Approximation Methods for Fixed Points of Quasi-contractive Mappings in Arbitrary Real Banach Spaces

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Abstract: Owojori and Imoru [1] introduced some iteration methods and investigated its convergence for pseudocontractive and accretive operators in arbitrary Banach spaces. In this work, strong convergence results are established for fixed points of a general type of quasincontractive mappings in arbitrary real Banach spaces. Our results here represent some improvements on the earlier results of Owojori and Imoru [1], Nampally and Singh [2], Chidume [3, 4], Qihou [5], Ganguly and Bandgypalnay [6], Chidume and Osilike [7], Rhoades [8] and Osilike [9] for quasincontractive operators in Banach spaces.

Key words: Quasincontractive Mappings, Arbitrary Real Banach Spaces, Iteration Methods

INTRODUCTION

Let $K$ be a nonempty subset of an arbitrary real Banach space $X$ and $T$ a selfmapping of $K$. An operator $T: K \to K$ is called quasincontractive, in the general sense, if the following inequality:

$$T_x - T_y \leq s_1 |x - y| + s_2 \{ |x - T_x| + |y - T_y| \} \quad \quad (1.1)$$

holds for all $x, y \in K$, where $s_1, s_2, s_3$ are real constants in $[0,1]$ satisfying $s_1 + 2s_2 + 2s_3 = 1$.

Several authors including the authors, Rhoades [8], Chidume [3, 4], Osilike [9], Chidume and Osilike [7], Qihou [5] among others, have investigated the fixed points of quasincontractive mappings by the Mann and Ishikawa iteration methods (including their modifications) and fixed point results have been established.

Recently, Owojori and Imoru [1] introduced a more acceptable three-step iteration method which contains the previous iteration methods of Mann and Ishikawa as special cases. It is defined for arbitrary $x_0 \in K$ - a closed bounded convex subset of a Banach space $B$, by:

$$\begin{align*}
x_{n+1} &= a_n x_n + b_n T x_n + c_n S x_n \\
y_n &= a_n' x_n + b_n' T x_n + c_n' v_n \\
z_n &= a_n'' x_n + b_n'' T x_n + c_n'' v_n
\end{align*} \quad \quad (1.2)$$

where, $S, T$ are nonlinear uniformly continuous self-mappings of $K$ satisfying some contractive definitions and $\{v_n\}, \{w_n\}$ are bounded sequences in $K$. Also $\{a_n\}, \{a'_n\}, \{a''_n\}, \{b_n\}, \{b'_n\}, \{b''_n\}, \{c_n\}, \{c'_n\}$, $\{c''_n\}$ are real sequences in $[0,1]$ satisfying:

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n = 1$,
(ii) $\sum_{n=0}^{\infty} \rho_n = \infty$

Two special cases of (1.2) are given respectively by:

$$\begin{align*}
x_{n+1} &= a_n x_n + b_n T x_n + c_n T x_n \\
y_n &= a_n' x_n + b_n' T x_n + c_n' v_n \\
z_n &= a_n'' x_n + b_n'' T x_n + c_n'' v_n
\end{align*} \quad \quad (1.3)$$

and

$$\begin{align*}
x_{n+1} &= a_n x_n + b_n T x_n + c_n w_n \\
y_n &= a_n' x_n + b_n' T x_n + c_n' v_n \\
z_n &= a_n'' x_n + b_n'' T x_n + c_n'' v_n
\end{align*} \quad \quad (1.4)$$

where, the parameters satisfy the same conditions as for (1.2).

We observed that the iteration schemes given by (1.2), (1.3) and (1.4) are all well defined and are generalizations of the Mann and Ishikawa iteration schemes (with or without errors). Also, (1.3) is a slight generalization of (1.4). Our purpose in this manuscript is to establish the convergence of the iteration methods (1.4), (1.3) and (1.2) to the fixed points and common fixed points of the general quasincontractive operator given by (1.1), in arbitrary real Banach space.

RESULTS

In the sequel, we shall require the following result due to Weng [10].

Lemma 2.1: Let $\{\rho_n\}$ be a nonnegative sequence of real numbers satisfying:
\[ \rho_{n+1} \leq (1 - \delta_n) \rho_n + \sigma_n \]  \hspace{1cm} (2.1) \\

where \( \delta_n \in [0,1] \), \( \sum \delta_i = \infty \), and \( \sigma_n = \alpha(\delta_n) \). Then \( \lim_{n \to \infty} \rho_n = \rho \).

