Study of Families of Curves in the Euclidian Plan

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Abstract: Non-standard analysis techniques are more considered in approaching complex mathematical domains. By using some concepts of non-standard analysis methods such as regionalization method, we deal with a family of curves in an Euclidian plan. The solutions of the algebraic equations representing these curves in a plan have an hyperbolic forms.

Key words: Non-standard analysis, regionalization, unlimited number, infinitesimal, appreciable

INTRODUCTION

Our recent work deals with a family of curves in the Euclidian plan by using some concepts of non-standard analysis given by Robinson, A. [1] and axiomatized by Nelson, E. [2]. More precisely, under some conditions concerning domains we show that the solutions of the algebraic curves have geometrical forms (hyperbolic).

We start our study with the algebraic curve $E(m,n,a)$ defined in $\mathbb{R}^2$ by the set

$$E(m,n,a)=\{(x,y)\in\mathbb{R}^2/ \left(\frac{1}{x^m}+\left(\frac{1}{y}\right)^2\right)=a, m\geq n>0,a>0\}$$

where $(x,y)$ verify the following equation $x^{-2m}y^{-2n}a=y^{-2n}+x^{-2m}. a>0$ real $x>0, y>0$ by using the regionalization method [3].

This curve allows us to define two sets

$$Q\left(\frac{1}{2m}, a^{-\frac{1}{2n}}\right)$$

$$Q\left(\frac{1}{2m}, a^{-\frac{1}{2n}}\right)$$

the quadrant defined by

$$(a^{-\frac{1}{2m}}, a^{-\frac{1}{2n}})$$

and

$$(a^{-\frac{1}{2m}}, a^{-\frac{1}{2n}})$$

the vertex

$$(a^{-\frac{1}{2m}}, a^{-\frac{1}{2n}})$$

and

$$(a^{-\frac{1}{2m}}, a^{-\frac{1}{2n}})$$

the quadrant defined by $x\geq a^{-\frac{1}{2m}}$ and $y\geq a^{-\frac{1}{2n}}$ and the vertex

$$(a^{-\frac{1}{2m}}, a^{-\frac{1}{2n}})$$

and

$$(a^{-\frac{1}{2m}}, a^{-\frac{1}{2n}})$$

the vertex

$$(a^{-\frac{1}{2m}}, a^{-\frac{1}{2n}})$$

and

$$(a^{-\frac{1}{2m}}, a^{-\frac{1}{2n}})$$

which allow us to cover the curve $E(m,n,a)$.

From the equation $\left(\frac{1}{x^m}+\left(\frac{1}{y}\right)^2\right)=a$ which defines the curve $E(m,n,a)$ and we can write the function $f_{m,n,a}$ defined from $a^{-\frac{1}{2m}}, +\infty$ into $\mathbb{R}$, such that

$$f_{m,n,a}(x)=\frac{x^{-\frac{m}{n}}}{(ax^{-2m}-1)^{-\frac{1}{2n}}}$$

Proposition 1: The function $f_{m,n,a}$ has the following properties:

1°) $f_{m,n,a}(x)$ is strictly decreasing on $\left[a^{-\frac{1}{2m}}, +\infty\right]$.

2°) $f_{m,n,a}(x)$ has $y=a^{-\frac{1}{2n}}$ as horizontal asymptote

3°) $f_{m,n,a}(x)$ has $x=a^{-\frac{1}{2m}}$ as vertical asymptote

Proof of the proposition 1: We show that $f_{m,n,a}(x)$ is strictly decreasing; we study the sign of its derivable form:
Given: \[ f_{m,n,a}^*(x) = \frac{-m x^m}{n (ax^{2m} - 1)^{1/n}} \]

Since \( x \) belongs to \( a^{-2m}, +\infty \), \( x^m \) is equivalent to \( \frac{1}{a} \).

\[ x > a^{-2m} \text{ equivalent } x^{2m} > \frac{1}{a} \]

Since \( \frac{m}{n} < 0 \) then \( \frac{-m x^m}{n (ax^{2m} - 1)^{1/n}} < 0 \)

However, \( x(ax^{2m} - 1)^{1/(2m+1)} \neq 0 \) where \( f^* \neq 0 \) then the function \( f \) is decreasing.

2- We are going to verify that \( f \) has a horizontal asymptote, for this we compute

\[ \lim_{x \to 0} f_{m,n,a}^*(x) = \lim_{x \to \infty} \frac{x^n}{(ax^{2m} - 1)^{1/n}} = \frac{a^{-1/m}}{a^{-1/2n}} = a \]

Then \( y = a \) is an horizontal asymptote.

