Behavior of the Dedekind’s Function over First Order Theta Function According to Conditions Modular Form

İsmet Yıldız
University of Bahcesehir Vocational School
Beşiktaş-İstanbul/TURKEY

Abstract: The effect Dedekind’s eta function on theta functions is analyzed according to the characteristics of theta functions under modular group conditions.

Key words: Theta functions, characteristic values, modular group, dedekind functions

INTRODUCTION

By $\text{SL}_2$ we mean the group of 2x2 matrices with determinant 1. We write $\text{SL}_2(\mathbb{R})$ for those elements of $\text{SL}_2$ having coefficients in a ring $R$. In practice, the ring $R$ will be integers $\mathbb{Z}$, rational numbers $\mathbb{Q}$ and real numbers $\mathbb{R}$. We call $\text{SL}_2(\mathbb{Z})$ the modular group $G$.

If $L$ is lattice in complex numbers $\mathbb{C}$, then we can always select a basis, $L = (e_1, e_2)$ such that $T = \begin{pmatrix} e_1 & e_2 \\ 0 & 1 \end{pmatrix}$ is an element of the upper half-plane $\mathbb{H}$, i.e. has $\text{Im}(T) > 0$ which is not real. If $\mathcal{D}$ consist of all $u \in \mathbb{H}$ such that $1 \leq \text{Re}(u) < \frac{1}{2}$ and $T = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \in G$, then $S,T \in G$ generate modular group $G$.

We define characteristic $\epsilon$ and $\epsilon'$ where $\epsilon, \epsilon'$ are integers according to characteristics $\begin{pmatrix} e \\ e' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (mod 2) but $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for theta function $\Theta$. If $n$ is any positive integer we define $\Gamma_0(n)$ to be the set of all matrices $\Lambda = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in $G$ with $\gamma \equiv (\text{mod } n)$ and but $\Gamma_0(n) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G(1) : \gamma \equiv (\text{mod } n) \right\}^{[2,3]}$

It is easy to verify that $\Gamma_0(n)$ is a subgroup $G$. If we consider the congruence subgroup $\Gamma(2)$, then $\Gamma(2) = \{ W \in \Gamma(1) : W \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} (\text{mod } 2) \}$ where $I$ is the unit matrix and for matrices $X = S$, $X = T$, $X = U$ the three subgroups $\Gamma_s(2)$, $\Gamma_t(2)$, $\Gamma_u(2)$ are conjugate subgroups of $\Gamma(1)$ for $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

We shall need to study such groups when we introduce theta functions.

We note that the above matrices, defined $V = ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ and $V' = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} = I$, $U = TST$ form a set coset representatives of $\Gamma(1)$ modulo $\Gamma(2)$. The subgroup $\varphi(n)$ of $\Gamma(1)$ is generated by $V^2$ and $S$ where $k$ is an odd positive integer and the set of elements in $\varphi(k)$ of the form $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

Theorem 1: Let $n$ be any prime and $S\tau = -\frac{1}{\tau}$, $T\tau = \tau + 1$ be the generations of the full modular group $G$, then for every $W = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, $W \not\in \Gamma_0(n)$ there exists an element $K = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in \Gamma_0(n)$.

Proof: If $W = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ where $c$ is not $c \equiv 0(\text{ mod } n)$ then we wish to find $W = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, with $s \equiv 0(\text{ mod } n)$ and an integer $q$, $0 \leq q \leq n$, such that

$W = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

All matrices here are nonsingular so we can solve for $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ to get
\[
\begin{bmatrix}
    p & r \\
    s & t \\
\end{bmatrix} = \begin{bmatrix}
    a & b \\
    c & d \\
\end{bmatrix} \begin{bmatrix}
    0 & -1 \\
    1 & 0 \\
\end{bmatrix}
= \begin{bmatrix}
    a & b \\
    c & d \\
\end{bmatrix} \begin{bmatrix}
    w & 1 \\
    -1 & 0 \\
\end{bmatrix} = \begin{bmatrix}
    qa - b & a \\
    qc - d & c \\
\end{bmatrix}.
\]

Choose \( q \) to be that solution of the congruence \( qc \equiv d \pmod{m} \) with \( 0 \leq q \leq n \). This is possible since \( c \) is not \( c \equiv 0 \pmod{m} \), now take \( s = qa - t \), \( p = wp - r \), \( r = a \), \( t = c \), then \( s \equiv 0 \pmod{m} \) so \( K = \begin{bmatrix}
    p & r \\
    s & t \\
\end{bmatrix} \in \Gamma(n) \).