**Theorem 2.2:** Let \( X \) be an arbitrary real Banach space and \( K \) a closed, convex, bounded subset of \( X \). Suppose \( T \) is a uniformly continuous selfmapping of \( K \) satisfying the quasi-contractive definition (1.1). Define sequence \( \{x_n\} \) iteratively for arbitrary \( x_1 \in K \) by:

\[
\begin{align*}
x_{n+1} &= a_n x_n + b_n Ty_n + c_n u_n \\
y_n &= a_n' x_n + b_n' Tz_n + c_n' y_n \\
z_n &= a_n'' x_n + b_n'' T\rho_n x_n + c_n'' w_n
\end{align*}
\]

where \( \{u_n\}, \{v_n\} \) and \( \{w_n\} \) are arbitrary sequences in \( K \) and \( \{a_n\}, \{a_n'\}, \{a_n''\}, \{b_n\}, \{b_n'\}, \{b_n''\}, \{c_n\}, \{c_n'\}, \{c_n''\} \), are real sequences in \([0,1]\) satisfying:

(i) \[ a_n + b_n + c_n = a_n' + b_n' + c_n' = 1, \]

(ii) \[ \sum a_n = 1, \]

(iii) \[ a_n = b_n + c_n = b_n' + c_n' \]

and \( k_1 \leq \frac{1 - \alpha_n}{\alpha_n' \beta_n' \gamma_n} \)

(iv) \[ \frac{s_1 - s_3}{1 - s_3} < 1 \]

Then, the sequence \( \{x_n\} \) converges strongly to a fixed point of \( T \).

**Proof:** Since \( T \) is uniformly continuous on the closed bounded convex set \( K \), then, by Deimling [11], \( F(T) \) is nonempty. Let \( x^* \in F(T) \). Then, from our hypothesis.

\[
x_{n+1} - x^* = a_n x_n + b_n Ty_n + c_n u_n - x^* = (1 - \alpha_n)(x_n - x^*) + \alpha_n (Ty_n - x^*) + c_n (u_n - Ty_n)
\]

\[
\leq (1 - \alpha_n) x_n - x^* + \alpha_n Ty_n - x^* + c_n (u_n - Ty_n)
\]

\[(2.2)\]

Since \( T \) is quasi-contractive in the sense of (1.1), then

\[ Ty_n - x^* \leq s_3 y_n - x^* + s_1 [ u_n - Ty_n ] + x_n - Ty_n \]

Therefore,

\[ (1 - s_3) Ty_n - x^* \leq (s_1 + s_3) y_n - x^* + s_2 y_n - Ty_n \]

Therefore, \( Ty_n - x^* \leq s_1 y_n - x^* + s_3 [ y_n - Ty_n ] \)

\[(2.3)\]

Continuity of \( T \) implies that there exists a real number \( M_1 < \infty \) such that:

\[ u_n - Ty_n \leq M_1, \quad y_n - Ty_n \leq M_1. \]

Also \( c_n \leq \alpha_n \). Substituting into (2.3), we have

\[ x_{n+1} - x^* \leq (1 - \alpha_n) x_n - x^* + k_1 y_{n+1} - x_n + \alpha_n (k_2 + 1) M_1 \]

\[(2.4)\]

We also have the following estimates:

\[ y_n - x^* = a_n x_n + b_n Ty_n + c_n v_n - x^* \leq (1 - \beta_n)(x_n - x^*) + \beta_n (Ty_n - x^*) + c_n (v_n - Ty_n) \]

\[(2.5)\]

Continuity of \( T \) implies that there exists a real number \( M_2 < \infty \) such that:

\[ Tz_n - x^* \leq k_2 z_n - x^* + k_3 z_n - Tz_n \]

\[(2.6)\]

Therefore, substituting (2.6) into (2.5), we have:

\[ y_n - x^* \leq (1 - \beta_n) x_n - x^* + \beta_n k_1 z_n - x^* + \beta_n M_2 + \beta_n M_2 \]

\[(2.7)\]

Substitute (2.7) into (2.4), we obtain:

\[ x_{n+1} - x^* \leq (1 - \alpha_n) x_n - x^* + k_2 k_3 (1 - \beta_n) x_n - x^* \]

\[ = \beta_n k_1 z_n - x^* + \beta_n k_2 (M_2 + M_2) + \alpha_n (k_2 + 1) M_1 \]

\[(2.8)\]

We also have the following estimates:

\[ z_n - x^* \leq [1 - k_2 (k_2 - 1)] x_n - x^* + \gamma_n k_1 (k_2 + 1) M_1 \]

\[(2.9)\]
where, \( M_5 < \infty \) is a real number such that
\[
x_n - T x_n \leq M_5 \quad \text{and} \quad T x_n - w_n \leq M_5.
\]

Substitute (2.9) into (2.8) we obtain:
\[
x_{n+1} - x^* \leq \left(1 - \alpha_n(1-\lambda_n(1-\beta_n))\right) x_n - x^* + \alpha_n \beta_n(1-\lambda_n(1-\beta_n)) x_n - x^*
+ \gamma_n(1-\lambda_n(1-\beta_n)) M_5 + \alpha_n \beta_n(1-\lambda_n(1-\beta_n)) M_5
+ \alpha_n \beta_n(1-\lambda_n(1-\beta_n)) M_5
= \left(1 - \alpha_n(1-\lambda_n(1-\beta_n))\right) x_n - x^*
+ \alpha_n \beta_n(1-\lambda_n(1-\beta_n)) M_5 + \alpha_n \beta_n(1-\lambda_n(1-\beta_n)) M_5
+ \alpha_n \beta_n(1-\lambda_n(1-\beta_n)) M_5
\]
(2.8)

\[
\frac{s_1 - s_3}{1 - s_3} < 1
\]

Then \( \{ x_n \} \) converges strongly to the fixed point of \( T \).