3- We are going to verify that \( f \) has a vertical asymptote for this we compute

\[ \lim_{x \to a^{-2m}} f_{m,n,a}^*(x) = \lim_{x \to \frac{1}{a^{1/2n}}} \frac{x^n}{(ax^{2m} - 1)^{1/n}} = +\infty \]

Then \( x = a^{-1/2m} \) is a vertical asymptote.

**Lemma d’encadrement:** We have the relations:

\( \left\{ \left( \frac{a}{2} \right)^{1/m}, \frac{a}{2} \right\} \subseteq Q \left( \frac{a}{2}^{1/m}, \frac{b}{2}^{1/m} \right) \) with \( Q^0 \) the interior part of \( Q \).

**Proof of the lemma d’encadrement:** The vertex \( Q \left( \frac{a}{2}^{1/m}, \frac{b}{2}^{1/m} \right) \) belongs to \( E(m,n,a) \)

because \( \left\{ \frac{1}{a^{1/m}}, \frac{b}{a^{1/m}} \right\} = a^{1/2} \). However

\( \left\{ \frac{a}{2}^{1/m}, \frac{b}{2}^{1/m} \right\} \in E(m,n,a) \)

\( (x_0, y_0) \in Q^0 \left[ a^{-1/2n}, a^{-1/2n} \right] \subset Q \left[ a^{-1/2n}, a^{-1/2n} \right] \)

following the definition of an interior of a set.

(ii) we show that \( E(m,n,a) \Rightarrow \left( \frac{1}{x_0} \right)^{2n} + \left( \frac{1}{y_0} \right)^{2n} = a \)

It remains to be shown that

\( (x_0, y_0) \notin Q^0 \left( \frac{a}{2}^{1/m}, \frac{a}{2}^{1/m} \right) \)

If \( (x_0, y_0) \notin Q \left( \frac{a}{2}^{1/m}, \frac{b}{2}^{1/m} \right) \) then

\( (x_0, y_0) \notin Q^0 \left( \frac{a}{2}^{1/m}, \frac{b}{2}^{1/m} \right) \)

By contradiction we suppose that

\( (x_0, y_0) \in Q^0 \left[ a^{-1/n}, b^{-1/n} \right] \)

If we take a point

\( (x_0, y_0) \in Q^0 \left[ a^{-1/n}, b^{-1/n} \right] \)

Then

\( x_0 \left( \frac{a}{2}^{1/n} \right) \text{ and } y_0 \left( \frac{a}{2}^{1/n} \right) \) or \( x_0 \left( \frac{a}{2}^{1/n} \right) \text{ and } y_0 \left( \frac{a}{2}^{1/n} \right) \)

Imply \( (x_0, y_0) \notin E(m,n,a) \), hence contradiction.
Lemma of general framing: We have the following relations:
\[
\left( \frac{a}{k} \right)^{\frac{1}{2m}}, \left( \frac{a(k-1)}{k} \right)^{\frac{1}{2m}} \in E(m,n,a) \subset Q \left( \frac{1}{2m}, \frac{1}{2m} \right) - Q \left( \frac{a}{k} \right)^{\frac{1}{2m}}, \left( \frac{a(k-1)}{k} \right)^{\frac{1}{2m}}
\]

Proof of the lemma of general framing:
Let \( (x_0, y_0) \in E(m,n,a) \Rightarrow \left( \frac{1}{x_0} \right)^{2m} + \left( \frac{1}{y_0} \right)^{2m} = a \)
\[
\left( \frac{1}{x_0} \right)^{2m} \langle a \ and \ \left( \frac{1}{y_0} \right)^{2m} \langle a \ then \ x_0, a^{\frac{1}{2m}} \text{ and } y_0, a^{\frac{1}{2m}}
\]

Implies \( (x_0, y_0) \in Q \left( \frac{1}{2m}, a^{\frac{1}{2m}} \right) \subset Q \left( \frac{1}{2m}, a^{\frac{1}{2m}} \right) \)

It remains to be shown that \( (x_0, y_0) \notin Q \left( \frac{1}{2m}, \left( \frac{a(k-1)}{k} \right)^{\frac{1}{2m}} \right) \)

By contradiction we suppose \( (x_0, y_0) \in Q \left( \frac{1}{2m}, \left( \frac{a(k-1)}{k} \right)^{\frac{1}{2m}} \right) \)

if we take a point \( (x_0, y_0) \in Q \left( \frac{1}{2m}, \left( \frac{a(k-1)}{k} \right)^{\frac{1}{2m}} \right) \)

\[ x_0 \left( \frac{a}{k} \right)^{\frac{1}{2m}} \text{ and } y_0 \left( \frac{a(k-1)}{k} \right)^{\frac{1}{2m}} \]

then \( (x_0, y_0) \notin E(m,n,a) \)

where \( x_0 \geq \left( \frac{a}{k} \right)^{\frac{1}{2m}} \text{ and } y_0 \left( \frac{a(k-1)}{k} \right)^{\frac{1}{2m}} \)

