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Cost Elasticities of Reliability and *MTTF* for *k*-out-of-*n* Systems

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Abstract: The application of the concept of cost elasticity of reliability was extended from the parallel system to the partially redundant or *k*-out-of-*n*:G(F) system (or *k*-out-of-*n* system, for short). An expression for the cost elasticity of reliability was derived for a general *k*-out-of-*n* system. The expression yielded acceptable results for a wide range of values for *k*, *n*, and component reliability. For systems of practical interest characterized by good components, the expression became highly susceptible to round-off errors, and catastrophic cancellations took place. These numerical problems seemed unavoidable as they were inherently associated with the definition of the cost-elasticity-of-reliability metric itself. We introduced another metric, the cost elasticity of the Mean Time To Failure (*MTTF*), which measures the relative change in the life expectancy that can be obtained for a given relative change in cost. We believe the cost-elasticity-of-*MTTF* metric is a more tangible and a more cumulative measure than the cost-elasticity-of-reliability metric. We derived a very simple expression for the cost elasticity of *MTTF* for a *k*-out-of-*n* system and showed that it is a function of only *k* and *n*, i.e. it is independent of component characteristics such as component failure rate or component reliability. This expression is insensitive to round-off errors since it is a purely additive formula. We provided charts for the cost elasticity of *MTTF* that can be used to assess the cost incurred in achieving a certain life expectancy for a *k*-out-of-*n* system. These charts can be used with any coherent system, since the *MTTF* for a coherent system can be approximated by that of a *k*-out-of-*n* system.

Key words: Cost, Reliability, Mean time to failure, Parallel system, Partially redundant or *k*-out-of-*n*:G(F) system

INTRODUCTION

An important goal for reliability engineering is to achieve cost minimization [1]. However, this goal has rarely been achieved, primarily because of the lack of suitable mathematical models or metrics [2]. A recently introduced metric that captures the value of reliability from a financial viewpoint is the cost elasticity of reliability [3], defined as

$$\in_{R,C} = (\Delta R / R)/(\Delta C / C). \qquad (1)$$

This metric measures the relative change in reliability R that can be obtained for a given relative change in cost C. As its name indicates, this metric mimics a well-known material constant, viz., the modulus of elasticity which relates an applied stress to the resulting strain or relative change in length [4]. However, the metric $\in_{R,C}$ is more analogous, from a cause-effect point of view, to the price elasticity of demand or supply, a concept well known in microeconomics [5].

The new metric $\in_{R,C}$ was studied in [3] in the case of a parallel system. We extend this study by applying this new metric to a *k*-out-of-*n*:G(F) system. The *k*-out-of-*n*:G(F) system is a system of n components that functions (fails) if at least k out of its n components function (fail). Situations in which this system serves as a useful model are frequently encountered in practice [6]. The *k*-out-of-*n* system plays a central role for the general class of coherent systems, as it can be used to approximate the reliability of such systems [7]. While virtually all nontrivial network reliability problems are known to be NP-hard for general networks, the regular structure of the *k*-out-of-*n* system allows the existence

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of efficient algorithms for its reliability analysis that are of quadratic-time linear-space complexity in the worst case [6]. The *k*-out-of-*n*:G system covers many interesting systems as special cases. These include the perfectly reliable system (k = 0), the parallel system (k = 1), the voting or N-modular redundancy (NMR) system ($k = \lceil (n+1)/2 \rceil$), the fail-safe system ($k=n-1$), the series system ($k=n$), and the totally unreliable system ($k=n+1$). For $1<k<n$ the *k*-out-of-*n* system is sometimes called a partially-redundant system [6], as it lies somewhere between the extreme cases of the (non-redundant) series system and the (fully-redundant) parallel system. The *k*-out-of-*n*:G system and the *k*-out-of-*n*:F system are mirror images of each other; their successes are dual switching functions. The *k*-out-of-*n*:G system is exactly equivalent to the (n-k+1)-out-of-*n*:F system [6].