**Proof:** Let \( p \) be a fixed point of \( T \). Then
\[
x_{n+1} = a_n x_n + b_n T x_n + c_n T^2 x_n - p
= \left(1 - \alpha_n(\gamma_n p + \alpha_n(\gamma_n p) + \beta_n(\gamma_n p)) \right)
+ \alpha_n \beta_n(\gamma_n p + \beta_n(\gamma_n p))
+ \alpha_n \beta_n(\gamma_n p + \beta_n(\gamma_n p))
\]
(2.11)

Since \( T \) satisfies the condition (1.1) above, we have:
\[
T y_n - p \leq k_1 y_n - p + k_2 y_n - T y_n
\]
(2.12)

where, \( k_1, k_2 \) are the same as in Theorem 2.2 and by continuity of \( T \), there exists \( M_5 < \infty \)
such that \( y_n - T y_n \leq M_5 \) and \( T y_n - T x_n \leq M_5 \).

Substituting into (2.12), yields
\[
x_{n+1} - p \leq (1-\alpha_n) x_n - p + \alpha_n(\gamma_n p + \beta_n(\gamma_n p))
+ \alpha_n(\gamma_n p + \beta_n(\gamma_n p))
\]
(2.13)

We also have
\[
y_n - p = a_n x_n + b_n T x_n + c_n x_n - p
\leq (1-\beta_n) x_n - p + \beta_n(\gamma_n p + \beta_n(\gamma_n p))
+ \beta_n(\gamma_n p + \beta_n(\gamma_n p))
\]
(2.14)

Quasi-contractive condition on \( T \) implies that:
\[
T y_n - p \leq k_1 y_n - p + k_2 y_n - T x_n
\]
(2.15)

Continuity of \( T \) implies that there exists a real number \( M \) such that:
\[ u_n - Tz_n \leq M_7 \text{ and } z_n - Tz_n \leq M_7 \]

Substituting (2.15) into (2.14), we obtain:
\[
y_n - p \leq (1 - \beta_n) x_n - p + \beta_n k_1 z_n - p + \beta_n (k_2 + 1) M_7\tag{2.16}
\]

We also have:
\[ z_n - p \leq (1 - \gamma_n) x_n - p + \gamma_n Tz_n - p + c_n v_n - Tz_n \]

and
\[ Tz_n - p \leq k_1 x_n - p + k_2 z_n - Tz_n \]

Furthermore, continuity of T on the and bounded set K implies that there exists \( M_8 < \infty \) such that \( v_n - Tz_n \leq M_8 \) and \( x_n - Tz_n \leq M_8 \). So that
\[ z_n - p \leq (1 - \gamma_n) x_n - p + \gamma_n k_1 x_n - p + \gamma_n (k_2 + 1) M_8 \tag{2.17}\]

Substitute (2.17) into (2.16), we obtain:
\[
y_n - p \leq (1 - \beta_n) x_n - p + \beta_n k_1 (1 - \gamma_n (1 - \gamma_n) x_n - p + \beta_n k_1 (1 - \gamma_n (1 - \gamma_n) M_8 + \beta_n (k_2 + 1) M_7)\tag{2.18}
\]

Substituting (2.18) into (2.13) yields:
\[
x_n - p \leq (1 - \alpha_n) x_n - p + \alpha_n k_1 (1 - \beta_n (1 - \gamma_n (1 - \gamma_n) x_n - p + \beta_n k_1 (1 - \gamma_n (1 - \gamma_n) M_8 + \beta_n (k_2 + 1) M_7) + \alpha_n (k_2 + 1) M_8\tag{2.19}
\]

where, \( M_8 = \max\{M_6, M_7, M_8\} \).

Since \( k_1^2 < k_1 \), we have \( \alpha_n k_1 < \alpha_n k_2 \).

Let \( \sigma_n = \alpha_n k_1 (1 - \beta_n y_n - k_1) + \beta_n + 1) M_8 \). Then (2.19) reduces to
\[
x_n - p \leq [1 - \alpha_n + \alpha_n k_1 - \alpha_n k_1 (1 - \beta_n y_n - k_1)] x_n - p + \sigma_n = [1 - \alpha_n (1 - k_1 (1 - \beta_n y_n - k_1))] x_n - p + \sigma_n\tag{2.20}
\]

Put
\[ t_n = \alpha_n (1 - \beta_n y_n - k_1) \text{ and } \rho_n = x_n - p \]

Then (2.20) becomes
\[ \rho_n - 1 = (1 - \tau_n) \rho_n + \sigma_n \]

Clearly, \( 0 \leq \tau_n \leq 1 \) and \( \Sigma \tau_n = \infty \). Also \( \sigma_n = o(\tau_n) \).

Hence, by Lemma 2.1, \( \rho_n \to 0 \) as \( n \to \infty \). This implies that \( \{x_n\} \) converges strongly to \( p \). The proof is complete.