We define the first order theta function with characteristic \( \begin{bmatrix}
    \varepsilon \\
    \varepsilon' \\
\end{bmatrix} \), \( u \in \mathcal{C} \) and theta period \( \tau \) by
\[
\theta_{\begin{bmatrix}
    \varepsilon \\
    \varepsilon' \\
\end{bmatrix}}(u, \tau) = \sum_{n \in \mathbb{Z}} \exp \left( (N + \frac{\varepsilon + 2m}{2} \pi i \tau + 2\pi i (N + \frac{\varepsilon'}{2}) (u + \frac{\varepsilon'}{2}) \right)
\]
where \( N \) is a integer and \( \varepsilon \), \( \varepsilon' \) are general real numbers. As \( \varepsilon, \varepsilon' \) are residue classes (mod2) it is natural to concentrate attention on the four functions \( \theta_{\begin{bmatrix}
    \varepsilon & 1 \\
    \varepsilon' & 1 \\
\end{bmatrix}}(u, q), \theta_{\begin{bmatrix}
    0 & 1 \\
    \varepsilon & 1 \\
\end{bmatrix}}(u, q), \theta_{\begin{bmatrix}
    0 & 1 \\
    \varepsilon' & 1 \\
\end{bmatrix}}(u, q) \) and the characteristic \( \mu \) which we shall call the four principal theta functions. For any integers \( m,n \), when \( \varepsilon, \varepsilon' \) are integers, we have
\[
\theta_{\begin{bmatrix}
    \varepsilon & 1 \\
    \varepsilon' & 1 \\
\end{bmatrix}}(u, q) = \sum_{n \in \mathbb{Z}} q \left( \frac{n + \varepsilon}{2} \right)^{2m} e^{2\pi i (n + \varepsilon / 2)} u^{\frac{\varepsilon'}{2} - \frac{n + \varepsilon}{2}}.
\]

When \( \varepsilon, \varepsilon' \) are integers, the theta series defined by
\[
\theta_{\begin{bmatrix}
    \varepsilon \\
    \varepsilon' \\
\end{bmatrix}}(u, \tau) = \sum_{n \in \mathbb{Z}} q \left( \frac{n + \varepsilon}{2} \right)^{2m} e^{2\pi i (n + \varepsilon / 2)} u^{\frac{\varepsilon'}{2} - \frac{n + \varepsilon}{2}}
\]
can be converted into fourier series by pairing off the terms which \( n + \varepsilon / 2 \) has equal and opposite values, \( n \) with -n if \( \varepsilon = 0 \) and \( n \) with -\varepsilon, leaving in the former case an unpaired term for \( n = 0 \), whose values is \( 1 \). The terms so paired have a common factor \( q \) and the sum of their remaining factors is 2Cos(2n+\varepsilon)(u - \frac{\varepsilon'}{2} \pi). Therefore we have the four series
\[
\theta_{\begin{bmatrix}
    1 & 1 \\
\end{bmatrix}}(u, q) = 2 \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{\varepsilon'}{2}} \sin(2n+1)u
\]
and
\[
\theta_{\begin{bmatrix}
    1 & 1 \\
\end{bmatrix}}(u, q) = 2 \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{\varepsilon'}{2}} \cos(2n+1)u
\]
Moreover,
\[
\theta_{\begin{bmatrix}
    1 & 1 \\
\end{bmatrix}}(u, \tau) = \theta_{\begin{bmatrix}
    1 & 1 \\
\end{bmatrix}} \left( u + \frac{\pi}{2} \right)
\]
\[
\theta_{\begin{bmatrix}
    0 & 1 \\
\end{bmatrix}}(u, \tau) = -iq^\tau e^{\mu} \theta_{\begin{bmatrix}
    1 & 1 \\
\end{bmatrix}} (u + \frac{\pi}{2})
\]
\[
\theta_{\begin{bmatrix}
    0 & 1 \\
\end{bmatrix}}(u) = q^\tau e^{\mu} \theta_{\begin{bmatrix}
    1 & 1 \\
\end{bmatrix}} (u + \frac{\pi}{2})
\]

If \( N \) is a positive integer then theta function order \( n \) or \( n^b \) is defined by
\[
\theta_{\begin{bmatrix}
    \mu & 1 \\
    \mu' & 1 \\
\end{bmatrix}}(u, \tau) = \sum_{m \in \mathbb{Z}} C_{\mu, \mu'} \frac{2(M + \mu)}{N} \left( Nu, N \tau \right)
\]
where \( 0 \leq M \leq N - 1 \).

