Kernel Density Estimation for Interdeparture Time of GI/G/1 Queues

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Abstract: The departure process of a single queue has been studied since the 1960s. Due to its inherent complexity, closed form solutions for the distribution of the departure process are nearly intractable. In this study, kernel type estimators of the density of interdeparture time in a GI/G/1 queue are studied. Uniform strong consistency of the estimators in a GI/G/1 queue and their rates of convergence are obtained. The stochastic processes are shown to satisfy the strong mixing condition with random instants of sampling. With the analysis presented, we provide a novel analytic tool for studying the departure process in a general queuing model.

Key words: Strong Mixing, Density Estimators, Kernel, Bandwidth, GI/G/1, Departure Processes

INTRODUCTION

Queueing theory has been extensively used in today's applications in communication systems and flexible manufacturing networks. In order to obtain a good performance estimate of the system, instead of solving the optimal problem at the whole network, it is often preferable to study the waiting distribution at each station (isolated node). Describing the characteristic of the departure process at a node is thus one important issue in studying queuing networks, because the departure processes at one node may be considered as the arrival process at subsequent nodes. Many literatures have been studied in this field. For example, Daley [1] investigated departure processes from a GI/M/1 queue and studied the correlation structure. Bertsimas [2] pointed out the difficulty of analysis of a G/G/1 queueing system and derived an algorithm of a relatively low order of complexity for the system-size, prearrival and post-departure probability distributions. Chang [3] showed that the Poisson process is the only stationary and ergodic process that induces identical distributions on the interdeparture times when the service times are exponentially distributed. Luh [4] provided an analytic tool for studying the departure process in a GI/G/1 queueing system.

Since network traffic is composed of complex random processes which may not conform to any known Markovian model as commonly adopted in queuing analysis, the sequence of interdeparture times may be nonstationary, and even have a long history. In the application of most real cases, at least certain kinds of (weakly) dependent should be considered in the process. Instead of conventional queuing approaches, many researchers have paid exceptional attention on the covariance structure. For example, Melamed et. al. [5] captured the autocorrelated traffic by TES (Transform-Expand-Sample). Hwang and Li [6] developed a statistical-match queuing (SMaQ) tool to study measurement-based traffic management problem. In advance queuing analysis, recent real-life traffic measurement indicates the significance of traffic macrodynamics to network performance. The macrodynamics, versus microdynamics, is defined for characterizing the traffic behavior on the coarse, versus refined, time scales at which the process is observed. Consequently, the Autoregressive/Moving Average (ARMA) process has been used to model the macrodynamic behavior of the arrival process in a queueing system. Kulkarni and Li [7] show the second-order statistics of the microdynamics are well captured by white noise in the arrival process with power spectrum which can have a significant impact on the queuing performance.

In the present research, we study kernel type estimators of the density of interdeparture time in a GI/G/1 queue. Uniform strong consistency of the estimators in a GI/G/1 queue and their rates of convergence are obtained. The stochastic processes are shown to satisfy the strong mixing condition with random instants of sampling.

Kernel Estimate of the Interdeparture Time:
Statistically, the description of the departure process is usually written in terms of the interdeparture intervals \( \{ D_n \} \), where \( D_n \) is the time between the \( n \)th and \( (n+1) \)th departure epochs, \( n = 1, 2, \ldots \). We shall confine our discussion to a more general GI/G/1 queuing system that implies a stationary and weak dependent sequence of positive random variables \( \{ D_n \} \) with finite mean \( \mathbb{E}(D_n) \). Let \( P(\cdot) \) be the probability measure defined in Definition 1. In view of stationarity, define the distribution function
of interdeparture times as:

\[ f(t) = P(D_1 \leq t) . \]

This study is concerned with the estimation of the probability density function \( f(t) \) for an interdeparture times process \( \{D_n, n = 1, 2, \ldots\} \) on the basis of the discrete time samples \( \{D(t_k)\}, \quad 1 \leq k \leq n \), where the sampling instants \( \{t_k\} \) are random. As an estimator of \( f(t) \) we shall consider the kernel estimate defined by:

\[ f_n(t) = \left( n b_n \right)^{-1} \sum_{j=1}^{n} K \left( \frac{t - D(t_j)}{b_n} \right), \tag{1} \]

where \( K \) is a kernel function and \( \{b_n\} \) is a sequence of bandwidths tending to zero as \( n \) tends to infinity. Here \( f_n \) takes values in \( \mathbb{R}_+ = [0, \infty) \).

Density estimation has been studied extensively since the works of Rosenblatt [8] and Parzen [9]. Under dependent situations, kernel type density estimators have been investigated by Masry [10, 11], Robinson [12], Roussas [13] and Tran [14, 15] for various weakly dependent processes. Gyorfi et al. [6] studied the uniform convergence and the \( L_1 \) convergence under different mixing conditions.

The purpose of this study is to establish weak conditions under which \( f_n \) converges uniformly on \( \mathbb{R}_+ \) to \( f \) a.s. We also obtain sharp rates of convergence of \( f_n \) to \( f \).