**Remark:** Theorem 2.3 is an improvement on Theorem 2.2. Hence, it also generalizes the results of Chidume [4], Chidume and Osilike [7], Rhoades [8] and Osilike [9].

We now consider the general iteration method (1.2) above and establish its convergence to the fixed point of operator satisfying (1.1) in the following.

**Theorem 2.4:** Let \( K \) be a nonempty closed bounded convex subset of a Banach space \( B \). Suppose \( T \) and \( S \) are uniformly continuous self-mappings of \( K \) and \( T \) satisfies the quasi-contractive definition (1.1). Define sequence iteratively for arbitrary by:
\[ x_{n+1} = a_n x_n + b_n T x_n + c_n S x_n \]

\[ y_n = a_n x_n + b_n T x_n + c_n w_n \]

where, \( \{a_n\}, \{w_n\} \) are arbitrary sequences in \( K \) and \( \{a_n\}, \{a_n\}, \{a_n\}, \{b_n\}, \{b_n\}, \{b_n\} \), \( \{b_n\}, \{c_n\}, \{c_n\}, \{c_n\}, \{c_n\} \), are real sequences in \( [0,1] \) satisfying

(i) \( a_n + b_n + c_n = a_n + b_n + c_n = a_n + b_n + c_n = 1 \),

(ii) \( \sum b_n = \infty \), \( \lim b_n = \lim b_n = \lim b_n = 0 \)

(iii) \( \alpha_n := b_n + c_n \beta_n := a_n + b_n + c_n Y_n := b_n + c_n + \).

(iv) \( k_1 = \frac{s_1 - s_3}{1 - s_3} < \frac{1}{2} \)

Then the sequence \( \{x_n\} \) converges strongly to a fixed point of \( T \).

**Proof:** Let \( p \) be a fixed point of \( T \). Then
\[ x_{n+1} - p = a_n x_n + b_n T y_n + c_n S x_{n+1} - p \]
\[ = (1 - \alpha_n) x_n - p + \alpha_n T y_n - p + c_n S x_{n+1} - T y_n - p \]
\[ \leq (1 - \alpha_n) x_n - p + \alpha_n T y_n - p + c_n S x_{n+1} - T y_n - p \]
\[ \leq (1 - \alpha_n) x_n - p + \alpha_n T y_n - p + c_n S x_{n+1} - T y_n - p \]
\[ = (1 - \alpha_n) x_n - p + 2 \alpha_n T y_n - p + c_n S x_{n+1} - T y_n - p \]

Now, by quasi-contractive property of \( T \), we have:
\[ T y_n - p \leq k_1 y_n - p + k_2 y_n - T y_n \]

By continuity of \( T \) and \( S \) on \( K \), there exists a real number \( M_1 < \infty \) such that \( y_n - T y_n \leq M_1 \) and \( S x_{n+1} - p \leq M_1 \). Substitute into (2.21), we have:
\[ x_{n+1} - p \leq (1 - \alpha_n) x_n - p + 2 \alpha_n (k_1 y_n - p) + k_1 M_1 + \alpha_n M_1 \leq (1 - \alpha_n) x_n - p + 2 \alpha_n k_1 y_n - p + \alpha_n (1 + 2k_2) M_1 \]  
\[ (2.22) \]

From our hypothesis and applying similar procedure as above, we have:
\[ y_n - p \leq (1 - \beta_n) x_n - p + \beta_n S x_n - p + \beta_n S x_n - p \]  
\[ (2.23) \]

since \( c_n \leq \beta_n \). Continuity of \( S \) on \( K \) implies that there exists a real number \( M_2 < \infty \) such that \( S x_n - p \leq M_2 \) and \( S x_n - y_n \leq M_2 \). Substituting into (2.23), we have:
\[ y_n - p \leq (1 - \beta_n) x_n - p + 2 \beta_n M_2 \]  
\[ (2.24) \]

substituting (2.24) and (2.22) yields
\[ x_{n+1} - p \leq (1 - \beta_n) x_n - p + 2 \alpha_n k_1 M_2 + \alpha_n (1 + 2k_2) M_1 \leq (1 - \alpha_n (1 + 2k_1 (1 - \beta_n))) x_n - p + \alpha_n (1 + 2k_2 + 1) M \]  
\[ (2.25) \]

where, \( \beta = \max\{M_1, M_2\} \). Now, put \( \rho_n = \alpha_n (1 - 2k_1 (1 - \beta_n)) \), and \( \alpha_n = \alpha_n (1 + 2k_2) M_1 \) and let \( \rho_n = x_n - p \), then (2.25) becomes
\[ \rho_{n+1} = (1 - \alpha_n) + \alpha_n \]  
Clearly \( \alpha_n > 0 \). Therefore, by Lemma 2.1, \( \rho_n \rightarrow 0 \), as \( n \rightarrow \infty \). This implies that \( \{x_n \} \) converges strongly to \( p \) – the fixed point of \( T \). The proof is complete.

**Remark:** Theorem 2.4 generalizes both Theorem 2.2 and 2.3 above, hence it is a generalization of the results of Rhoades [8], Chidume [3, 4], and Osilike [7] and Osilike [9].