Contradiction

Proposition 2: When \( m \geq n \geq 0 \) are integers the geometric place of the vertex of the quadrants
\[
Q \left( \frac{1}{2m}, a^{\frac{1}{2n}} \right) \text{ and } Q \left( \frac{a}{k}^{\frac{1}{2m}}, a^{\frac{1}{2m}} \right)
\]
is the curve of equation \( y = x^n \)

Proof of the proposition 2: Since the vertex of the quadrants
\[
Q \left( \frac{1}{2m}, a^{\frac{1}{2n}} \right) \text{ and } Q \left( \frac{a}{k}^{\frac{1}{2m}}, a^{\frac{1}{2m}} \right)
\]
are \( \left( \frac{1}{2m}, a^{\frac{1}{2n}} \right) \) and \( \left( \frac{a}{k}^{\frac{1}{2m}}, \frac{a}{k}^{\frac{1}{2m}} \right) \)

the writing \( a \left( \frac{1}{2m} \right)^m \) and \( a \left( \frac{1}{2m} \right)^m \)
shows that the vertex verify the equation \( y = x^n \).

Proposition 3: When \( m \geq n \geq 0 \) are fixed integers and a real fixed \( \alpha = 2x_0^{2m} = 2y_0^{2n} \) the geometric place of the vertex of the quadrants
\[
Q \left( \frac{1}{2m}, a^{\frac{1}{2n}} \right) \text{ and } Q \left( \frac{a}{k}^{\frac{1}{2m}}, a^{\frac{1}{2m}} \right)
\]
is the curve of equation \( y = x^n \) ( resp \( y = \left( k-1 \right)^{\frac{1}{2m}} x^n \) ) when \( a \) ranges over \( 0, + \infty \)

Proof of the proposition 3: The vertex of the quadrants
\[
Q \left( \frac{1}{2m}, a^{\frac{1}{2n}} \right) \text{ and } Q \left( \frac{a}{k}^{\frac{1}{2m}}, a^{\frac{1}{2m}} \right)
\]
are \( S \left( \frac{1}{2m}, a^{\frac{1}{2n}} \right) \) and \( S \left( \frac{a}{k}^{\frac{1}{2m}}, a^{\frac{1}{2m}} \right) \).

The writing \( a \left( \frac{1}{2m} \right)^m \) shows that the coordinates of \( S \left( \frac{1}{2m}, a^{\frac{1}{2n}} \right) \). Verify the equation \( y = x^n \).
The writing \( \left( \frac{a(k-1)}{k} \right)^{-\frac{1}{2n}} = (k - 1)^\frac{1}{2n} \left( \frac{a}{k} \right)^{\frac{1}{2n}} \)
shows that the coordinates of \( S \left( \frac{a}{k}, \frac{a(k-1)}{k} \right) \) verify the equation
\( y = (k-1)^\frac{1}{2n} x^\frac{m}{2n} \)

Reciprocally: A point \( (x_0, y_0) \) of the curve \( y = (k-1)^\frac{1}{2n} x^\frac{m}{2n} \) is the vertex of the quadrant
\( Q \left( \frac{a}{k}, \frac{a(k-1)}{k} \right) \) and \( Q \left( \frac{a\log2}{2n}, \frac{a\log2}{2n} \right) \).

From the equality
\( k\frac{x_0^{2m}}{k} = a \), we obtain \( a = k.x_0^{-2m} \).

From
\( \frac{k-1}{k} \), we obtain \( a = k^{-1}y_0^{2n} \).

which give us:
\( a = k.x_0^{-2m} = \frac{k}{k-1}y_0^{-2n} \); hence the point \((x_0, y_0)\) is a curve point.

Parameters monitoring the shape of the curves:

Let \( V = (a\frac{1}{2n})^\frac{1}{2n} - a^{-\frac{1}{2n}} = \left( 1 - 2^{\frac{1}{2n}} \right) \left( \frac{a}{2} \right)^\frac{1}{2n} \)
Vertical thickness of the «main band of encadrement».

Let \( h = (a\frac{1}{2n})^\frac{1}{2n} - a^{-\frac{1}{2n}} = \left( 1 - 2^{\frac{1}{2n}} \right) \left( \frac{a}{2} \right)^\frac{1}{2n} \)
Horizontal thickness of the «main band of encadrement».

