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Folding of Trefoil Knot and its Graph

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Abstract: In this study, we introduce the effect of the folding, conditional folding, retraction and conditional retraction on trefoil knot and its graph. We introduce the sheeted trefoil knot and the folding and retraction of this type and present the limit of foldings. We study how to obtain the types of link graph by adjacency matrix for any number of vertices and how to calculate Jones polynomial for this by its connection with Tutte polynomial. The link graph which represents a trefoil knot applied as an example.

Key words: Folding, retraction, knot, link graph, trefoil knot

INTRODUCTION

It is difficult to say who first showed a mathematical interest in what we now call knot theory and when. However, in modern times it is known that the famous C.F. Gauss (1777-1855) had some interest in this field, the American mathematician J.W. Alexander (1888-1971) was the first to show that knot theory is extremely important in the study of 3-dimensional topology. the German mathematician H. Seifert from the late 1920s to the 1930s.

In the 1970s knot theory was shown, among other things, to be connected to algebraic number theory, by virtue of the solution of Smith's conjecture concerning periodic mappings. At the beginning of the 1980s, due to the discovery by V.F.R. Jones of his epochal knot invariant, knot theory moved from the realm of topology to mathematical physics. This was further underlined when it was shown that knot theory is closely related to the solvable models of statistical mechanics. As knot theory grows and develops, its boundaries continue to shift[1].

A knot is a subset of 3 - space that is homeomrphic to the circle, a link is a set of finitely many disjoint knots that are called its components, a singular knot is a knot with self-intersections. If K is a knot (or link), we shall say that $\hat{P}\ (K) = \ \hat{K}$ is the projection of K . $\hat{P}$ The map that projects the point $\hat{P}(x,y,z)$ in E^3 onto the point $\hat{P}\ (x.y,0)$ in the xy - plane. However $\hat{K}$ is not a simple closed curve lying on the plane, since $\hat{K}$ possesses several points of intersection. In fact $\hat{K}$ is a graph represents K and has some properties:-

1. $\hat{K}$ has at most a finite number of points of intersection.
2. If Q is a point of intersection of $\hat{K}$ then $P^{-1}(Q)\bigcap K$ in K has exactly two points, which make a cross in K[1].

In general a graph not necessary represents a knot in E^3 by embedding. Now we can impose some condition on a graph to represent a knot (or a link) we call it a link graph. G is a link graph if

1. G is finite connected graph.
2. G is planer.
3. G has the homogeneous vertex degree 4[2].

For this every link graph G with n vertices has 2n edges and n + 2 countries.

Let f be a polygonal embedding of a link graph G into E^3. We call the image f(G) a representation of G which is a knot (in general a link).

To obtain the types of link graphs we look for the adjacency matrix.

The adjacency matrix of the link graph is square matrix of size n x n which is symmetric has integer values and for every row and every column the sum is 4 and the main diagonal are zeros.

For n=2 we have only one adjacency matrix also for n=3we have only one adjacency matrix. For any n we get all possible adjacency matrices and draw the link graph for all of them and calculate Jones polynomial $V_l(t)$ to know which knot (or link) the link graph is represented it[2].

Definitions and background: In this section we will summarize some definitions and theorems which we will be used in the main results.

1. An oriented knot is one with a chosen direction along the string[3,4].
2. Let K be an oriented knot (or link) diagram define the writhe of K, $\omega(K)$, by the equation $\omega(K) = \sum_P \varepsilon(P)$ where P runs over all crossings in K and $\varepsilon(P)$ is the sign of the crossing[1,3,4].