**Theorem 2.5:** Let \( K \) be a nonempty closed bounded convex subset of an arbitrary real Banach space \( B \). Suppose \( S \) and \( T \) are uniformly continuous quasi-contractive selfmappings of \( K \) satisfying (1.1). Define sequence \( \{x_n \} \) iteratively by (1.2) above, where, \( \{a_n \}, \{a_n \}, \{a_n \}, \{b_n \}, \{b_n \}, \{b_n \}, \{c_n \}, \{c_n \}, \{c_n \}, \{c_n \} \) are real sequences in \( [0, 1] \) satisfying the following conditions:

(i) \( a_n + b_n + c_n = a_n + b_n + c_n = a_n + b_n + c_n = 1 \),
(ii) \( \sum b_n = \infty \), \( \lim_{n \to \infty} b_n = \lim_{n \to \infty} b_n = \lim_{n \to \infty} b_n = 0 \)

(iii) \( a_n := b_n + c_n \), \( a_n := b_n + c_n \), \( a_n := b_n + c_n \), \( T_n := b_n + c_n \)
(iv) \( k_1 \leq \frac{1 - \alpha_n}{\alpha_n \beta_n} \gamma_n \)
(v) \( k_1 \leq \frac{s_i - s_j}{1 - s_i} < \frac{1}{3} \)

Suppose \( S, T \) have a common fixed point in \( K \). Then \( \{x_n \} \) converges strongly to the common fixed point of \( S \) and \( T \).

**Proof:** Let \( p \) be the common fixed point of \( S \) and \( T \). Then
\[ x_{n+1} - p = a_n x_n + b_n (T y_n - p) + c_n S x_n - p \]
\[ = (1 - \alpha_n) x_n - p + \alpha_n (T y_n - p) + \alpha_n (S x_n - T y_n) \]
\[ \leq (1 - \alpha_n) x_n - p + \alpha_n (T y_n - p) + \alpha_n (S x_n - T y_n) \]
\[ \leq (1 - \alpha_n) x_n - p + \alpha_n (T y_n - p) + \alpha_n (S x_n - T y_n) \]
\[ \leq (1 - \alpha_n) x_n - p + 2 \alpha_n (T y_n - p) + \alpha_n (S x_n - T y_n) \]
\[ \leq (1 - \alpha_n) x_n - p + 2 \alpha_n (T y_n - p) + \alpha_n (S x_n - T y_n) \]
By quasi-contractive property of \( T, S \), we have:
\[ T y_n - p \leq k_1 y_n - p + k_2 y_n - T y_n \]
and
\[ S x_n - p \leq k_1 x_n - p + k_2 x_n - S x_n \]
By continuity of \( S \) and \( T \) on the bounded set \( K \), there exists a real number \( M_1 < \infty \) such that \( y_n - T y_n \leq M_1 \) and \( x_n - S x_n \leq M_1 \). Substituting into (2.26), we have:
\[ x_{n+1} - p \leq (1 - \alpha_n) x_n - p + 2 \alpha_n k_1 y_n - p + 2 \alpha_n k_2 M_1 + \alpha_n k_1 x_n - p + \alpha_n k_2 M_1 \]
\[ = (1 - \alpha_n (1 - k_1)) x_n - p + 2 \alpha_n k_1 y_n - p + 3 \alpha_n k_2 M_1 \]
From our hypothesis,
\[ y_n - p \leq (1 - \beta_n) x_n - p + \beta_n S x_n - p + c_n S x_n - p \]
Since \( S \) satisfies (1.1), we have:
\[ S x_n - p \leq k_1 x_n - p + k_2 x_n - S x_n \]
Therefore,
\[ y_n - p \leq (1 - \beta_n) x_n - p + \beta_n k_1 x_n - p + \beta_n S x_n - p + \beta_n S x_n - p \]
Continuity of \( S, T \) on \( K \) implies that there exists a real number \( M_2 < \infty \) such that \( x_n - S x_n \leq M_2 \) and \( S x_n - p \leq M_2 \). Therefore:
\[ y_{n+1} \leq (1 - \beta_n)x_n + \beta_n \kappa_1 z_n + (\beta_n(k_2 + 1)M_2) \tag{2.28} \]

We also have the following estimates.

\[ z_n \leq (1 - \gamma_n) x_n + \gamma_n T x_n + c_n T x_n - w_n \leq (1 - \gamma_n) x_n + \gamma_n [k_1 x_n + (k_2 + 1)M_2] + k_2 \gamma_n x_n - T x_n - w_n = [1 - \gamma_n (1 - k_1)] x_n + \gamma_n (k_2 + 1)M_2 \tag{2.29} \]

where, \( M_2 < \infty \) is a real number such that:

\( x_n - T x_n \leq M_2 \) and \( T x_n - w_n \leq M_2 \).