In fact, An theta function order \( n \) may be found by taking the product of \( n \) first theta functions. Its characteristic \( \mu \) is given by the matrix sum of the \( n \) characteristic \( \varepsilon \). The \( C_M \) is independent of \( u \) and may depend on \( \tau \). \( C_M \) satisfy \( C_{NK+M} = C_M \cdot \exp(N \tau) \cdot \Phi(K) \)
where
\[
\Phi(K) = N \tau \left( K + \frac{(M + \frac{\varepsilon}{2})^2}{2} + (K + \frac{(M + \frac{\varepsilon'}{2})^2}{2} \right)
\]

Functions \( \theta_{\begin{bmatrix}
    0 & 1 \\
\end{bmatrix}}(u, \tau) \) has zeros at the points
\[
u = \frac{1}{2} - \frac{1}{2} \tau + r \tau
\]
These points form a lattice, that \( 1 - \exp(N \tau) \) has zeros at points \( u \) where \( (2k - 1) \tau \equiv 1 \pmod{2} \) or equivalently. Hence function theta order \( n \) defined by
\[
\Phi(u, \tau) = \prod_{1} \exp \pi i ([2k - 1] - 2u] \}
\]
\[
\prod_{1} \left( 1 + \exp \pi i ([2k - 1] - 2u] \right)
\]
has precisely the same zeros as first order theta 
\[ \theta[0]_0(u, \tau) \] provided the product converges. Thus we have absolute and uniform convergence of the first infinite product for \( im \tau > 0 \). For periods \( 1 \) and \( \tau \), we may write

\[ \Phi(u+1, \tau) = \prod \{1 + \exp\pi i [(2k-1)\tau + 2u + 2]\} \\
\Phi(u+\tau, \tau) = \prod \{1 + \exp\pi i [(2k-1)\tau + 2u]\} \\
\Phi(u+\tau^2, \tau) = \prod \{1 + \exp\pi i [(2k-1)\tau - 2u - 2]\}
\]

Setting \( q = \exp\pi i \tau \) we may write

\[ \Phi(u, \tau) = \prod \{1 + q^{2k+1} \exp 2\pi i u\} \]

It was introduced by Dedekind function \( \eta(\tau) \) and is defined by the equation

\[ \eta(\tau) = e^{\frac{\pi i}{12}} \prod_{n=1}^{\infty} \left(1 - e^{2\pi in\tau}\right) \]

The Dedekind function \( \eta(\tau) \) is cups form of weight \( \frac{1}{2} \) on \( \Gamma(1) \) and satisfy

\[ \eta(A\tau) = \nu(A)(\gamma + \delta)^\frac{1}{2} \eta(\tau) \]

for all \( A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma(1) \).

Dedekind proved the following law of transformation of logarithm \( \log \eta(\tau) \) under the action of the elliptic modular group. If \( A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma \) then we have

\[ \log \eta(A\tau) = \log \eta(\tau) + \frac{1}{12} \pi i \beta \] for \( \gamma = 0 \) and

\[ \log \eta(A\tau) = \log \eta(\tau) + \frac{1}{2} \log \left( \frac{\gamma \tau + \delta}{i} \right) \]

\[ \frac{1}{12} \pi i (\alpha + \delta) - \pi i g(\gamma, \delta), \] for \( c > 0 \)

where \( A(\tau) = \frac{\alpha \tau + \beta}{\gamma \tau + \delta} \), all logarithms are taken with respect to the principal branch and \( g(\gamma, \delta) \) is a Dedekind sum.