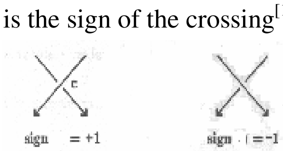

Fig. 1

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3. Kauffman's braket polynomial: Let L, $\hat{L}$ and $\hat{L}_0$ be the skein diagrams given below

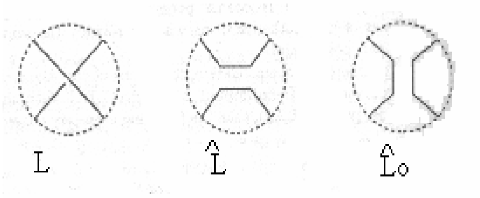

Fig. 2

Then the following equality holds:

$P_L(a) = aP_{\hat{L}}(a) + \overset{-1}{a} P_{\hat{L}_0}(a)$.

if L is a trivial diagram O of a trivial knot, than $P_0(a) = 1$

if $P_L(a)$ is the Kauffman bracket polynomial of the " unoriented " diagram L and $\omega(L)$ is the writhe of L then define

$\hat{P}_L(a) = (-\overset{-3}{a})^{\omega(L)} p_L(a)$. Then $\hat{P}_L(a)$ is an invariant of an oriented knot (or link) denoted by $\hat{P}_K(a)$ [1,3,4].

4. Jones polynomial: Suppose L is an oriented link, then the Jones polynomial $V_L(t)$ can be defined (uniquely) from the following two axioms.

Axiom (1): If L is the trivial knot ,then $V_L(t) = 1$.

Axiom (2): Suppose L_+, L_-, L_0 are skein diagram given bellow:

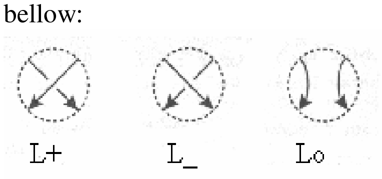

Fig. 3

Then the following skein relation holds,

$\frac{1}{t}V_{L_+}(t) - tV_{L_-}(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})V_{L_0}$.

The polynomial itself is a Laurent polynomial in $\sqrt{t}$ i.e it may have terms in which $\sqrt{t}$ has a negative exponent[1,5-7] .

5. Tutte polynomial: let $G = (V, E)$ be a given graph and e any arbitrary one of its edges. The Tutte polynomial of G is a polynomial in two variables $T(G; x, y)$ which satisfies the following properties:

1. $T(G; x, y) = 1$ for $G = (V, E)$ with $E = \phi$.
2. $T(G; x, y) = x$ if e is a bridge.
3. $T(G; x, y) = y$ if e is a loop.
4. $T(G; x, y) = T(G \setminus e; x, y) + T(G / e; x, y)$ where $G \setminus e$ (respectively G / e) denotes the graph obtained from G by deleting (resp. ,contracting) e .
5. if G has at least two edges and e is a bridge (resp., loop) of G, $T(G; x, y) = xT(G / e; x, y)$ (resp. $T(G; x, y) = yT(G \setminus e; x, y)$.

Not that properties (1),……,(5) allow the computation of $T(G; x, y)$ for any graph G [4,2,8].

6. Knot graph: Every planar graph gives a decomposition of the plane in connected regions the so called countries. By coloring these countries chessboard like manner with the two colors black and white such that the unbounded country is white. From the black countries (resp. white countries) one gets the corresponding graph of these black countries (resp. white countries),which called knot graph[2,8].

Example:

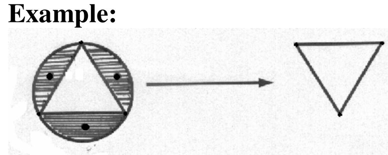

Fig. 4

Link graph and graph of black countries (knot graph).

7. Folding: Let (X, τ) be a topological space .A map $f : (X, \tau) \to (X, \tau)$ is said to be folding of a topological space into itself if $f(X) \subset X$,and either

i. $\forall G \in \tau, F(G) = U \subset G, U \in \tau$.

or ii. $\forall G \in \tau, f(G) = U \in \tau$ [9]. Many types of foldings are discussed in[10-13].

8. Retraction: Let A be a subset of a topological space X .A continuous map $r : X \to A$ is a retraction if $r(a) = a$ for all $a \in A$ [14].