Let \( r = \frac{m}{n} \) Parameter monitoring the curve \( C(m,n) \) of the vertex of the quadrants.

\( Q \left( a\frac{1}{2n}, a^{-\frac{1}{2n}} \right) \) and \( Q \left( a^{\frac{1}{2n}}, a^{-\frac{1}{2n}} \right) \).

And let \( C(m,n) \) : The curve of equation \( y = x^\frac{m}{n} \)

Comparaison of the thickness: We have two situations
(i) \( \frac{m}{n} \approx 1 \) the \( C(m,n) \) curve has the shape of the right-line \( y = x \)
(ii) \( 1 \ll \frac{m}{n} \ll \infty \) the \( C(m,n) \) curve has the shape of the right-line \( y = x^\frac{m}{n} \)

Proposition 4: If \( n \gg 0 \) is infinitely big, then the vertical thickness \( V \) is substantially positive if and only if: \( a \in \left( \frac{A_n}{2n} \right)^2 \)

Proof of proposition 4: We show that
\( V = \left( 1 - 2^{\frac{1}{2n}} \right) \left( a\frac{1}{2} \right)^\frac{1}{2n} \in A_n \) is equivalent to \( a \in \left( \frac{A_n}{2n} \right)^2 \)
Indeed:
\( \left( 1 - 2^{\frac{1}{2n}} \right) \left( a\frac{1}{2} \right)^\frac{1}{2n} \in A_n \) is equivalent to \( a \in \left( \frac{A_n}{2n} \right)^2 \)

We apply the limited development of \( 2^{\frac{1}{2n}} = e^{-\frac{1}{2n} \log2} \)
\( 2^{\frac{1}{2n}} = e^{-\frac{1}{2n} \log2} = 
= 1 - \frac{1}{2n} \log2 + \frac{1}{2!} \left( \frac{1}{2n} \log2 \right)^2 - \frac{1}{3!} \left( \frac{1}{2n} \log2 \right)^3 + .... 
= 1 - \frac{1}{2n} \log2 \left[ 1 - \frac{1}{2!} \left( \frac{1}{2n} \log2 \right)^2 + \frac{1}{3!} \left( \frac{1}{2n} \log2 \right)^2 + .... \right] 
we take \( \gamma = 1 - \frac{1}{2!} \left( \frac{1}{2n} \log2 \right)^2 + \frac{1}{3!} \left( \frac{1}{2n} \log2 \right)^2 + .... \)
\[ \gamma \approx 1 \quad \text{. If } n > 0 \text{ is infinitely big, then} \]

\[ 2^{\frac{1}{2n}} = 1 - \left( \frac{1}{2n} \log 2 \right) \gamma \approx 1 - \frac{1}{2n} \log 2 \]

Therefore

\[ \left( \frac{a}{2} \right)^{\frac{1}{2n}} = \frac{A_+}{1 + 1 - \frac{1}{2n} \log 2} = \frac{A_+}{\frac{1}{2n} \log 2} \]

\[ a \in \left( \frac{A_+}{2n} \right)^{2n} \]

Is equivalent

\[ a \in \left( \frac{2nA_+}{\log 2} \right)^{2n} \]

Is equivalent

\[ a \in \left( \frac{\log 2}{2nA_+} \right)^{2n} \]

As \( \frac{\log 2}{A_+} \approx A_+ \), therefore

\[ \left( \frac{P}{2n} \right)^{2n}, \left( \frac{G_+}{2n} \right)^{2n} \quad a \in \left( \frac{A_+}{2n} \right)^{2n}, \text{ c.q.e.d.} \]

Are the complements of

\[ \left( \frac{A_+}{2n} \right)^{2n} \]

**The graphical representation:**

\[ G_\wedge \quad A_\wedge \quad P \quad A_+ \quad G_+ \]

**Proposition 5:** If \( m > 0 \) is infinitely big, then the horizontal thickness \( h \) is substantial positive if and only if

\[ a \in \left( \frac{A_+}{2n} \right) \]

**Proof of proposition 5:** The same proof as for the proposition 4 by substituting \( m \) for \( n \).

**Proposition 6:** If \( m = n\delta \), \( \delta \in 1 + \frac{L}{\ln 2n} \) then we have

\[ \left( \frac{A_+}{2m} \right)^{2m} = \left( \frac{A_+}{2n} \right)^{2n} \]

with \( L \) : limit, \( A_+ \) : substantial positive.