METHODOLOGY

The methodology adopted combines analysis and simulation. We derive an expression for $\in_{R,C}$ for a parallel system based on the "continuous" limit $\Delta n \rightarrow 0$. We also derive an expression for $\in_{R,C}$ for a general *k*-out-of-*n*:G system, but since it cannot be based on the continuous limit, we base it on what we call the best discrete increment $\Delta n = 1$, since this is the nearest possible increment to the continuous limit. We also introduce another metric, viz., $\in_{T,C}$ or the cost elasticity of the *MTTF*, which we believe is a more tangible and a more accumulative measure than the $\in_{R,C}$ metric. We derive a very simple expression for $\in_{T,C}$ for a *k*-out-of-*n*:G system and show that it is a function of only k and n, i.e. it is independent of component characteristics such as component failure rate or component reliability. Furthermore, we present our experience and observations on computing $\in_{R,C}$ and $\in_{T,C}$.

Cost-reliability characterization for a parallel system: The unreliability of a parallel system with n identical but independent components of component reliability R_0 is given by:

$$1-R = (1-R_0)^n, \tag{2}$$

and hence, the reliability is given by:

$$R = 1-(1-R_0)^n. \tag{3}$$

Following Saleh et al. [3], we let the cost of a single component be C_0, and assume that the cost of n components, C, scales linearly with the number of components, i.e.,

$$C = nC_0. \tag{4}$$

The change ΔR in reliability due to a change Δn in the number of components is:

$$\Delta R = (\Delta R/\Delta n)\Delta n \approx (\partial R/\partial n)\Delta n$$

$$= -(1-R_0)^n \ ln\,(1-R_0)\,\Delta n,$$

$$= (1-R_0)^n \ ln\,\left(\frac{1}{1-R_0}\right)\Delta n. \tag{5}$$

The change in cost ΔC due to a change Δn in the number of components is

$$\Delta C = C_0\,\Delta n. \tag{6}$$

Finally, the cost elasticity of reliability is given by

$$\in_{R,C} = \frac{(1-R_0)^n \ ln \ \frac{1}{(1-R_0)}}{[1-(1-R_0)^n]}\, n. \tag{7}$$

Equation (7) for $\in_{R,C}$ is based on the use of

$$(\partial R/\partial n) = \lim_{\Delta n \rightarrow 0} (\Delta R/\Delta n),$$

i.e., it is obtained for the "continuous" limit $\Delta n \rightarrow 0$. Figure 1 shows a cost-reliability characterization of a parallel structure with n redundant components of a relatively good component reliability $R_0 = 0.9$. Figure 1(a) is a plot of unreliability *(1-R)* versus cost (expressed in units of C_0), which is essentially *(1-R)*, versus n, while Fig. 1(b) depicts cost elasticity of the structure's reliability versus n. Note that the system unreliability curve becomes quickly indistinguishable from the horizontal axis of value 0.

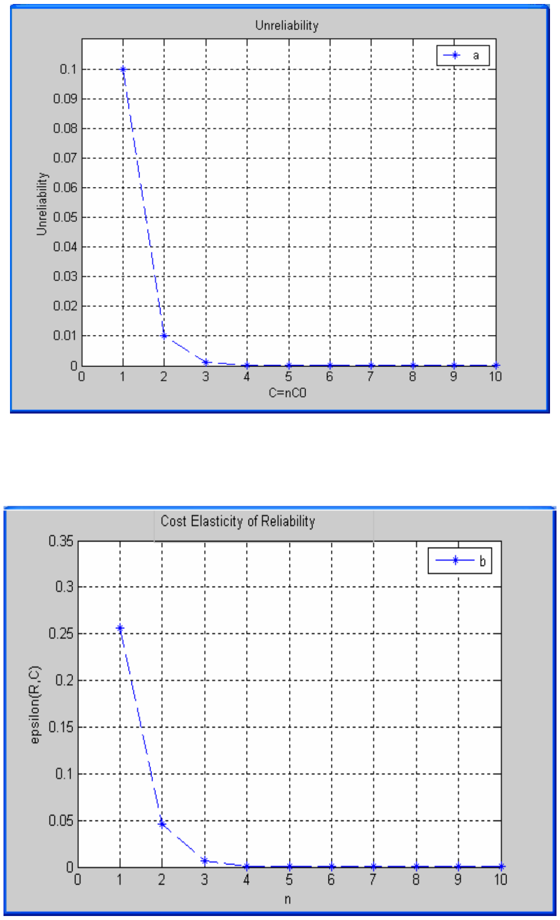

Fig.1: Cost-reliability characterization of a typical parallel structure

Cost-reliability characterization for a *k*-out-of–*n* system: The reliability of a *k*-out-of-*n*:G system (with independent components of identical reliabilities R_0) is given by [6]:

$$R = \sum_{m=k}^{n} c\,(m,n) R_0^{\ m} (1 - R_0)^{n-m} \quad , \qquad \text{(8a)}$$

$$= \sum_{m=k}^{n} (-1)^{m-k} c\,(k-1, m-1)\, c\,(m,n) R_0^{\ m} , \qquad \text{(8b)}$$

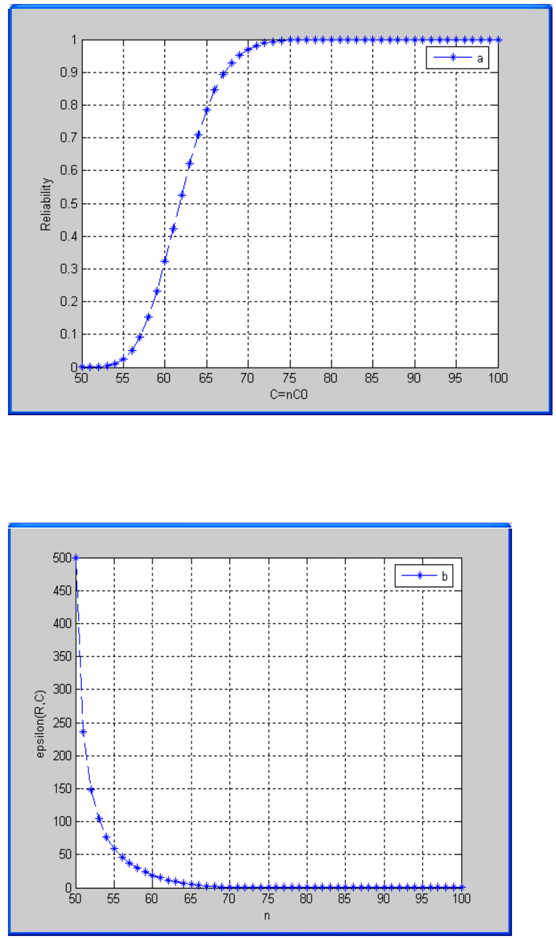

Fig. 2: Cost-reliability characterization of a 50-out-of-n:G system with component reliabilities R_0 = 0.8

where $c(m,n)$ is the combinatorial or binomial coefficient (n choose m). If we let the number of components n change to $(n + \Delta n)$, then the reliability R in (8a) changes to $(R + \Delta R)$ given by

$$R + \Delta R = \sum_{m=k}^{n+\Delta n} \Big(c\,(m, n + \Delta n) R_0^{\ m} (1 - R_0)^{n+\Delta n - m} \Big). \qquad \text{(9)}$$

From (8a) and (9), we can express the change ΔR in reliability due to a unit change Δn=1 in the number of components as

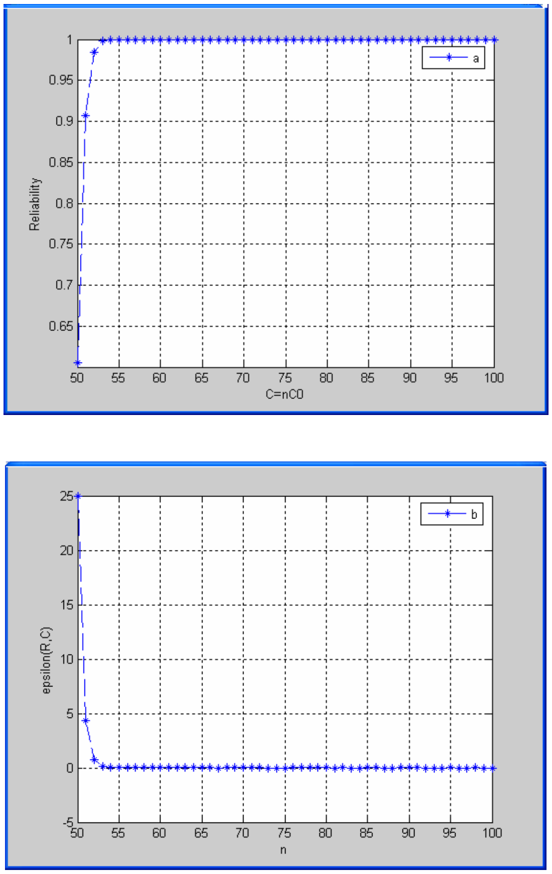

Fig. 3: Cost-reliability characterization of a 50-out-of-n:G system with component reliabilities $R_0 = 0.99$.