Substitute (2.29) into (2.28), we obtain:

\[ y_{n+1} \leq (1 - \beta_n)x_n + \beta_n \gamma_n [k_1 (1 - k_1)] x_n + \beta_n \kappa_2 (k_2 + 1)M_2 + \beta_n \gamma_n (k_2 + 1)M_2 \tag{2.29} \]

Now, let \( M = \max[M_1, M_2, M_3] \) and then substitute (2.30) into (2.27). We have:

\[ x_{n+1} \leq (1 - \alpha_n (1 - k_1)) x_n + 2\alpha_n \kappa_1 [1 - \beta_n (1 - k_1) (1 - \gamma_n (1 - k_1))] \]

\[ + 2\alpha_n \beta_n \kappa_2 (k_2 + 1)M_2 + \alpha_n \gamma_n (k_2 + 1)M_2 \tag{2.31} \]

Let

\[ \alpha_n = +2\alpha_n \beta_n \kappa_1 [1 - \beta_n (1 - k_1) (1 - \gamma_n (1 - k_1))] M_3 + 3\alpha_n \kappa_2 M \]

and

\[ t_n = \alpha_n (1 - k_1) - 2\alpha_n \kappa_1 (1 - \beta_n (1 - k_1)) \gamma_n (1 - k_1) \]

\[ = \alpha_n (1 - k_1) - 2\alpha_n \kappa_1 (1 - \beta_n (1 - k_1)) \gamma_n (1 - k_1) \]

\[ = \alpha_n (1 - k_1) - 2\alpha_n \kappa_1 (1 - \beta_n (1 - k_1)) \gamma_n (1 - k_1) \]

\[ \geq 0 \]

Since, \( k_1 \leq \frac{1}{3} \)

Observe that \( k_1 \leq \frac{1}{3} \) implies that \( k_1 \leq \frac{1}{2} \) and using the fact that \( 1 - k_1 < 1 \), we have:

\[ t_n \leq \alpha_n [1 - 2\kappa_1 (1 - \beta_n (1 - k_1))] \]

Clearly,

\[ 1 - \beta_n (1 - k_1) (1 - \gamma_n (1 - k_1)) \leq 1 \]

Therefore,

\[ 2\kappa_1 (1 - \beta_n (1 - k_1) (1 - \gamma_n (1 - k_1))) \leq 1 \]

Hence

\[ t_n \leq \alpha_n [1 - 2\kappa_1 (1 - \beta_n (1 - k_1) (1 - \gamma_n (1 - k_1)))] < 1 \]

Thus \( 0 \leq t_n \leq 1 \). Observe that since \( \sum \alpha_n = \infty \), then \( \sum t_n = \infty \). Also, \( \alpha_n = o(t_n) \).

Put \( \rho_n = x_n - y_n \). Then (2.31) reduces to

\[ \rho_{n+1} = (1 - t_n) \rho_n + \sigma_n \]

Hence, by Lemma 2.1, \( \rho_n \to 0 \), as \( n \to \infty \). This implies that \( \{x_n\} \) converges strongly to the common fixed point of \( S \) and \( T \). The proof is complete.

**Remark:** Theorem 2.5 is an improvement on the result of Owojori and Inoro [11] which itself is an extension of the previous results of Rhoades [8], Qiho [5] and Osilike [9] to the generalized Ishikawa type iteration method and to the common fixed point of the operators involved.

We now consider the situation when the two nonlinear operators in the iteration scheme satisfy different contractive definitions. In particular, we investigate the common fixed point of \( S \) and \( T \) when \( T \) is a uniformly continuous quasi-contractive operator in the sense of (1.1) and \( S \) is a \( k \)-contractive operator, where \( k \in (0,1) \), an arbitrary real Banach space. Our result is the following.

**Theorem 2.6:** Let \( T, K, B \) and \( \{x_n\} \) be as defined in Theorem 2.5 above. Suppose \( S: K \to K \) is \( k \)-contractive, \( k \in (0,1) \) and the condition (iv) on the parameters in Theorem 2.5, is replaced with \( k_1 \leq \frac{1}{2} \) \((1 - k)\) and all other conditions in Theorem 2.5 are satisfied. Suppose \( S, T \) have a common fixed point in \( K \). Then, \( \{x_n\} \) converges strongly to the common fixed point of \( S \) and \( T \), if it exists.

**Proof:** Let \( p \) be the common fixed point of \( S \) and \( T \). By similar procedure as in the proof of Theorem 2.5, we have the following estimates:

\[ x_{n+1} \leq (1 - \alpha_n) x_n + \alpha_n T x_n - S x_n \leq (1 - \alpha_n) x_n + \alpha_n k_1 y_n \]

\[ + \alpha_n k_2 y_n - T y_n + \alpha_n T y_n - S x_n \]

\[ = (1 - \alpha_n) x_n + \alpha_n k_1 y_n + \alpha_n k_2 y_n - T y_n + \alpha_n T y_n - S x_n \]

\[ + 2\alpha_n k_2 y_n - T y_n + \alpha_n T y_n - S x_n + 2\alpha_n k_1 y_n - S x_n \]

\[ + \alpha_n k_2 \]

\[ = (1 - \alpha_n (1 - k)) x_n + 2\alpha_n k_1 y_n + 2\alpha_n k_2 y_n \]

\[ + \alpha_n k_2 \]