3. Relation between polynomials

Theorem (1-3): The Kuafman bracket polynomial can be computed from Tutte polynomial by $P_L(a) = a^{V-C}T(-a^{-4}, -a^4)$, where V is the number of vertices and C is the number of countries of the knot graph[2,5,7].

Corollary(1-3): The Kaufman bracket polynomial of oriented link diagram is computing by:

$\hat{P}_L(a) = (-a^3)^{-\omega(L)} P_L(a)$ [2,3].

Theorem (2-3): The Jones polynomial is computed from kaufman brackt polynomial by:

$V_L(t) = \hat{P}_L(t^{-\frac{1}{4}})$ [5,7].

Theorem (3-3): The Jones polynomial is computed from Tutte polynomial of knot graph by:

$V_L(t) = -t^{\frac{c-v+3\omega(L)}{4}} T(-t, -t^{-1})$.

Appling to n=3, the only one adjacency matrix is:

0	2	2
2	0	2
2	2	0

The link graph corresponding of this adjacency matrix is:

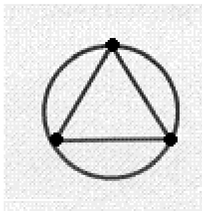

Fig. 5

The corresponding graph of black countries (knot graph) is:

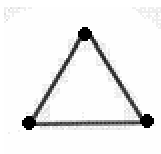

Fig. 6

And its Tutte polynomial is $T(G;x,y) = x^2 + x + y$.

By represent the link graph G in E^3 and take orientation .

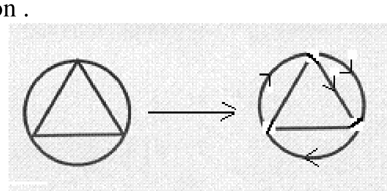

Fig. 7

Hence $\omega(L) = 3$.

$$V_L(t) = -t^{\frac{c-v+3\omega(L)}{4}} T(-t,-t^{-1}).$$

$$= -t^{\frac{2-3+9}{4}} T(-t,-t^{-1})$$

$V_L(t) = t + t^3 - t^4$ which is the Jones polynomial of trefoil knot.

RESULTS

Folding of trefoil knot and its graph

1-4. Folding of trefoil knot: Let K be a trefoil knot and f be a folding from K into itself . then we have some types of folding :

(1) $f(a) = f(b) = c$

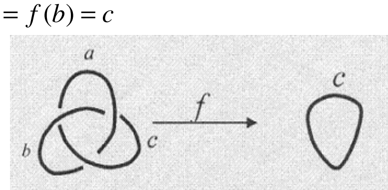

Fig. 8

Theorem (1-1-4): The folding of a knot is not necessary a knot.

2. Crossing folding: A folding which folds a point of upper arc crossing on a point of lower crossing is said to be crossing folding.
In this case we have:

a)

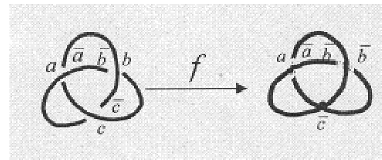

Fig. 9

$f(c) = c'$.

b)

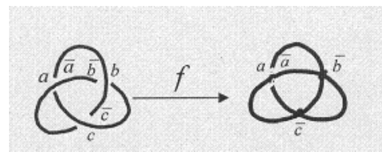

Fig. 10

c) $f(b) = b'$, $f(c) = c'$

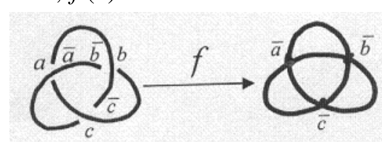

Fig. 11

$f(a) = a', f(b) = b', f(c) = c'$.

Theorem (2-1-4): Every crossing folding of a knot into itself gives a singular knot.