**Study of the tangents:** Let \( E(m,n,a) \) a family of curves of the plan \( \mathbb{R}^2 \) defined by the equation

\[ \left( \frac{1}{x} \right)^{2m} + \left( \frac{1}{y} \right)^{2n} = a \]

\( (m \geq n \geq 0) \) infinitely big integer, \( a > 0 \) real. the tangent in point \((x_0, y_0)\) of the \( E(m,n,a) \) curve has the equation

\[ n(y - y_0)x_0^{2m+1} + m(x-x_0)y_0^{2n+1} = 0. \]

Indeed: the equation of the tangent in a point \((x_0, y_0)\) is:

\[ y - y_0 = f'_{m,n,a}(x_0)(x-x_0) \quad (*) \]

Where

\[ y = f_{m,n,a}(x) = \frac{x^m}{(ax^{2m} - 1)^{\frac{1}{2n}}} \]

\[ - \frac{m}{n} x^{n-1} \]

And since \( y^{2n} = \frac{x^{2m}}{ax^{2m} - 1} \), we replace in (*) we obtain

\[ y - y_0 = \frac{m}{n} x_0^{n-1} \]

\[ (ax_0^{2m} - 1)^{\frac{1}{2n}} (x-x_0) \]

Hence

\[ (y - y_0)(ax_0^{2m} - 1) = \frac{m}{n} x_0^{n-1} (x-x_0) \]

\[ (y - y_0)x_0^{2m+1} \]

\[ y_0^{2n} \]

\[ y_0 (x-x_0) \]

imply

\[ n(y - y_0)x_0^{2m+1} + my_0^{2n+1} (x-x_0) = 0 \]

As \( x_0 > 0, y_0 > 0 \) we have the equation

\[ y - y_0 + m \frac{y_0^{2n+1}}{x_0^{2m+1}} (x-x_0) = 0 \]
Situation where the slope is infinitely small:
\[ \frac{m}{n} \left( \frac{y_0^{2n+1}}{x_0^{2m+1}} \right) \in P \text{ is equivalent } y_0^{2n+1} \in \left( \frac{n}{m} \right)^{2m+1} P \]

is equivalent \( y_0 \in \left( \frac{n}{m} P \right)^{\frac{1}{2m+1}} x_0^{\frac{2m+1}{2n+1}} \)

**Result 1:** the slope is infinitely small \((x_0, y_0) \in \mathcal{R}_+\)
as \(y_0 \in \left( \frac{n}{m} P \right)^{\frac{1}{2m+1}} x_0^{\frac{2m+1}{2n+1}} \)

Situation where the slope is substantial positive:
\[ \frac{m}{n} \left( \frac{y_0^{2n+1}}{x_0^{2m+1}} \right) \in A_s \text{ is equivalent } y_0 \in \left( \frac{n}{m} A_s \right)^{\frac{1}{2m+1}} x_0^{\frac{2m+1}{2n+1}} \]

if the slope is appreciable positive.

**Result 2:** the slope is substantial positive \((x, y) \in \mathcal{R}_+\)
as \(y_0 \in \left( \frac{n}{m} A_s \right)^{\frac{1}{2m+1}} x_0^{\frac{2m+1}{2n+1}} \)

Situation where the slope is infinitely big positive:
\[ \frac{m}{n} \left( \frac{y_0^{2n+1}}{x_0^{2m+1}} \right) \in G_s \text{ is equivalent } y_0 \in \left( \frac{n}{m} G_s \right)^{\frac{1}{2m+1}} x_0^{\frac{2m+1}{2n+1}} \]

if the slope is infinitely great then :

**Result 3:** the slope is substantial positive as if \((x, y) \in \mathcal{R}_+\)
as \(y_0 \in \left( \frac{n}{m} G_s \right)^{\frac{1}{2m+1}} x_0^{\frac{2m+1}{2n+1}} \)

Parameters monitoring the shape of the curves
( general case): In the general case \(V_k\) and \(h_k\) are equal:
\[ V_k = \left( 1 - \left( \frac{k-1}{k} \right)^{\frac{1}{k}} \right) \left( \frac{ak}{k} \right)^{\frac{1}{k}} \]

k>1 Vertical thickness.
\[ h_k = \left( 1 - \left( \frac{k-1}{k} \right)^{\frac{1}{k}} \right) \left( \frac{a}{k} \right)^{\frac{1}{k}} \]

k>1 horizontal thickness bands

The curve \(C_k(m, n)\) correspond to \(E(m, n, a)\).

**CONCLUSION**

In this study we have introduced a non-standard analysis technique and regionalization for resolving algebraic curves formalized by algebraic equations

**REFERENCES**