We also have the following estimates:

\[ y_{n+1} \leq (1 - \beta_n) x_n + \beta_n S x_n + c_n S x_n \]

\[ + \beta_n S x_n - v_n \]

\[ \leq (1 - \beta_n) x_n + \beta_n k_1 y_n + \beta_n k_2 y_n + \beta_n S x_n - v_n \]

\[ + \beta_n S x_n - v_n \]

\[ = (1 - \beta_n (1 - k)) x_n + \beta_n k_1 y_n + \beta_n k_2 y_n + \beta_n S x_n - v_n \]

\[ + \beta_n S x_n - v_n \]

\[ = (1 - \beta_n (1 - k)) x_n + \beta_n k_1 y_n + \beta_n k_2 y_n + \beta_n S x_n - v_n \]

\[ + \beta_n S x_n - v_n \]

\[ = (1 - \beta_n (1 - k)) x_n + \beta_n k_1 y_n + \beta_n k_2 y_n + \beta_n S x_n - v_n \]

\[ + \beta_n S x_n - v_n \]

\[ = (1 - \beta_n (1 - k)) x_n + \beta_n k_1 y_n + \beta_n k_2 y_n + \beta_n S x_n - v_n \]

\[ + \beta_n S x_n - v_n \]

\[ = (1 - \beta_n (1 - k)) x_n + \beta_n k_1 y_n + \beta_n k_2 y_n + \beta_n S x_n - v_n \]

\[ + \beta_n S x_n - v_n \]

\[ = (1 - \beta_n (1 - k)) x_n + \beta_n k_1 y_n + \beta_n k_2 y_n + \beta_n S x_n - v_n \]

\[ + \beta_n S x_n - v_n \]

\[ = (1 - \beta_n (1 - k)) x_n + \beta_n k_1 y_n + \beta_n k_2 y_n + \beta_n S x_n - v_n \]
By similar method as above, we have:

\[ z_n \rightarrow p \leq (1 - \gamma_n) x_n \rightarrow p + \gamma_n T x_n \rightarrow p + \gamma_n T x_n \rightarrow w_n \]
\[ \leq (1 - \gamma_n) x_n \rightarrow p + k \gamma_n x_n \rightarrow p + \gamma_n T x_n \rightarrow w_n \]
\[ + (1 - \gamma_n(1 - k_1)) x_n \rightarrow p + k \gamma_n x_n \rightarrow T x_n \]
\[ + \gamma_n T x_n \rightarrow w_n \quad (2.34) \]

Substitute (2.34) into (2.33) and observe that the continuity of S, T on bounded set K implies that there exists a positive real number \( N, 1 < \infty \) such that:

\[ x_n \rightarrow T x_n \leq N, \quad T x_n \rightarrow w_n \leq N \quad \text{and} \quad S x_n \rightarrow v_n \leq N \]

we have:

\[ x_{n+1} \rightarrow p \leq (1 - \beta_n) x_n \rightarrow p + \beta_n (1 - \gamma_n(1 - k_1)) x_n \rightarrow p \]
\[ + [\beta_n k \gamma_n x_n i + N] \leq [1 - \beta_n + \beta_n (1 - \gamma_n(1 - k_1))] \frac{x_n \rightarrow p}{x_n \rightarrow p} \]
\[ + [\beta_n k \gamma_n x_n i + N] \leq (2.35) \]

Substitute (2.35) into (2.32) and observe that continuity of T on K implies that there exists a positive real number \( N, 1 < \infty \) such that \( y_n \rightarrow T y_n \leq N \), we have:

\[ x_{n+1} \rightarrow p \leq (1 - \alpha_n(1 - k)) x_n \rightarrow p + \alpha_n k \gamma_n x_n i + N] \leq (2.36) \]

Let

\[ \alpha_n = 2 \alpha_n k \gamma_n x_n i + N] + 2 \alpha_n k \gamma_n x_n i + N] \]

and

\[ t_n = \alpha_n \left( (1 - k) \left( 2k_1 (1 - \beta_n (1 - \gamma_n(1 - k_1))) \left( 1 - \gamma_n(1 - k_1)) \right) \right) \]

Also, let \( p_n = x_n \rightarrow p \) then (2.36) reduces to

\[ \rho_{n+1} \leq (1 - \alpha_n) p_n + \sigma_n \]

Now, \( 0 < k < 1 \) implies that \( 0 < 1 - k < 1 \). Therefore,

\[ t_n \leq \alpha_n \left( 1 - k \left( 2k_1 (1 - \beta_n (1 - \gamma_n(1 - k_1))) \right) \right) \]

Observe that, \( (1 - \alpha_n(1 - k) \left( 1 - \gamma_n(1 - k_1)) \right) \leq 1 \) and from hypothesis \( k_1 = 1 - \frac{1}{2} k \) implies that \( k_1 \leq 1 \)

Thus, \( 2k_1 (1 - \beta_n (1 - \gamma_n(1 - k_1))) \) implies that \( k_1 \leq 1 \)

and

\[ 1 - 2k_1 (1 - \beta_n (1 - \gamma_n(1 - k_1))) \leq 1 \]