Proof: Let K be a knot and f be a crossing folding from K into itself, then *f(K)* is not a simple closed curve in E^3 (Fig. 9-11)),since *f(K)* possesses at lest one point of intersection .i.e. *f(K)* is a singular knot

Theorem (3-1-4): A folding of a singular knot not necessary a singular knot. The following examples show that:

1)

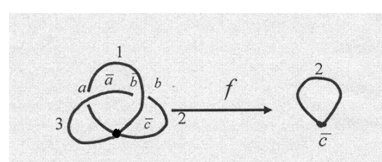

Fig. 12

$f(2) = f(3) = 1$.

2)

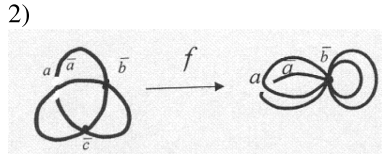

Fig. 13

$f(c') = b'$.

3)

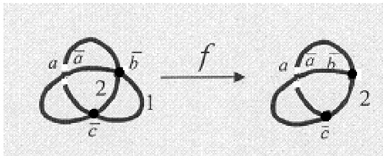

Fig. 14

$f(1) = 2$.

4)

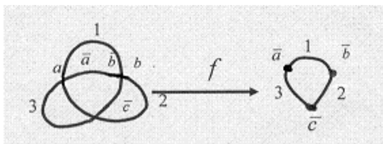

Fig. 15

3. Topological folding: Appling topological folding in a successive steps (Fig. 16) on a trefoil knot getting an equivalent knots until reaching the null knot which different of the successive knots ,it represents a limit folding .

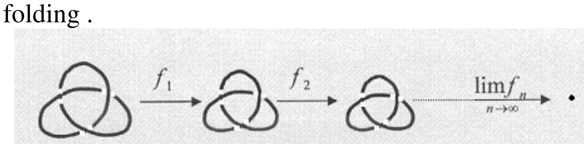

Fig. 16

Theorem (4-1-4): The limit of a topological foldings of any knot into itself is a point.

Proof

Let $f_1 : K \to K, f_2 : f_1(K) \to f_1(K), ...f_i :$
$f_{i-1}(f_{i-2}...(K)) \to f_{i-1}(f_{i-2}....(K))$

We see that $\lim_{i\to\infty} f_i(f_{i-1}(f_{i-2}(....(K).....)) = p$ is a point.

4. Special contraction folding

Definition: The folding which contracts the distance between the crossing is called contracting folding. In this type we fix one arc and contract the other arcs ,by successive steps (Fig. 17) getting an equivalent knots. The limit of this process the null knot ,which is exactly the limit of foldings.

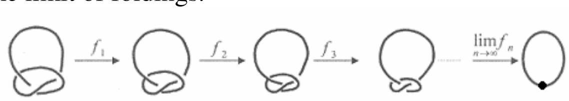

Fig. 17

Theorem (5-1-4): A limit of special contraction foldings of any knot is a loop.

Proof: Let K be a knot of m- crossings, then K has m arcs $a_1, a_2, ..., a_m$,and let L be the distance between crossings ,if we are fixed an arc a_1 and contracted another by sequence of foldings $f_i, i = 1,2,.....,n$ whenever $n \to \infty$ then $L \to zero$ i.e. The arc a_1 is a loop which exactly a limit of the special contraction foldings. See Fig. 17 as special case.

Corollary (1-1-4): If we fix two arcs of a knot, then the limit of special contraction folding of a knot is two loops common in a vertex.

Proof: The proof comes directly from theorem (5-1-4).

Theorem (6-1-4): The retraction of any knot by removing a point is a point.

Proof: Let $r : K - p_i \to K - p_i$ be a retraction, if $\dim r(k - p_i) = \dim K$, then it is not the minimum retraction. and $r(K - p_i) \equiv$ some topological folding $f \equiv r$, but if $\dim r(K - p_i) \neq \dim K \Rightarrow f \not\equiv r$ .the minimum retraction

Theorem (7- 1- 4): The limit of topological foldings of a knot is equivalent to the retraction of a knot by removing a point.

