{B_0, B_1, \ldots, B_{m-1}\}$ and $|B_i| = a_i (i = 1, 2, \ldots, m-1)$.

Since $P \in \mathfrak{S}_p$, then $a_i$'s are positive and even for $i = 1, 2, \ldots, m-1$ and also $a_1 \leq a_2 \leq \ldots \leq a_{m-1}$.

**Remark:** From the previous theorem, any SK-partition in $\mathfrak{S}_p$ was determined by the sizes of its classes, so the notation given below was used.

**Notation:** If $P \in \mathfrak{S}_p \Rightarrow P = \{B_0, B_1, \ldots, B_{m-1}\}$ and $|B_i| = a_i (i = 1, 2, \ldots, m-1)$, we shall denote $P$ by $\{(1, a_1, a_2, \ldots, a_{m-1})\}$, when it is convenient.

**THE LATTICES** $(\mathfrak{S}_p, \leq_s)$ and $(M_p, \leq)$

We presented the l.u.b and g.l.b of any two elements in $(\mathfrak{S}_p, \leq_s)$ and then established that it is a lattice.

**Lemma 4.1:** Let $P = \{B_0, B_1, \ldots, B_{m-1}\}$ and $Q = \{C_0, C_1, \ldots, C_{m-1}\}, (m \leq m')$ be any two elements of the poset $(\mathfrak{S}_p, \leq_s)$.

The l.u.b of $P$ and $Q$ is $G$ where $G = \{G_0, G_1, \ldots, G_{m-1}\}$, $|G_i| = \max \{|B_i|, |C_i|\}$; $i = 0, 1, \ldots, m-1$, and $|G_i| = \max \{|B_{m-1}|, |C_i|\}$; $i = m, m+1, \ldots, m' - 1$.

Also $G \in \mathfrak{S}_p$.

**Proof:** $|G_i| = \max \{|B_i|, |C_i|\} \leq \max \{|B_{i+1}|, |C_{i+1}|\} = |G_{i+1}|$ for $i = 1, 2, \ldots, m - 2$.

Also, for $i = m - 1, m, m + 1, m + 2, \ldots, m' - 2$ $|G_i| = \max \{|B_{m-1}|, |C_i|\} \leq \max \{|B_{m-2}|, |C_{i+1}|\} = |G_{i+1}|$

From Theorem 3.3, $G$ is an SK-partition. $G$ is an upper bound of $P$ and $Q$ since
If $K$ is another upper bound of $P$ and $Q$, then $G \leq K$, since $|G_i|$ is the smallest number satisfying $|B_i| \leq |G_i|; i = 0,1,\ldots,m-1$ and $|C_i| \leq |G_i|; j = 0,1,\ldots,m'-1$. 

$G$ is the l.u.b of $P$ and $Q$. 

$G_i$ is even for $(i = 1,2,\ldots,m'-1)$ and $G_i \leq |G_{i+1}|$ for $i = 1,2,\ldots,m'-2$. $G \in \mathcal{S}_p$.

**Lemma 4.2:** Let $P$ and $Q$ and $(\mathcal{S}_p,\leq)$ be as in Lemma 4.1. The g.l.b of $P$ and $Q$ is $H$, where $H = \{H_0,H_1,\ldots,H_m\}$ and $|H_i| = \max\{|B_i|,|C_i|\}; i = 0,1,\ldots,m-1$.

Also $H \in \mathcal{S}_p$.

**Proof:**

$|H_i| = \min\{|B_i|,|C_i|\} \leq \min\{|B_{i+1}|,|C_{i+1}|\} = |H_{i+1}|$

$s_i = 0,1,\ldots,m-2$.

$H$ is an SK-partition from Theorem 3.3. $H$ is a lower bound of $P$ and $Q$.

If $L$ is another lower bound of $P$ and $Q$, then $L \leq H$, since $|H_i|$ is the largest number satisfying $|H_i| \leq |B_i|; i = 0,1,\ldots,m-1$, and $|H_i| \leq |C_i|; j = 0,1,\ldots,m-1$.