An important connection between \( \theta[0]_0(0, \tau) \) and \( \eta(\tau) \) is given by

\[ \theta[0]_0(0, \tau) = \eta^2 \left( \frac{\tau + 1}{2} \right) / \eta(\tau+1) \]

The infinite product has the form \( \prod \{1 - u^n\} \) where \( u = e^{2\pi i \tau} \). If \( \tau \in \mathbb{C} \) then \( |u| < 1 \) so the product converges absolutely and non-zero.

Moreover, since the convergence is uniform on compact subsets of \( \mathbb{C} \), \( \eta(\tau) \) is analytic on \( \mathbb{C} \). This result and other properties of \( \eta(\tau) \) following from transformation formulas which describe the behavior of \( \eta(\tau) \) under elements of the modular group \( \Gamma \).

i. For the generator \( T\tau = \tau + 1 \) we have

\[ \eta(\tau+1) = e^{-\frac{\pi i}{12}} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n\tau}\right) \]

ii. For the other generator \( S\tau = -\frac{1}{\tau} \) we have the

\[ \eta(-\frac{1}{\tau}) = (-i\tau)^\frac{1}{2} \eta(\tau) \]

For proof, let \( \tau = iy \) where \( y > 0 \) and then extend the results to all \( \tau \in \mathbb{C} \) by analytic continuation. The transformation formula becomes

\[ \eta(iy) = y^{\frac{1}{2}} \eta(iy) \] for \( \tau = iy \)

and

\[ \log \eta(iy) = -\frac{1}{12} \pi y + \log \prod_{n=1}^{\infty} \left(1 - e^{-2\pi iny}\right) \]

\[ = -\frac{1}{12} \pi y + \sum_{n=1}^{\infty} \left(1 - e^{-2\pi iny}\right) = -\frac{1}{12} \pi y - \sum_{n=1}^{\infty} \frac{e^{-2\pi iny}}{m} \]

we obtained \( \eta(-\frac{1}{\tau}) = (-i\tau)^\frac{1}{2} \eta(\tau) \) since

\[ \sum_{n=1}^{\infty} \frac{1}{m} \left(1 - e^{-2\pi iny}\right) = \sum_{m=1}^{\infty} \frac{1}{m} \left(1 - e^{-2\pi in/\tau}\right) \]

\[ = -\frac{1}{12} \pi (y - \frac{1}{y}) = \frac{1}{2} \log y \]

**Lemma 1:** Let \( \Gamma \) be a subgroup of \( \Gamma(1) \). If \( \phi(\tau) \) is a modular form of weight \( \Gamma \) with multiplier system \( t \) then we write \( \phi(\tau) \in A(\Gamma, n, t) \). If \( \phi(\tau) \in A(\Gamma, n, t) \) then the \( \psi \)-transform \( \psi \phi \) of \( \phi \) is defined by

\[ \psi \phi(\tau) = \phi(\tau) / \psi = \left(\xi(\psi, \tau)\right)^{-1} \phi(\psi, \tau) \]
Here, $\xi(\gamma, \tau) = (\gamma \tau + \delta)^{n}$ where $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$

If $\varphi_1(\tau) \in A(\Gamma, n_1, t_1)$ and $\varphi_2(\tau) \in A(\Gamma, n_2, t_2)$ then we have $\varphi_1(\tau) \varphi_2(\tau) \in A(\Gamma, n_1 + n_2, t_1 + t_2)$.

$\Phi(\tau) = \begin{pmatrix} \eta(\kappa, \tau) \eta(\lambda) \\ \eta(\lambda) \eta(\lambda) \end{pmatrix}^\rho$ is a modular function on the group $\Gamma_0(k)$. The multiplier system $\rho$ of $\Phi(\tau)$ is given by $\rho(A) = \begin{pmatrix} \delta_k \end{pmatrix}$ where $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(k)$ and $\begin{pmatrix} \delta_k \end{pmatrix}$ is Legendre’s symbol.

**Lemma 2:** The functions $\theta_0(0, \tau)$, $\theta_0(0, \tau)$ and $\theta_1(0, \tau)$ are entire modular form of weight $\frac{1}{2}$ for the groups $\Gamma_2$, $\Gamma_2$ and $\Gamma_2$, respectively. Further, $\theta_0(0, \tau) | \mathbb{K} = e^{-\frac{\pi i}{2} \theta_0(0, \tau)}$

Also, for $n \geq 0$

The functions $\theta_0(0, \tau)$, $\theta_0(0, \tau)$ and $\theta_0(0, \tau)$ are entire modular form of weight $\frac{n}{2}$ for the groups $\Gamma_2$, $\Gamma_2$ and $\Gamma_2$, respectively.