Proof: Let $f_i : K \to K$ be a sequence of topological folding of the knot into itself then $\lim_{i\to\infty} f_i(K) = p$ theorem (4-1-4).

Assume $r : K - q \to K - q$ a retraction by removing a point, $r(K - q) = p$ theorem (6 -1-7). i.e. $r(K - q) = \lim_{i\to\infty} f_i(K)$.

2-4. Folding of a link graph: Let L be a link graph which represent a trefoil knot (Fig. 5) and let f a folding from L into itself ,then we have some types of folding:

(1)

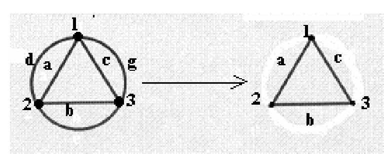

Fig. 18

$f(d) = a, f(e) = b, f(g) = c$.

This folding gives a complete graph K_3, which is a knot graph of a trefoil knot ,but not represent a knot.

Theorem (1-2-4): A folding of a link graph is not necessary a link graph.

(2) (a)

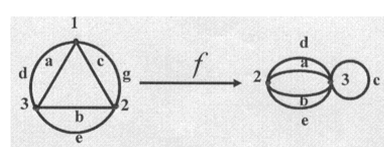

Fig. 19

$f(g) = c, f(1) = 3$

(b)

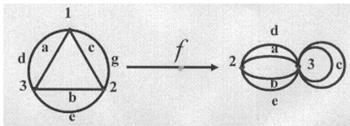

Fig. 20

$f(1) = 3.$

3. Topological folding: Appling topological folding in a successive steps (Fig. 21) of a link graph , represent a trefoil knot getting an equivalent link graph until reaching the null link graph which different of the successive link graph ,it represents the limit of foldings.

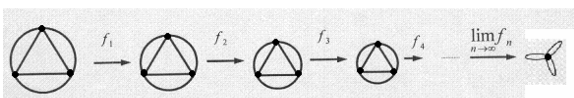

Fig. 21

Theorem (2-2-4): The limit of topological foldings for any link graph of n vertices is a graph with only one vertex having n loops.

Proof: The proof is clear from the above discussion.

Theorem (3-2-4): There are some types of foldings of a link graph decrease the degree of Y in Tutte polynomials and there is another types increase the degree of Y in Tutte polynomials.

Proof: Let L be a link graph Fig. 20 The Tutte polynomial of L is

$T(L; x, y) = y^4 + 2y^3 + 3y^2 + (1+3x)y + x + x^2$ The degree of Y is 4 and the Tutte polynomial of $f(L)$ is $T(f(L); x, y) = y^5 + y^4 + y^3 + xy^2$ The degree of Y is 5, i.e. degree of y was increased.

Look Fig. 18 The Tutte polynomial of $f(L)$ is $T(f(L); x, y) = y + x + x^2$ the degree of Y is 1,i.e. degree of y was decreased. We have two types of retractions of a link graph:

The first types:

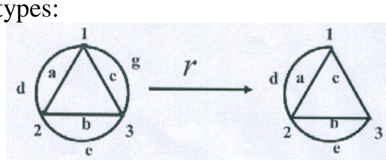

Fig. 22

$r : K - \{v\} \to \bar{K}, v \in g, v \neq 1, v \neq 3$

the second types:

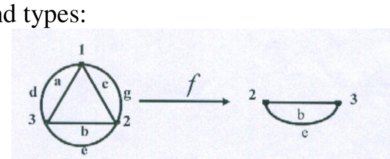

Fig. 23

$r : K - \{1\} \to \bar{K}$.

Corollary (1-2-4): Every retraction of a link graph decreases the degree of y in Tutte polynomials.