Hence $H$ is the g.l.b of $P$ and $Q$. Clearly $|H_i|$ is even for $i = 1,2,\ldots,m-1$ and $|G_i| \leq |G_{i+1}|$ for $i = 1,2,\ldots,m-2$. Hence $H \in \mathcal{S}_p$.

From the foregoing the following result was obvious.

**Theorem 4.3:** $(\mathcal{S}_p,\leq)$ is a lattice.

**Corollary 4.4:** $(M_{p},\leq)$ is a lattice. 

$(\mathcal{S}_p,\leq)$ is not distributive, as shown:

**Example:** Let $P, Q$ and $R$ be elements of $(\mathcal{S}_p,\leq)$ such that $P = (1,2,6,6,8,8,10,14)$, $Q = (1,2,4,4,4,6,6,10)$ and $R = (1,2,8,10)$.

Now, $Q \vee R = (1,2,8,10,10,10,10,10)$ and $P \wedge (Q \vee R) = (1,2,2,2,6,6,8,8,10)$.

Hence $P \wedge (Q \vee R) \neq P \wedge Q \vee P \wedge R$.

Also, the following example showed that $(\mathcal{S}_p,\leq)$ is not modular.

**Example:** Let $P = (1,2,4,4,4,8,10,12)$, $Q = (1,6,8,10)$ and $R = (1,2,4,6,6)$.

We have, $P \geq R$

$Q \vee R = (1,6,8,10,10,10)$

$P \wedge (Q \vee R) = (1,2,2,4,4,4,6,8)$

$P \wedge Q = (1,2,4,4,4)$

$(P \wedge Q) \vee R = (1,2,4,6,6,6)$

Hence $P \wedge (Q \vee R) \neq (P \wedge Q) \vee (P \wedge R)$ and so $(\mathcal{S}_p,\leq)$ is not modular.

**THE SUBLATTICES** $(\mathcal{S}_{p,m},\leq)$ and $(M_{p,m},\leq)$

**Notation:** For a fixed integer $m \in \{2,3,\ldots\}$, let $\mathcal{S}_{p,m} = \{P \in \mathcal{S}_p | P has m classes\}$ and let $M_{p,m} = \{\mu \in M_p | P has m classes\}$.

The following result was deduced.

**Theorem 5.1:** $(\mathcal{S}_{p,m},\leq)$ is a sub lattice of $(\mathcal{S}_p,\leq)$ for each $m \in \{2,3,\ldots\}$.

**Remark:** For any natural number $n$ all codes over $(F_p)^n$ had the same maximum weight with respect to any element to any element of $\mathcal{S}_{p,m}$. All codes have the same maximum distance with respect to any element of $\mathcal{S}_{p,m}$.
Theorem 5.2: \( (\mathcal{S}_{p,m}, \leq_s) \) is a distributive lattice.

Proof: Let \( P, Q \) and \( R \) be fixed, arbitrary elements of \( \mathcal{S}_{p,m} \).

We show that \( P \land (Q \lor R) = (P \land Q) \lor (P \land R) \).

\[
\|P \land (Q \lor R)\| = \|P \land (Q \lor R)\| = \|P \land Q \lor (P \land R)\|,
\]

\( i = 0,1,\ldots,m-1 \).

Let:

\[
H_i = \left\lceil P \land (Q \lor R) \right\rceil = \min \left\{ \left\lceil P \right\rceil, \left\lceil Q \right\rceil, \left\lceil R \right\rceil \right\}, i = 0,1,\ldots,m-1.
\]

\[
K_i = \left\lceil (P \land Q) \lor (P \land R) \right\rceil = \max \left\{ \left\lceil P \right\rceil, \left\lceil Q \right\rceil, \left\lceil R \right\rceil \right\}, i = 0,1,\ldots,m-1.
\]

\( H_i = K_i, i = 0,1,\ldots,m-1 \).