**Theorem 2:** Let $k$ be a prime number greater than 3 and $\sigma$ is an even integer such that $\sigma(k - 1) \equiv 0 (\text{mod } 24)$ and put $r = n \rho$ where $n$ is a positive integer. If the characteristics $\begin{pmatrix} e & \mu \\ e^* & \mu^* \end{pmatrix}$ and $\begin{pmatrix} e & \mu \\ e^* & \mu^* \end{pmatrix}$ are equivalent modulo 2, the $\Phi_s(\tau)$ is a modular function on the group $\Gamma_0(k)$. The multiplier system $\rho$ of $\Phi(\tau)$ is given by $\rho(A) = \begin{pmatrix} \delta_k \end{pmatrix}$ where $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(k)$ and $\begin{pmatrix} \delta_k \end{pmatrix}$ is Legendre’s symbol.

**Proof:** $\phi_0(\tau) \neq 0$ is regular in $\mathbb{R}$. If each positive integers $\sigma$, $n$ and even positive integer $r = n \rho$. Therefore, the characteristics $r^{th}$ order theta functions are $\begin{pmatrix} r e^* \\ r e^* \end{pmatrix}$ and $\begin{pmatrix} e^* \\ e^* \end{pmatrix}$ (mod 2), then we have

$\phi_s(\tau) = \begin{pmatrix} \eta(kt) \eta(kt) \\ \eta(kt) \eta(kt) \end{pmatrix}^\rho = \begin{pmatrix} \eta(\frac{kt + 1}{2}) \eta(\frac{kt + 1}{2}) \\ \eta(\frac{kt + 1}{2}) \eta(\frac{kt + 1}{2}) \end{pmatrix}^\rho$.

Setting $\lambda = \frac{\tau + 1}{2}$ and observing that

$\Phi(\lambda) = \begin{pmatrix} \eta(\lambda) \eta(\lambda) \\ \eta(\lambda) \eta(\lambda) \end{pmatrix}^\rho$, we obtained $\phi_0(\tau) = \begin{pmatrix} \Phi_s(\lambda) \Phi_s(2\lambda) \\ \Phi_s(2\lambda) \Phi_s(2\lambda) \end{pmatrix}^\rho$.

By Lemma1 and from the equation

$\Phi(A \tau) = \begin{pmatrix} \delta_k \end{pmatrix} \Phi(\tau)$, we have

$\phi_0(A \tau) = \begin{pmatrix} \Phi_s(A\lambda) \Phi_s(2\lambda) \\ \Phi_s(2\lambda) \Phi_s(2\lambda) \end{pmatrix}^\rho = \begin{pmatrix} \delta_k \end{pmatrix} \phi_0(\tau)$.

Finally, we consider the expansions of $\phi_0(\tau)$ at the parabolic cusp $\infty$ and 0. Hence, We have

$\Phi(\lambda) = \pi (\lambda - 1)^2 12 \sum\limits_{n=1}^{\infty} C_n e^{-2\pi n / \lambda} \ln$ as the Fourier expansion of the $\Phi(\lambda)$ function at $\infty$. $\phi_0(\tau)$ has the Fourier expansion at $\infty$ of the form

$\phi(\lambda) = \sum\limits_{n=1}^{\infty} C_n e^{-2\pi n / \lambda}$

$\Phi(\lambda) = k \pi (\lambda - 1)^2 12 \sum\limits_{n=1}^{\infty} H_n e^{-2\pi n / \lambda}$ as the Fourier expansion at 0. Hence

$\phi_0(\tau) = \exp \begin{pmatrix} \pi (\lambda - 1)^2 8 \sum\limits_{n=1}^{\infty} R_n e^{-2\pi n / \lambda} \end{pmatrix}$.

It follows that $\phi_0(\tau)$ is a modular function on $\Gamma_0(k)$.