Theorem (4-2-4): For any foldings of link graph which decrease the degree of y in Tutte polynomials there exist a retraction is equivalent to it.

Proof: Let L be a link graph and *f* be a folding from L into itself s.t. *f* decreases the degree of y in Tutte polynomials, this means one edge or more was folded to another this equivalent to remove this edge hence there is a retraction r equivalent to *f*.

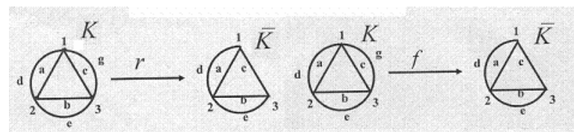

Fig. 24

Now we see the foldings and retraction with Tutte polynomials:

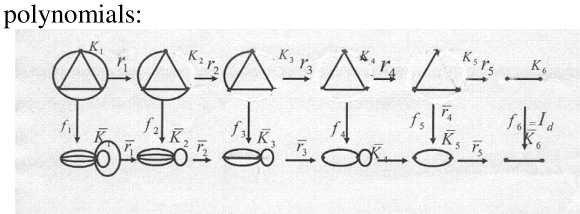

Fig. 25

From the chin of foldings and retractions we get $\bar{r}_i \circ f_i = f_{i+1} \circ r_i$ and

$T(K; x, y) = T(K_1; x, y) + T(\bar{K}_1; x, y)$

$T(K_{1;}; x, y) = T(K_2; x, y) + T(\bar{K}_2; x, y)$

$T(K_2; x, y) = T(K_3; x, y) + T(\bar{K}_3; x, y)$

$T(K_4; x, y) = T(K_4; x, y) + T(\bar{K}_4; x, y)$

this means

$T(K; x, y) = T(\bar{K}_1; x, y) + T(\bar{K}_2; x, y)$
$+ T(\bar{K}_3; x, y) + T(K_4; x, y) + T(\bar{K}_4; x, y)$

3-4. Folding of sheeted trefoil knot: Suppose R be a sheeted in E^3, Fig. 25, so that we form a trefoil knot $V(R)$

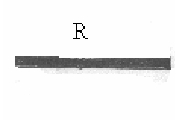

Fig. 25

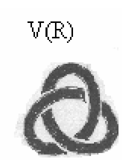

Fig. 26

by it, Fig. 26, called the sheeted trefoil knot ,hence $V(R)$ takes four cases:

Case 1: $V(R)$ with two boundaries $\partial\, V(R)$={L,M}, which is exactly a sheeted bound knots in normal case

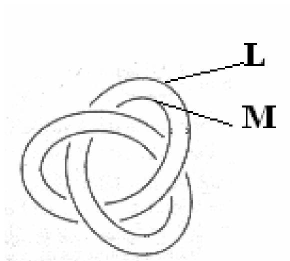

Fig. 27

Case 2: $V(R)$ with no boundaries $\partial\, V(R)$=0 ,we call a sheeted stripe trefoil knot.

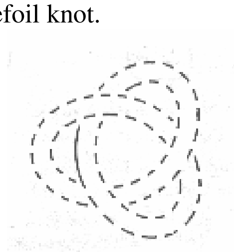

Fig. 28

Case 3: $V(R)$ with external boundary $\partial\, V(R)$=L..

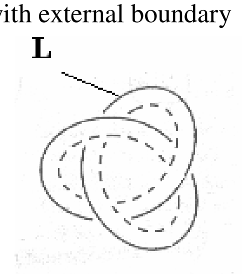

Fig. 29

Case 4: $V(R)$ with internal boundary $\partial\, V(R)$=M.

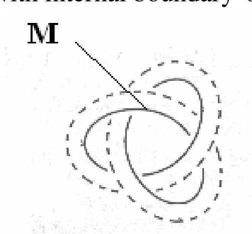

Fig. 30

Now we introduce some types of foldings for all above cases:

For case 1

a. Let K_1 be a sheeted trefoil knot with boundaries $\partial\, K_1$={L,M} and f be a folding from K_1 into itself such that f twists L on itself in a successive steps until reaching a boundary M so that we get a trefoil knot which is the limit of folding. Fig. 31, i.e.