$$(\Delta R)_{\Delta n=1} = [c\,(n+1,n+1)R_0^{\;n+1} +$$

$$\sum_{m=k}^{n}(c\,(m,n+1)R_0^{\;m}(1-R_0)^{n+1-m} - c\,(m,n)R_0^{\;m}(1-R_0)^{n-m})]. \quad (10)$$

Using the binomial identifies

$$c\,(n+1,n+1) = 1, \quad (11)$$

$$c\,(m,n+1) - c\,(m,n) = c\,(m-1,n), \quad (12)$$

we reduce (10) to

$$(\Delta R)_{\Delta n=1} = R_0^{\;n+1} + \sum_{m=k}^{n} R_0^{\;m}(1-R_0)^{n-m}\left[c\,(m-1,n) - c\,(m,n+1)R_0\right], \quad (13)$$

and finally, obtain the cost elasticity of reliability as

$$\in_{R,C} = \frac{n}{R}\left[\; R_0^{\;n+1} + \sum_{m=k}^{n} R_0^{\;m}(1-R_0)^{n-m}c\,(m-1,n)\,(1-\frac{n+1}{m}R_0)\;\right]. \quad (14)$$

Formula (8a) is a purely additive formula; it expresses R as the sum of nonnegative terms. Formula (14) is not an additive formula unless $1 > \frac{n+1}{k}R_0$, i.e., unless $R_0 < \frac{k}{n+1}$. Additive formulas have the distinguishing characteristic that they are less prone to the inaccuracies (and never subject to the catastrophic cancellation) caused by round-off errors.

Figures 2 and 3 present a cost-reliability characterization for a 50-out-of-n:G system with component reliabilities R_0 = 0.8 and R_0 = 0.99, respectively.

Formula (14) for the cost elasticity of reliability $\in_{R,C}$ gives satisfactory results up to R_0 = 0.8 (Fig. 2(b)) and then starts to exhibit some unacceptable negative values (values of -0 rather than +0), i.e. it exhibits erratic behavior for very small or negligible values of $\in_{R,C}$ (Fig. 3(b)).

We must stress that the erratic behavior obtained is solely due to aggravated cumulative round-off error and is definitely not a result of some error in formulation or programming. Formula (14) gives acceptable and verifiable results for a wide range of values of k, n, and R_0. However, it fails to assess $\in_{R,C}$ properly for systems having good components (i.e., for systems of practical interest). Anyhow, for such systems $\in_{R,C}$ diminishes and becomes indistinguishable from zero.

Equation (14) for $\in_{R,C}$ is obtained by using the smallest possible nonzero discrete increment Δn = 1. This increment is the best discrete increment since it is the nearest one to the "continuous" limit $\Delta n \to 0$. For a parallel system (k = 1), we have two estimates for $\in_{R,C}$, one based on the continuous limit $\Delta n \to 0$ in equation (7) and another based on the best discrete increment Δn = 1 in equation (14). Our computational experience reveals that there is no significant difference between these two estimates for large n, and small R_0, i.e. when (14) is not spoiled by accumulated round-off errors.

Cost elasticity of *MTTF* for a *k*-out-of-*n* system: From cost considerations, *MTTF* seems to be a more tangible and cumulative measure than reliability itself.