If \( \|P\| \geq \|Q\| \geq \|R\| \), then

\[
H_i = \min \left\lceil P \right\rceil, \left\lceil Q \right\rceil = \left\lceil Q \right\rceil = H_i
\]

For other relative sizes of \( \left\lceil P \right\rceil, \left\lceil Q \right\rceil \) and \( \left\lceil R \right\rceil \), similarly \( H_i = K_i, i = 0,1,\ldots,m-1 \).

Hence \( P \land (Q \lor R) = (P \land Q) \lor (P \land R) \).

Corollary 5.3: \( (M_{p,m}, \leq \mu) \) is a distributive lattice.

**GENERATOR SETS OF \( (\mathcal{S}_{p,m}, \leq_s) \) AND \( (M_{p,m}, \leq \mu) \)**

Notation: Let \( G_{p,m} = \bigcup_{i=1}^{m-1} G_{p,m,i} \)

Lemma 6.1. \( G_{p,m} \) is a generator set of \( (\mathcal{S}_{p,m}, \leq_s) \) where \( m = 2, 3, \ldots \)

Proof: We showed that any element of \( \mathcal{S}_{p,m} \) is the upper bound of elements of \( G_{p,m} \).

Let \( (\{a_1, a_2, \ldots, a_{m-1}\}) \) for \( m = 2, 3, \ldots \)

be fixed, arbitrary elements of \( \mathcal{S}_{p,m} \).

each SK-partition on the R.H.S is in \( G_{p,m} \).

In view of the above, we inferred the following result:

Theorem 6.2: \( G_{p} \) is a generator set for \( (\mathcal{S}_{p}, \leq_s) \), where \( G_{p} = \bigcup_{m=2}^{\infty} G_{p,m} \).

Notation: Let \( G_{p,m} = \{ \mu \in M_p \mid P \in G_{p,m} \} \) and \( G_{p,m} = \{ \mu \mid P \in G_{p} \} \)

Corollary 6.3: \( G_{p,m} \) is a generator set for \( (M_{p}, \leq \mu) \).

Notation: Let \( K \) be infinite, increasing sequence of positive, even numbers, namely, \( k_1, k_2, \ldots \).

Also

\[
H_{p,m,k,i} = \left\lceil (1, 2, 3, \ldots, a_{m-i}, a_{m-i+1}, \ldots, a_{m-1}) \right\rceil
\]

for \( i \in \{ 1, 2, \ldots, m-2 \} \),

\[
H_{p,m,k,m-1} = \left\lceil (a_1, a_2, \ldots, a_{m-1}) \right\rceil
\]

and

\[
H_{p,m,K} = \bigcup_{i=1}^{m-1} H_{p,m,K,i}
\]

Theorem 6.4: \( G_{p,m}, H_{p,m,K}, H_{p,m,K}, \ldots \) form an infinite chain of generators for \( (\mathcal{S}_{p,m}, \leq_s) \), where \( K \) is an infinite, increasing sequence of positive, even
numbers and \( K_{h+1} \) is an infinite subsequence of \( K_h (h = 1,2,...) \).

**Proof:** Let \( j \) be a fixed, arbitrary element of \( \{1,2,...\} \). We showed that \( H_{p,m,K_j} \) is a generator of \( \left( \mathfrak{Z}_p, \leq \right) \) by proving that any element of \( G_{p,m} \) was obtained by performing lattice operations on elements of \( H_{p,m,K_j} \).

Let \( K_j = k_{j_1}, k_{j_2}, ... \) be an increasing sequence of positive, even numbers and let \( A = (1,2,2,...,2,a,a,...,a) \) be an arbitrary element of \( G_{p,m} \) for some fixed, positive integer such that \( A \notin G_{p,m,m-1} \). Then \( A \in G_{p,m,j} \) for some \( i \in \{1,2,...,m-2\} \).

\[ A = ((1,2,2,...,2,a,a,...,a)) \]

\[ \leftarrow \begin{array}{c} \text{times (m-1) times} \\ \text{where b \in \{k_{j_1}, k_{j_2}, \ldots \} \geq b \geq a.} \end{array} \]

\[ ((1,2,2,...,2,b,b,...,b)) \in H_{p,m,k_i,j} \]

\[ \leftarrow \begin{array}{c} \text{times (m-1) times} \\ \text{where a \in \{1,2,2,...,2,a,a,...,a\}} \end{array} \]

\[ ((1,2,2,...,2,a,a,...,a)) \in H_{p,m,K_j,m-1} \]

We have shown that \( A \) is the g.l.b of elements of \( H_{p,m,K_j} \).

