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The Extended Laplace Transform

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Abstract: The space of new generalized functions has been constructed. The operation of associative multiplication has been defined on this space. The Extended Laplace Transform has been defined.

Key words: New generalized functions, extended Laplace Transform, associative multiplication, semi norms, topology. 44A10,46F12

INTRODUCTION

If $f(t)$ is defined for $t \geq 0$, then the improper integral $\int_0^\infty K(s,t)f(t)dt$ has many important applications. The choice $K(s,t) = e^{-st}$ gives us an especially important integral transform said to be Laplace transform of f, provided the integral converges.

In the linear mathematical model for a physical system such as a spring /mass system or a series electrical circuit, the right-hand member of the differential equation

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = f(t) \text{ or}$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

is a driving function and represents either an external force $f(t)$ or an impressed voltage $E(t)$. Solving these differential equations in general is difficult but not impossible . The Laplace transform is an invaluable tool in solving problems such as these .

Moreover the Laplace transform can be used for solving the Voltera integral equations $f(t) + \lambda\int_a^t K(t,\tau)f(\tau)d\tau = g(t)$, and integro differential equations $Dy = f(t) + \lambda\int_a^t y(\tau)K(t,\tau)d\tau$, where D is the differential operator.

In (1) we defined the space $\zeta(E)$ as a factor space $T^*(E)/I^*(E)$ [5] and we proved many Important results for this space. Also we have defined the extended Fourier Transform $\overline{F}$: $\zeta(S(R)) \rightarrow \zeta(S(R))$.

Also in algebras $\xi(E)$ constructed in(1-5) all the operations of multiplication convolution, differentiation are defined.

There arises a natural question : How is to define the Laplace transform in those algebras ?

In (6) the spaces $\prod(R)$,and the space of New Generalized functions $\zeta(\prod(R))$ were constructed so that

$$\prod{}'(R) \subset S'(R) \subset \xi(S(R)) \subset \xi(\prod(R)),$$

where $\zeta(S(R))$ - the space of New Generalized functions constructed in (1); $S(R)$ – the space of test functions of rapid decay; $S'(R)$ - the space of tempered distributions.

We also use the definitions ,and some results in(6) .Lat us repeat some of them which are used throughout this paper .

Define the space $\prod(R) = \prod_1(R) \cup \prod_3(R)$ where

$$\prod{}_1 = \left\{\eta(t) \in C^\infty(R) : \lim_{t\to\infty} t^n\eta^{(k)}(t) = 0 \ , \ \forall \ n, \ k \in Z\right\}$$

$$\prod{}_2 = \left\{\eta(t) \in C^\infty(0,\infty) : \lim_{t\to\infty} t^n\eta^{(k)}(t) = 0 \ , \ \forall \ n, \ k \in Z\right\}$$

$$\prod{}_3 = \left\{ g(t) = \eta(|t|) : \eta(t) \in \prod{}_2(R)\right\}.$$

We define topology on $\prod(R)$ by the following semi-norms:

$P_\alpha(\eta(t)) = P_{n,l}(\eta(t)) = \sup\limits_{k\le n, m\le l} q_{k,m}(\eta(t))$ *where* $q_{k,m}(\eta(t)) = \sup\limits_{t\in(0,\infty)} \left|t^k \eta^{(m)}(t)\right|$

The embedding of algebra $\zeta(S(R))$ in to the algebra $\zeta(\prod(R))$ is defined by the following mapping:

$$J_\pi : (\lambda_k) + I(S(R)) \to (\lambda_k) + I(\prod(R))$$

So we get the following results :

$$\prod{}^l(R) \subset S^l(R) \subset \zeta(S(R)) \subset \zeta(\prod(R)) .$$

In algebra $\zeta(\prod(R))$ we define the associative multiplication for $\lambda = (\lambda_k) + I^*(\prod(R))$, $\gamma = (\gamma_k) + I^*(\prod(R))$ by

$$\lambda \Theta \gamma = (\lambda_k . \gamma_k) + I^*(\prod(R)) .$$

Theorem 1.1: The operation of multiplication Θ is independent of a representative .