**Theorem 3:** $\eta(\alpha \tau + \beta \gamma + \delta) \equiv \frac{\pi \gamma}{\tau + \delta} \eta(\tau)$ where $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$, $\gamma > 0$ and $P(\alpha, \beta, \gamma, \delta) = \exp \begin{pmatrix} \alpha \gamma + q(\beta, \gamma) \end{pmatrix}$

and
Theorem 4: The set of modular forms, the entire modular forms and the cups forms each of same dimension for \( \Gamma(1) \), form vector space over the complex field.

Let \( g \) be a homogeneous modular form of dimension \(-k\) for the group \( \Gamma \) in the variables \( \omega_1, \omega_2 \).

We write this in the form \( g \left[ \begin{array}{c} \omega_1 \\ \omega_2 \end{array} \right] \) and consider \( \left( \begin{array}{c} \omega_1 \\ \omega_2 \end{array} \right) \) as a matrix. We define the function \( g_B \) by

\[
g_B \left[ \begin{array}{c} \omega_1 \\ \omega_2 \end{array} \right] = g \left[ \begin{array}{c} \omega_1 \\ \omega_2 \end{array} \right] g^{B} \left[ \begin{array}{c} \omega_1 \\ \omega_2 \end{array} \right] \epsilon \subset C , \lambda \neq 0
\]

where \( M = \left( \begin{array}{cc} \alpha & \beta \\ 0 & \delta \end{array} \right) \), \( \alpha \neq 0 \), \( \alpha \gamma = n \) and \( \text{Im} \frac{\omega_1}{\omega_2} > 0 \)

and call it a transform of \( g \) of order \( n \). It satisfies the following equations

\[
g_B \left[ \begin{array}{c} \omega_1 \\ \omega_2 \end{array} \right] = \lambda^{-k} g_B \left[ \begin{array}{c} \omega_1 \\ \omega_2 \end{array} \right] \quad \text{for} \quad \lambda \epsilon \subset C \quad , \quad \lambda \neq 0
\]

\[
g_B \left[ \begin{array}{c} \omega_1 \\ \omega_2 \end{array} \right] = g_B \left[ \begin{array}{c} \omega_1 \\ \omega_2 \end{array} \right] \quad \text{for} \quad M = \left( \begin{array}{cc} \alpha & \beta \\ 0 & \delta \end{array} \right) \epsilon \subset \Gamma_B
\]

\[
g_B \left[ \begin{array}{c} \omega_1 \\ \omega_2 \end{array} \right] = g_B \left[ \begin{array}{c} \omega_1 \\ \omega_2 \end{array} \right] \quad \text{for} \quad M = \left( \begin{array}{cc} \alpha & \beta \\ 0 & \delta \end{array} \right) \epsilon \subset \Gamma_B
\]

Therem 5: \( \Delta(\tau) = (2\pi)^{12} \eta^2(\tau) = (2\pi)^{12} \sum_{n=1}^{\infty} (1-x^n)^{24} \)

Proof: Let \( f(\tau) = \Delta(\tau)/\eta^2(\tau) \). Then \( f(\tau + 1) = f(\tau) \)

and \( f\left( \frac{-1}{\tau} \right) = f(\tau) \), so \( f \) is invariant under every transformation in \( \Gamma \) because \( \Delta(\tau) \) is analytic and non-zero in \( \Re \) because \( \eta(\tau) \) never vanishes in \( \Re \). Next we examine the behavior of at \( \iota = \infty \). We have

\[
\eta^2(\tau) = e^{2\pi i \tau} \prod_{n=1}^{\infty} (1-e^{2\pi i n\tau})^{24} = \sqrt{x} \sum_{n=1}^{\infty} (1-x^n)^{24} = \sqrt{x}(1+I(x))
\]

where \( I(x) \) denotes a power series in \( x \) with integer coefficients. Thus, \( \eta^2(\tau) \) has a first order zero at \( x = 0 \).

At first we see the infinite products

\[
\theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (u, \tau) = \prod_{n=1}^{\infty} (1-e^{2\pi i nu}) \prod_{n=1}^{\infty} (1+e^{2\pi i nu}) \prod_{n=1}^{\infty} (1+e^{2\pi i nu})
\]

which converges absolutely.