If \( B \in G_{p,m,m-1} \), then \( B \in H_{p,m,K_j} \).

Any element of \( G_{p,m} \) either already was an element of \( H_{p,m,K_j} \) or could be obtained by taking the g.l.b. of two elements of \( H_{p,m,K_j} \).

**Notation:** Let \( H_{\mu,P,m,k} = \{ P = H_{p,m,k} \mid P \in H_{p,m,k} \} \).

**Corollary 6.5:** \( H_{\mu,P,m,k,1}, H_{\mu,P,m,k,2}, ... \) form an infinite chain of generators for \( \left( M_{p,m} \leq \mu \right) \).

We next considered SK-partitions with bounded class sizes. In the next theorem, we showed that some sets of such partitions are ideals of \( \left( \mathfrak{Z}_p, \leq \right) \).

**Theorem 7.1:** Every ideal \( H \) of \( \left( \mathfrak{Z}_p, \leq \right) \) has one of the forms:

\( (i) H = H_{p,m,K_j} \)

\[ \begin{align*} &P \in \mathfrak{Z}_p \\
&\{ P = \{ B_0, B_1, ..., B_{m-1} \}, m \leq d, \\
&\text{and either (1) } \\
&\text{or (2) } |B_i| \leq b_i (i = 1,2,...,h), \text{if } h \leq m - 1 \\
&\text{and } b_1, b_2, ..., b_h \text{ are fixed, arbitrary, positive integers such that } h \leq d - 1 \end{align*} \]

\( (ii) \) \( H \) has the same form as in (ii), except \( m \) is not bounded.

**Proof:** Let \( H \) be an ideal of \( \left( \mathfrak{Z}_p, \leq \right) \), \( d = \max \{ m : P = \{ B_0, B_1, ..., B_{m-1} \} \in H \} \)

\[ \begin{align*} &P = \{ B_0, B_1, ..., B_{m-1} \} \in H, \text{ and either} \\
&\text{or (2) } |B_i|, |B_{i+1}|, ..., |B_{m-1}| \text{ are bounded above if } r > m - 1 \end{align*} \]

\( H \) takes different forms, depending on whether \( h \) exists and \( d \) exists.

\( (i) \) If \( h \) does not exist, then \( d \) does not exist and \( H = \mathfrak{Z}_p \).

\( (ii) \) (a) If \( h \) and \( d \) exist, and \( h < d - 1 \), let

\[ b_i = \max \left\{ |B_i| : P = \{ B_0, B_1, ..., B_{m-1} \} \right\} \in H, i = 1,2,...,h \]

\[ Q_i = b_i (i = 1,2,...,h) \]

From the definition of \( b_i \) \( (i = 1,2,...,h) \), \( \exists Q \in H \ni Q = \left\{ Q_0, Q_1, Q_2, ..., Q_{m-1} \right\}, m - 1 \geq i \)

and \( |Q_i| = b_i (i = 1,2,...,h) \).
Let $D = Q_1 \lor Q_2 \lor \ldots \lor Q_h$.

From property (1) of an ideal, $D \in H$.

Clearly $D = \{D_0, D_1, D_2, \ldots, D_{n-1}\}$, for some integer $n \geq 1$, and $|D_i| = b_i (i = 1, 2, \ldots, h)$.

\[
\therefore b_1 \leq b_2 \leq \ldots \leq b_n.
\]

Let $K = \{P \in \mathcal{S}_p \mid (1) P \leq b_i (i = 1, 2, \ldots, h), \text{if } h \leq m-1, \text{or } (2) P \leq b_i (i = 1, 2, \ldots, m-1 \text{if } h > m-1)\}$.

We proved that $H = K$.