Construction of $K(C)$ AND $\zeta(K(C))$

Let $K(a)$ be the set of all complex –valued functions analytic in the half plane $\mathrm{Re}(z) \ge a > 0$, and $\lim\limits_{z\to\infty} F(z) = 0$ in $-\dfrac{\pi}{2} + \delta < \arg(z) < \dfrac{\pi}{2} - \delta$ for each $\delta > 0$.

Define the space $K(C) = \bigcup\limits_{a\ge 0} K(a)$, and topology in this space define by the family of the following semi norms :

$$\rho_k(F(z)) = \sup_{\substack{\mathrm{Re}(z)\ge 0 \\ i\le k}} \left|F^{(i)}(z)\right|$$

By T(E) we denote the set of all possible sequences in E, where E be separated locally -convex algebra with topology defined by family of semi norms $(P_\alpha)_{\alpha\in A}$ such that for $\alpha \in A$, there exist $\beta \in A$ a constant $C_\alpha > o$ for which (2.1)

$$\rho_\alpha(\lambda.\gamma) \le C_\alpha P_\beta(\lambda) P_\beta(\gamma) \quad \forall\ \lambda,\ \gamma \in E$$

Let $T^*(E)$ be the set of all sequences $(\lambda_k)_{k=n}^{\infty} \in E$ satisfy the following conditions there is a number m such that for each $\alpha \in A$, there is a nonnegative $\chi_\alpha > 0$ such that $P_\alpha(\lambda_k) \le \chi_\alpha k^m$ for each k. And $I^*(E)$ be the set of all sequences $(\lambda_k)_{k=n}^{\infty} \in E$ satisfy the following conditions for each number m and for each $\alpha \in A$, there is a nonnegative $\chi_\alpha > 0$ such that $P_\alpha(\lambda_k) \le \chi_\alpha k^{-m}$ for each k. The following results are true:

Theorem 2.2: Let E be an algebra satisfies (2.1) then $T^*(E)$ is a sub algebra of algebra T(E) and $I^*(E)$ is an Ideal in $T^*(E)$.

proof. see the prove in(1)

Theorem 2.2: The Space $K(C)$, with the topology given by the semi norms $\rho_k(F(z)) = \sup\limits_{\substack{\mathrm{Re}(z)\ge 0 \\ i\le k}} \left|F^{(i)}(z)\right|$ satisfy the inequality (2.1).

proof: Consider $\rho_k(F(z)G(z)) = \sup\limits_{i\le k} \left|(FG)^{(i)}(z)\right|$

$$= \sup_{i\le k}\left|\sum_{j=0}^{i} c_i^j F^{(j)}(z).G^{(i-j)}(z)\right| \le$$

$$\le \sup_{i\le k}\left|(F)^{(i)}(z)\right| \sup_{i\le k}\left|(G)^{(i)}(z)\right| \sup_{i\le k}\left|\sum_{j=0}^{i} c_i^j\right| \le$$

$$2^i \rho_k(F(z))\rho_k(G(z)) .$$

We construct the algebra $\zeta(K(C))$ as a factor space $\zeta(K(C)) = T^*(K(C)) / I^*(K(C))$.

If L is the Laplace transform , then the image of the space $\prod(R)$ is included in the space $K(C)$ (10), so we can write $L : \prod(R) \to K(C)$.

Theorem2.3: a. If $\lambda \in T^*(\prod(R))$, then $L(\lambda(x)) \in T^*(K(C))$;

b. If $\lambda \in I^*(\prod(R))$, then $L(\lambda(x)) \in I^*(K(C))$.

Proof: Let $\lambda \in T^*(\prod(R))$, then there is a number m such that for each $\alpha \in A$, there is a nonnegative χ_1 such that (2.2)

$$p_\alpha(\lambda_k) \le \chi_1 k^m \text{ for each } k$$

From the continuity of the Laplace transform $L : \prod(R) \to K(C)$ implies that for each j there is i and a constant $c_i > 0$, such that (2.3)

$$p_j(L(\lambda_k)) \le c_{j1} p_i(\lambda_k), \quad \forall k .$$

Now from the inequalities (2.2), and (2.3) we see that there is m for each j there $c_j = c_{j1}\chi_1$ so that

$p_j(L(\lambda_k)) \le c_j k^m, \ \forall k$ .So we conclude that $L(\lambda_k) \in T^*(\prod(R))$.