Therefore, we introduce the concept of the cost elasticity of the *MTTF*, which we define as

$$\in_{T,C} = \frac{(\Delta T/T)}{(\Delta C/C)} = \frac{(\Delta T/T)}{(\Delta n/n)}, \qquad (15)$$

where

$$T = MTTF = \int_0^\infty R(t)\,dt. \qquad (16)$$

For a k-out-of-n:G system having components subject to a common constant failure rate (CFR) λ, the component reliability is

$$R_0(t) = e^{-\lambda t}, \qquad t \geq 0, \qquad (17)$$

and the *MTTF* of the system is obtained from equations (8b), (16), and (17) as

$$T = \int_0^\infty \sum_{m=k}^{n} (-1)^{m-k} c\,(k-1, m-1)\, c\,(m,n) (e^{-\lambda t})^m \,dt,$$

$$= \sum_{m=k}^{n} \frac{(-1)^{m-k}}{\lambda m} c\,(k-1, m-1)\, c\,(m,n). \qquad (18)$$

A simpler and a purely additive expression for T, however, can be obtained form the state diagram for a k-out-of-n:G system [8]. When the system is in state $m \geq k$, it has m working components and it behaves exactly as a series system of a failure rate $m\lambda$. The system resides in this state for an average time ($1/(m\lambda)$), and hence the *MTTF* of the system is

$$T = \frac{1}{\lambda} \sum_{m=k}^{n} \frac{1}{m}. \qquad (19)$$

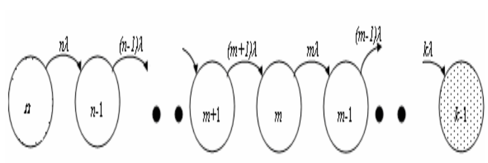

Fig. 4: State diagram for a k-out-of-n:G system, in which each state is indexed by the number of working components, and the shaded state (k-1) is one of catastrophic failure.

If we let the number of components n change to ($n + \Delta n$) in (19), then the *MTTF* changes to ($T + \Delta T$) given by

$$T + \Delta T = \frac{1}{\lambda} \sum_{m=k}^{n+\Delta n} \frac{1}{m}. \qquad (20)$$

From (19) and (20), we can express the change ΔT in *MTTF* due to a unit change $\Delta n = 1$ in the number of components as

$$(\Delta T)_{\Delta n=1} = \frac{1}{\lambda} \frac{1}{n+1}, \qquad (21)$$

and hence, we can express $\in_{T,C}$ as

$$\in_{T,C} = \frac{\frac{1}{\lambda}\frac{1}{n+1} n}{\frac{1}{\lambda}\sum_{m=k}^{n}\frac{1}{m}} = \frac{\frac{n}{n+1}}{\sum_{m=k}^{n}\frac{1}{m}}$$

$$= \frac{\frac{n}{n+1}}{\frac{1}{k} + \frac{1}{k+1} + \dots + \frac{1}{n-1} + \frac{1}{n}}. \qquad (22)$$

The cost elasticity $\in_{T,C}$ of the *MTTF* of a k-out-of-n:G system is a function of n and k only and is independent of the component reliability R_0 and the component failure rate λ.

Noting that the sum $S = \sum_{m=k}^{n} \frac{1}{m}$ satisfies the following inequalities for $k > 1$

$$S > \int_k^{n+1} \frac{dx}{x} = ln\left(\frac{n+1}{k}\right), \qquad (23)$$

$$S < \int_k^{n+1} \frac{dx}{x-1} = ln\left(\frac{n}{k-1}\right), \qquad (24)$$

we obtain the following tight bounds on $\in_{T,C}$

$$\frac{n/(n+1)}{ln\left(\frac{n}{k-1}\right)} < \in_{T,C} < \frac{n/(n+1)}{ln\left(\frac{n+1}{k}\right)}. \qquad (25)$$

RESULTS

Table 1 lists $\in_{T,C}$ values in proper-fraction form (exact integer arithmetic) for small k and n values. Figures 5 and 6 represent the cost elasticity of *MTTF* for a parallel system and for a 50-out-of-n:G system versus the number of components n. Figure 7 represents the cost elasticity of *MTTF* for a k-out-of-100:G system versus the number of components k required for system success. Table 1 and Figure 7 explain why the fail-safe ((n-1)-out-of-n:G system or 2-out-of-n:F system) is so popular. Among redundancy systems it has the best cost for added redundancy since it has the highest $\in_{T,C}$. Of course, the series system (n-out-of-n:G system or 1-out-of-n:F system) has an $\in_{T,C} = (n^2/(n+1))$ that is higher than that of the fail-safe system, but the series system has no redundancy at all and cannot tolerate even a single failure.