Let $L = \{L_0, L_1, \ldots, L_{n-1}\}$ be an arbitrary element of $H$, for some positive integer $n_2$.

Then either (1) $|L_i| \leq b_i (i = 1, 2, \ldots, h)$ if $h \leq n_2 - 1$ or (2) $|L_i| \leq b_i (i = 1, 2, \ldots, n_2 - 1)$ if $h > n_2 - 1$ from the definition of $b_i (i = 1, 2, \ldots, h)$.

\[
\therefore L \in H \Rightarrow L \in K, \quad (I)
\]

Let $M$ be an arbitrary element of $K \ni \|M\| \leq h$.

Then $M \leq_d D$.

Since $D \in H$, from property (ii) of an ideal $M \in H$.

Now, let $N = \{N_0, N_1, \ldots, N_{n_3 - 1}\}$ (where $n_3$ is a positive integer) be an arbitrary element of $K \ni \|N\| > h$.

Since only the first $(h + 1)$ classes of elements of $H$ are bounded (for those elements of $H$ having $h + 1$ or more classes),

\[
\exists T = \{T_0, T_1, \ldots, T_{n_3 - 1}\} 
\in H \ni (n_3 \leq n_2), \exists \|N_0\| \leq \|T_{n_3 - 1}\|.
\]

Also, $\exists U = \{U_0, U_1, \ldots, U_{n_4 - 1}\} \in H \ni n_4 \leq n_3$, from the definition of $d$.

Let $V = D \lor T \lor U$.

From property (i) of an ideal, $V \in H$.

Since $N \leq_s V$, from property (ii) of an ideal, $N \in H$.

\[
\therefore N \in K \Rightarrow N \in H
given that $K \subseteq H$.
\]

From (I) and (II)

\[
H = K
\]

(b) If $H = d - 1$, the proof is similar.

If $h$ exists and $d$ does not exist, the proof is also similar to the proof in (ii).

**Corollary 7.2:** If $H_{\mu_{\mu'}} = \{\mu \in M_p \mid P \in H\}$ where $H$ is an ideal of $(\mathcal{S}_p, \leq_s)$, then $H_{\mu_{\mu'}}$ is an ideal of $(M_p, \leq_\mu)$.

The next result was expressed in terms of filters of $(\mathcal{S}_p, \leq_s)$.

**Theorem 7.3:** Every filter $H$ of $(\mathcal{S}_p, \leq_s)$ has the form $H = \{P \in \mathcal{S}_p \mid P \geq D\}$ for some fixed element $D$ of $\mathcal{S}_p$.

**Proof:** Let $H$ be a filter of $(\mathcal{S}_p, \leq_s)$ and let $h + 1 = \min\{m \in N \mid P = \{B_0, B_1, \ldots, B_{m-1}\} \in H\}$.

Also let $b_i = \min\{b_i \in N \mid P = \{B_0, B_1, \ldots, B_{m-1}\} \in H\}$.

From the definition of $h + 1$, $\exists$ some element $C = \{C_0, C_1, \ldots, C_h\} \in H$, and from the definition of $b_i (i = 1, 2, \ldots, h)$, $\exists Q_i \in H \ni$

\[
Q_i = \{Q_{i0}, Q_{i1}, Q_{i2}, \ldots, Q_{i(m-1)}\},
\]

$|Q_i| = b_i (i = 1, 2, \ldots, h)$, and $h \leq m_i - 1$.

Let $D = C \lor Q_1 \lor Q_2 \lor \ldots \lor Q_h$.

From property (i) of a filter $D \in H$.

$D = \{D_0, D_1, D_2, \ldots, D_h\}$,

where $|D_i| = b_i (i = 1, 2, \ldots, h)$ and $H = \{P \in \mathcal{S}_p \mid P \geq D\}$.

**Corollary 7.4:** Every filter $K$ of $(M_p, \leq_\mu)$ has the form $K = \{\mu \in M_p \mid P \geq D\}$, where $D$ is a fixed element of $\mathcal{S}_p$.

**REFERENCES**