$f_1 : K_1 \to K_1, f_2 : f_1(K_1) \to f_1(K_1),$
$f_3 : f_2(f_1(K_1)) \to f_2(f_1(K_1)),........,$
$f_n(f_{n-1}(f_{n-2}(.......(f_1(K_1))...) \to$
$f_{n-1}(f_{n-2}(f_{n-3}(.......(f_1(K_1)).....)$

such that $f_i(L) = L$ and $\lim\limits_{n\to\infty} f_n = M$ which is a trefoil knot of one dimension.

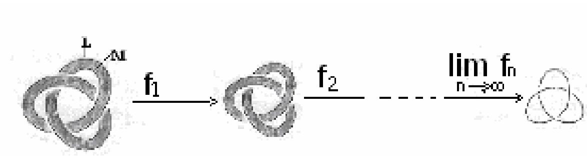

Fig. 31

b. Let K_2 be a sheeted trefoil knot with boundaries $\partial\, K_2$={L,M} and f be a folding from K_2 into itself such that f twists M on itself in a successive steps until reaching a boundary L so that we getting a trefoil knot which is the limit of folding . Fig. 32, i.e.

$f_1 : K_2 \to K_2, f_2 : f_1(K_2) \to f_1(K_2),$
$f_3 : f_2(f_1(K_2)) \to f_2(f_1(K_2)),........,$
$f_n(f_{n-1}(f_{n-2}(.......(f_1(K_2))...) \to$
$f_{n-1}(f_{n-2}(f_{n-3}(.......(f_1(K_2)).....)$

such that $f_i(M) = M$ and $\lim\limits_{n\to\infty} f_n = L$ which is a trefoil knot of one dimension.

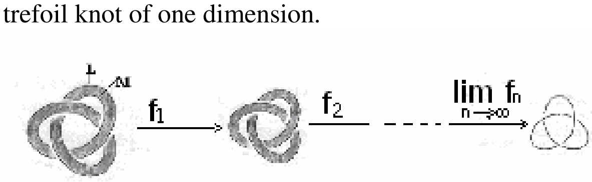

Fig. 32

c. Let K_3 be a sheeted trefoil knot with boundaries $\partial\, K_3$={L,M} and f be a folding from K_3 into itself such that f (L)= M we getting a tubular trefoil knot, which in fact is a hollow torus. Fig. 33, $f : K_3 \to K_3, s.t. f\ (L) = M$ and any generator $L_i, f(L_i) = \overline{L}_i$ and $\overline{L}_i$ homeomorphic or diffeomorphic to L_i. Any sequence of foldings $f_1, f_2, f_3, ..., f_n \Rightarrow \lim\limits_{n\to\infty} f_n = \text{torus}$. Also the unfoldings $unf_1, unf_2, unf_3, ...unf_n \Rightarrow \lim\limits_{n\to\infty} unf_n = \text{torus}$

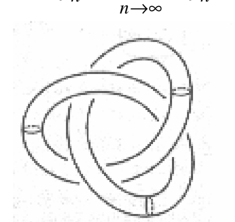

Fig. 33

For case 2: Let V be a sheeted trefoil knot with out boundaries $\partial V = 0$ and b is a rim line, Fig. 34 and f be a folding from V into itself such that f twists a sheeted around the rim line, so we getting a trefoil knot which exactly the limit of foldings (Fig. 35)

$f_1 : V \to V$, $f_2 : f_1(V) \to f_1(V)$,

$f_3 : f_2(f_1(V)) \to f_2(f_1(V))$,........,

$f_n(f_{n-1}(f_{n-2}(.......(f_1(V))...)$

$\to f_{n-1}(f_{n-2}(f_{n-3}(.......(f_1(V)).....)$

$f_i(\text{rim}) = \text{rim}, \lim\limits_{n\to\infty} f_n = \text{rim}$ which is a trefoil knot of one dimension,