Table 1: Value of $\in_{T,C}$ as a function of k and n for $1 \le k \le n \le 6$.

k \ n	1	2	3	4	5	6
1	1/2	4/9	9/22	48/125	50/137	120/343
2		4/3	9/10	48/65	50/77	120/203
3			9/4	48/35	50/47	120/133
4				16/5	50/27	360/259
5					25/6	180/77
6						36/7

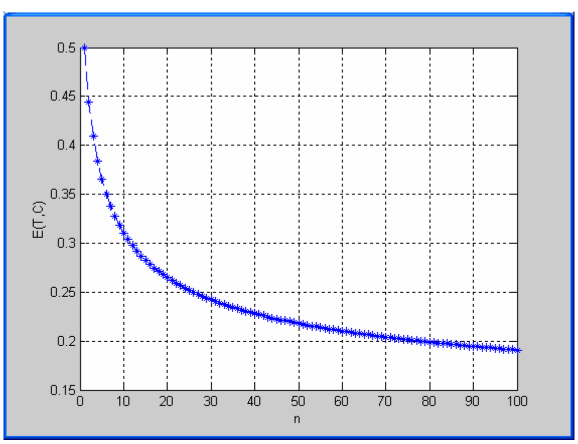

Fig. 5: Cost elasticity of *MTTF* for a parallel system versus its number of components n.

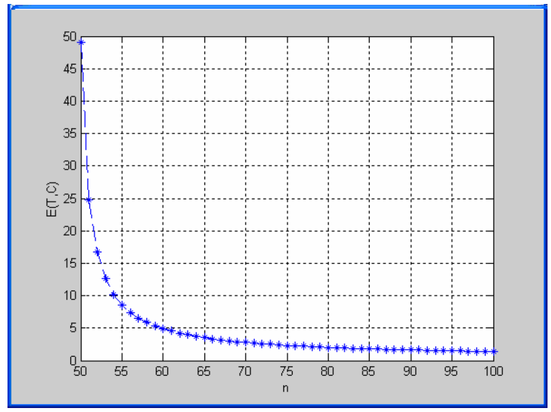

Fig. 6: Cost elasticity of *MTTF* for a 50-out-of-n:G system versus its number of components n.

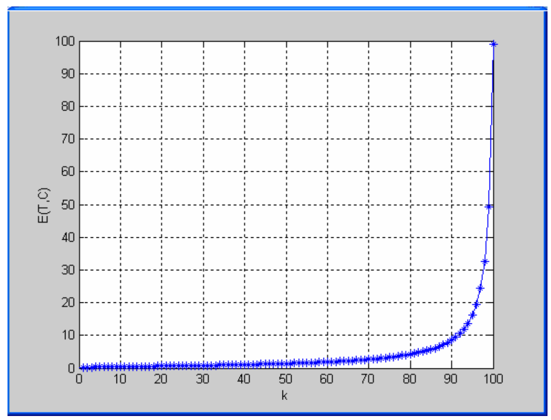

Fig. 7: Cost elasticity of *MTTF* for a k-out-of-100:G system versus the number of components k required for system success.

DISCUSSION

The concept of cost elasticity of *MTTF* $\in_{T,C}$ introduced herein is a novel concept and has several advantages when compared with the earlier competitive concept of cost elasticity of reliability $\in_{R,C}$. One advantage stems from the fact that the *MTTF* is a cumulative, integral, or averaging measure for reliability itself. For the wide class of *k*-out-of-*n*:G(F) systems $\in_{T,C}$ depends only on *k* and *n* while $\in_{R,C}$ depends on component characteristics in addition to its dependence on *k* and *n*. For such systems, it was possible to express $\in_{T,C}$ by a purely additive formula that is insensitive to round-off errors, while the $\in_{R,C}$ formula is very susceptible to round-off errors to the extent that catastrophic cancellations take place. Moreover, the $\in_{T,C}$ metric decreases but remains distinguishable from zero for large *n*, while the $\in_{R,C}$ metric diminishes and becomes indistinguishable from zero for large *n* and R_0. This fact imposes a limitation on the utility of both metrics for large ultra-reliable systems.

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