We can prove the second part in a similar way. Now from the proved theorem 2.3 , and from the properties of the Laplace transform we conclude that the extended Laplace transform

$\tilde{L} : \zeta(\prod(R)) \to \zeta(K(R))$ is defined in the following way : Let $\lambda \in \zeta(\prod(R))$,and let (λ_k) be any representative for λ , then

$$\tilde{L}(\lambda) = (L(\lambda_k)) + I^*(K(C)).$$

Theorem 2.4: The Laplace transform $\tilde{L}$ is independent of a representative .

Proof: Let $\lambda \in \zeta(\prod(R))$, and let (λ_k) , and (γ_k) be any two representatives for λ .That is $(\lambda_k - \gamma_k) \in I^*(\prod(R))$ which means that for each number m and for each $\alpha \in$ A , there is a nonnegative χ_1 such that (2.4)

$$p_\alpha(\lambda_k - \gamma_k) \le \chi_1 k^{-m} \text{ for each k}$$

From the continuity of the Laplace transform $L : \prod(R) \to K(C)$ implies that for each j there is i and a constant $c_i > 0$, such that (2.5)

$$p_j(L(\lambda_k - \gamma_k)) \le c_{j1} p_\alpha(\lambda_k - \gamma_k), \ \forall k$$

Now from the inequalities (2.4), and (2.5) we see that for each j and for each m there is i , and $c_j = c_{j1}\chi_1$ so that

$p_j(L(\lambda_k - \gamma_k)) \le c_j k^{-m}, \ \forall k$ .that is

$$\tilde{L}(\lambda_k - \gamma_k) \in I^*(K(C)).$$

It is worthy to point here that the extended Laplace transform $\tilde{L}$ satisfy all properties of the ordinary Laplace transform (10):

Theorem 2.5: If $f, \ g \in \zeta(\prod(R))$, and

$\tilde{L} f = \tilde{F} \in \zeta(K(C)), \ \tilde{L} g = \tilde{G} \in \zeta(K(C))$, then

1. $\tilde{L}\left[e^{p_0 t} f(t)\right] = \tilde{F}(p - p_0);$
2. $\tilde{L}[(f * g)(t)] = \tilde{F}(p)\tilde{G}(p);$
3. $\tilde{L}[f(\alpha t)] = \frac{1}{\alpha}\tilde{F}(\frac{p}{\alpha});$
4. $\tilde{L}[t^n f(t)] = (-1)^n \tilde{F}^{(n)}(p);$
5. $\tilde{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{\tilde{F}(p)}{p}.$

proof: 2. Let $f, g \in \zeta(\prod(R))$, then $f = (f_k) + I^*(\prod(R))$,and $g = (g_k) + I^*(\prod(R))$ .Consider

$$\tilde{L}[(f * g)(t)] = (L(f_k * g_k)) + I^*(\prod(R)) =$$
$$(L(f_k).L(g_k)) + I^*(\prod(R)) = \tilde{F}(p)\tilde{G}(p).$$

The prove of other properties is similar.
Moreover we get the following commutative diagram:

$$\begin{array}{ccc} & \alpha & \\ \prod(R) & \to & \zeta(\prod(R)) \\ L\downarrow & & \downarrow \tilde{L} \\ K(C) & \to & \zeta(K(C)) \\ & \beta & \end{array}$$

where α, β embeddings defined in the following way:

$$\alpha : \prod(R) \to \zeta(\prod(R)), \quad \alpha(f) = (f_k) \quad \forall k;$$
$$\beta : K(C) \to \zeta(K(C)), \quad \beta(g) = (g_k) \quad \forall k.$$

So the following commutative formula is true :

$$\tilde{L} o \alpha = \beta o L.$$

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