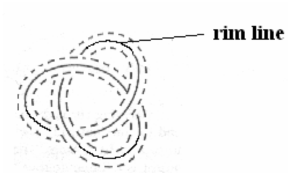

Fig. 34

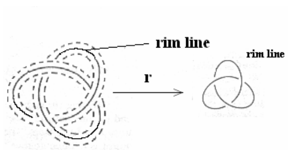

Fig. 35

For case 3: Let *V(R)* be a sheeted trefoil knot with boundaries ∂ *V(R)* = L and *f* be a folding from *V(R)* into itself such that *f* twists L on itself in a successive steps, so that we getting a trefoil knot which is the limit of foldings (Fig. 35).

In this case the limit of this conditional folding equivalent to the retraction of *V(R)* into L.

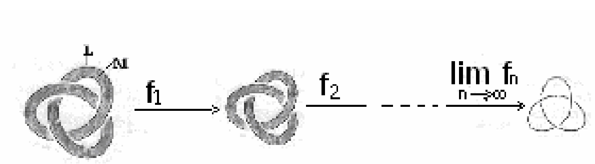

Fig. 36

For case 4: Let *V(R)* be a sheeted trefoil knot with boundaries $\partial V(R)$ = M and *f* be a folding from *V(R)* into itself such that *f* twists M on itself in a successive steps, so that we getting a trefoil knot which is the limit of foldings. Fig. 36.

In this case the limit of this conditional folding equivalent to the retraction of *V(R)* into M.

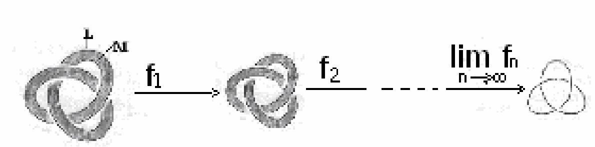

Fig. 37

Theorem (1- 3 - 4): The limit of foldings of a sheeted trefoil knot which twist a boundary on itself is a trefoil knot.

Proof: The proof is clear.

For all cases, if we apply the folding f from $V(R)$ into itself such that f (1)= f (2)= 3 we getting a sheeted unknot hereditary case of boundary, (Fig. 38)

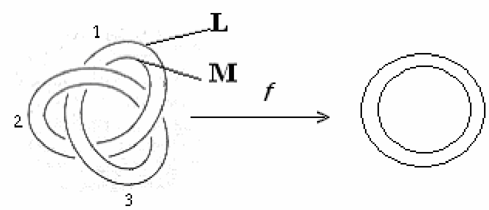

Fig. 38

Theorem (2-3-4): The limit of foldings of a sheeted knot $V(R)$ is equivalent to the retraction if $V(R)$ is open.

Proof: If

$f_1 : V(R) \to V(R)$, $f_2 : f_1((V(R)) \to f_1(V(R))$,

$f_3 : f_2(f_1(V(R))) \to f_2(f_1(V(R)))$ and

,........, $f_n(f_{n-1}(f_{n-2}(.......(f_1(V(R)))...)$

$\to f_{n-1}(f_{n-2}(f_{n-3}(.......(f_1(V(R))).....)$

$\lim\limits_{n\to\infty} f_n = \text{rim line}$ Fig. 34

by take a retraction $r : V(R) \to \text{rim line}$ s.t. $r(V(R)) = \text{rim line}$

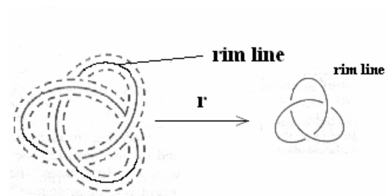

Fig. 39

hence we get $\lim\limits_{n\to\infty} f_n = \text{rim line} = r(V(R))$

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