But \( \alpha_n \leq 1 \), therefore,

\[ t_n \leq \alpha_n \left( 1 - 2k_1 (1 - \beta_n (1 - \gamma_n(1 - k_1))) \right) \leq 1 \]

We can also write

\[ t_n = \alpha_n \left( (1 - k) - 2k_1 (1 - \beta_n (1 - \gamma_n(1 - k_1))) \right) \]

So that \( t_n \geq 0 \). Thus \( 0 \leq \alpha_n \leq 1 \), for all \( n \rightarrow \infty \). It is also clear that \( \sum \alpha_n = \infty \) and \( \alpha_n = o(\alpha_n) \).

Therefore, by Lemma 2.1, \( p_n \rightarrow 0 \) as \( n \rightarrow \infty \). This implies that \( \{x_n\} \) converges strongly to \( p \) the common fixed point of \( S \) and \( T \). This completes the proof.

**Remark:** a generalized Mann type iteration method is defined for arbitrary \( x_1 \) in \( K \) by

\[ x_{n+1} = a_0 x_n + b_n T x_n + c_n S x_n \]

where, \( S, T \) are nonlinear self-mappings of nonempty compact convex subset of an arbitrary Banach space Band \( \{a_n\}, \{b_n\}, \{c_n\} \) are real sequences in \([0, 1] \) satisfying

\[ (i) \quad a_0 + b_n + c_n = 1, \quad (ii) \quad \sum b_n = \infty. \]

We now investigate the common fixed points of \( S \) and \( T \) for the generalized Mann type iteration method. Our result is the following.

**Theorem 2.7:** Let \( S, T, K, X \) be as in Theorem 2.5. Define sequence \( \{x_n\} \) for arbitrary \( x_1 \) in \( K \) by

\[ x_{n+1} = a_0 x_n + b_n T x_n + c_n S x_n \]

where, \( \{a_n\}, \{b_n\}, \{c_n\} \) are sequences in \([0, 1] \) satisfying

\[ (i) \quad a_0 + b_n + c_n = 1, \quad (ii) \quad \sum b_n = \infty. \]

Suppose \( S, T \) have a common fixed point in \( K \). Then the sequence \( \{x_n\} \) converges strongly to the common fixed point of \( T \) and \( S \).

**Proof:** Let \( p \) be a common fixed point of \( S \) and \( T \). Then, from the hypothesis, we have:

\[ x_{n+1} \rightarrow p = (a_0 x_n + b_n T x_n + c_n S x_n) \]
\[ = (1 - \alpha_n) x_n \rightarrow p + \alpha_n T x_n \rightarrow p + \alpha_n S x_n \rightarrow p \]
\[ \leq (1 - \alpha_n) x_n \rightarrow p + \alpha_n T x_n \rightarrow p + \alpha_n S x_n \rightarrow p \]
\[ + \alpha_n T x_n \rightarrow p + \alpha_n S x_n \rightarrow p \]
\[ \leq (1 - \alpha_n) x_n \rightarrow p + \alpha_n T x_n \rightarrow p + \alpha_n S x_n \rightarrow p \]
\[ + \alpha_n T x_n \rightarrow p + \alpha_n S x_n \rightarrow p \]
\[ \leq (1 - \alpha_n) x_n \rightarrow p + \alpha_n T x_n \rightarrow p + \alpha_n S x_n \rightarrow p \]
\[ + \alpha_n T x_n \rightarrow p + \alpha_n S x_n \rightarrow p \]
\[ = (1 - \alpha_n(1 - k_1)) x_n \rightarrow p + 3 \alpha_n k_2 N_3 \]

where, \( s_1, s_2 \) and \( s_3 \) are the constants in the general quasi-contractive definition (1.1), with
\[ k_2 = \frac{s_1 - s_3}{1 - s_3} < 0 \quad \text{and} \quad N_3 < \infty \] is a real number such that
\[ x_n - T x_n \leq N_3 \quad \text{and} \quad x_n - S x_n \leq N_3 \]
(by uniform continuity of S and T on the bounded set K).

Now, let \( t_n = \alpha_n (1 - 3k_2) \). Then from hypothesis, \( 0 < t_n < 1 \) and \( \sum t_n = \infty \). Let \( \sigma_n = 3 \alpha_n k_2 N_3 \). Then \( \sigma_n = o(t_n) \).

Put \( \rho_n = x_n - p \). Then (2.14) becomes
\[ \rho_{n+1} = (1 - t_n) \rho_n + \sigma_n \]
Therefore, by Lemma 2.1.
\[ \lim_{n \to \infty} \rho_n = 0. \]
This implies that \( \{x_n\} \) converges strongly to p.

The proof is complete.

**Remark:** Theorem 2.7 is an extension of the results of Ganguly and Bandyopadhyay [6], Rhoades [8] to the general quasi-contractive definition (1.1) and to the common fixed point of the two operators involved in the generalized Mann type iteration procedure.

**REFERENCES**