Theorem 6: We have the relations

\[
\eta(u) = e^{\frac{2\pi i}{12}u} \theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (\frac{u}{2} + 3u + 2k)
\]

\[
\theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (\frac{u}{12}, \frac{1}{12}u) = e^{-\frac{\pi i u}{12}} \eta(u) \prod_{n=1}^{\infty} (1-e^{2\pi i n\tau})
\]

between the functions \( \theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (u, \tau) \), \( \theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (u, \tau) \)

and Dedekind’s \( \eta \)-function which defined by the infinite product

\[
\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} (1-e^{2\pi i n\tau})
\]

where \( \text{Im} \tau > 0 \) and \( k \) is a integer.

Proof: a. Let us recall the formula

\[
\theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (u, \tau) = \prod_{n=1}^{\infty} (1-e^{2\pi i nu}) \prod_{n=1}^{\infty} (1+e^{2\pi i nu}) \prod_{n=1}^{\infty} (1+e^{2\pi i nu})
\]

If \( k \) integer, then we have

\[
\theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (\frac{u}{2}, 3u + 2k) = \prod_{n=1}^{\infty} (1-e^{2\pi i nu}) \prod_{n=1}^{\infty} (1+e^{2\pi i nu}) \prod_{n=1}^{\infty} (1+e^{2\pi i nu})
\]

\[
= \prod_{n=1}^{\infty} (1-e^{2\pi i nu}) \prod_{n=1}^{\infty} (1+e^{2\pi i nu}) \prod_{n=1}^{\infty} (1+e^{2\pi i nu})
\]

\[
= \prod_{n=1}^{\infty} (1-e^{2\pi i nu}) \prod_{n=1}^{\infty} (1+e^{2\pi i nu}) \prod_{n=1}^{\infty} (1+e^{2\pi i nu})
\]

If we set \( R = e^{2\pi i nu} \), then we obtain

\[
\theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (\frac{u}{12}, 3u + 2k) = \prod_{n=1}^{\infty} (1-R^n) \prod_{n=1}^{\infty} (1-R^{n+1}) \prod_{n=1}^{\infty} (1-R^{n+2})
\]

\[
= (1-R)(1-R^2)(1-R^3)(1-R^4) = \cdots
\]

\[
= \prod_{n=1}^{\infty} (1-R^n) = \prod_{n=1}^{\infty} (1-e^{2\pi i nu})
\]

According to above, we have

\[
\eta(\tau) = e^{\frac{\pi i \tau}{12}} \theta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (\frac{u}{2}, 3u + 2k)
\]

from the Dedekind’s \( \eta \)-function defined by the infinite product

\[ \eta(\tau) = e^{\frac{\pi i}{\tau}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau}) \]

where \( m = n' \).

b. According to the equation,
\[
\theta \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] (u, \tau) = \sum_{n=-\infty}^{\infty} (-1)^n \exp(n^3 \pi i \tau + 2n \pi i u)
\]

we have
\[
\theta \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] \left( \frac{m+1}{2}, \frac{1}{2} u \right) = \sum_{n=-\infty}^{\infty} (-1)^n \exp \left( \frac{1}{2} n(3n+1) \pi i u \right)
\]

\[= 1 + \sum_{n=1}^{\infty} (-1)^n \left\{ \exp \left( \frac{1}{2} n(3n-1) \pi i u \right) + \exp \left( \frac{1}{2} n(3n+1) \pi i u \right) \right\} = 1 \]

\[= 1 + \sum_{n=1}^{\infty} (-1)^n \left( x^{2n(3n-1)} + x^{2n(3n+1)} \right) = 1 - x - x^3 + x^5 + \cdots
\]

\[\theta \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] \left( \frac{m+1}{2}, \frac{1}{2} u \right) = (1-x)(1-x^3)(1-x^5) \cdots = \prod_{n=1}^{\infty} (1-x^{2n})
\]

where \( x = e^{\pi i u} \) for \( |x| < 1 \) and \( \frac{1}{2} n(3n+1) \) are known as the pentagonal numbers \( n = -1, -2, \ldots \).

As a result, the relation has been obtained between theta and Dedekind's \( \eta(\tau) \) functions by using the characteristic \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) and the variable \( \frac{u+4}{4} \) instead of the characteristic \( \begin{bmatrix} 0 \\ 0 \end{bmatrix} \) and the variable \( \frac{u+1}{2} \) which were previously used by Jacobi.

REFERENCES