**Assumptions and Preliminaries:** Consider the GI/G/1 queue, in which the arrivals form a renewal process and the service time are independently and identically distributed. Without loss generality, service times are assumed independent of the arrival process, and the arrival rates is strictly less than the service rate. Let \( A_n, S_n, T_n \) and \( W_n \) be the interarrival time, service time, flow time, waiting time of the \( n \)-th customer. Since the flow time equals service time plus waiting time, we have:

\[ T_n = S_n + W_n \]

Moreover, for each \( n \), we have \( D_n = S_{n+1} + (A_n - T_n)^+ \), where \( x^+ = \max(x, 0) \) which implies

\[ D_{n+1} = W_{n+1} - W_n + S_{n+1} - S_n + A_n \tag{2} \]

Let \( A, S, T, W \) and \( D \) be a generic interarrival time, service time, steady-state flow time, steady-state waiting time, and interdeparture time. Therefore in steady-state we have

\[ T = S + W = (T - A)^+ \]

and

\[ D = S + (A - T)^+ \]

where \( = \) means equal in distribution.

Because \( S \) and \( A \) are bounded, the density of \( D \) is bounded as well. Recall the strong mixing condition which is defined by Tran [15].

**Definition 1.** Let \( X_k, \quad k = \ldots, 1, 0, 1, \ldots \), be a strictly stationary sequence of random variables defined on a probability space \( (\Omega, F, \mathbb{P}) \) and taking values in \( \mathbb{R} \). Let \( F^0_{-\infty} \) and \( F^\infty_m \) denote, respectively, the \( \sigma \)-fields generated by \( X_k, \quad k \leq 0 \) and by \( X_k, \quad k \geq m \). Then \( X_k \) is strong mixing if

\[ \alpha(m) = \sup \{ |P(A \cap B) - P(A)P(B)| : A \in F^0_{-\infty}, B \in F^\infty_m \} \downarrow 0, \quad \text{as} \quad m \to \infty . \tag{3} \]

The strong mixing condition is well known to be weaker than many dependence conditions, for example, the absolutely regular condition or the \( \phi \)-mixing condition. For more information on strong mixing processes, see Rosenblatt [8], or Roussas [13].

We first write a weaker condition of (3) according to the evolutionary development of \( \{D_n\} \). From (2), the
covariance of $D$ gives:

$$
E(D_k, D_{k+h}) = E(D_k) E(D_{k+h}) \tag{4}
$$

$$
= E(S_{k+h-1}, W_{k-1}) - E(A_{k+h-1}, W_{k-1}) - E(S_{k+h-1}, W_k) + E(S_{k+h}, W_{k-1}) + E(A_{k+h}, W_k) + E(S_{k+h}, W_{k+h}) - E(S_k, W_{k+h})
$$

$$
+ E(W_{k+h-1}, W_{k+h}) - E(S_{k+h}, W_{k+h}) - E(S_{k+h-1}, W_{k+h}) + E(S_k, W_{k+h})
$$

$$
- E(S_{k+h-1}) E(W_{k-1}) - E(S_{k+h}) E(W_k) + E(A_{k+h-1}) E(W_{k-1}) - E(A_{k+h}) E(W_k) + E(S_{k-1}) E(W_{k+h-1}) + E(S_k) E(W_{k+h-1})
$$

$$
- E(S_{k+h-1}) E(W_{k-1}) + E(S_{k-1}) E(W_{k+h}) - E(S_k) E(W_{k+h})
$$

Note that $T_n, A_n$ and $S_{k+h}$ are independent of each other; therefore, $W_{k-1}, A_{k+h}$ and $S_{k+h}$ are independent of each other as well. Thus, (4) becomes

$$
E(S_{k+h-1}, W_{k-1}) - E(A_{k+h-1}, W_{k-1}) - E(S_{k+h-1}, W_k) + E(S_{k+h}, W_{k-1}) + E(A_{k+h}, W_k) + E(S_{k+h}, W_{k+h}) - E(S_k, W_{k+h})
$$

$$
+ E(W_{k+h-1}, W_{k+h}) - E(S_{k+h}, W_{k+h}) - E(S_{k+h-1}, W_{k+h}) + E(S_k, W_{k+h})
$$

$$
- E(S_{k+h-1}) E(W_{k-1}) - E(S_{k+h}) E(W_k) + E(A_{k+h-1}) E(W_{k-1}) - E(A_{k+h}) E(W_k) + E(S_{k-1}) E(W_{k+h-1}) + E(S_k) E(W_{k+h-1})
$$

$$
- E(S_{k+h-1}) E(W_{k-1}) + E(S_{k-1}) E(W_{k+h}) - E(S_k) E(W_{k+h})
$$

Since $\operatorname{Cov}(S_{k+h-1}, W_{k-1}) \rightarrow 0, \operatorname{Cov}(W_{k-1}, W_{k+h-1}) \rightarrow 0$ as $h \rightarrow \infty$, it has (5) approach to 0 and so does (4) as $h \rightarrow \infty$.

Thus, in order to have a general result for the density function $f$, we should give the following assumptions for the kernel $K$ and the process $f(t)$. Let the letter $C$ to denote a generic constant. All limits are taken as $n \rightarrow \infty$ unless indicated otherwise. To prove the main theorem, we need the following assumptions.

**Assumption 1:** The kernel $K$ is a probability density function satisfies $|K(x) - K(y)| < C|x - y|$.

We assume that the sampling instants $\{t_k\}$ are random, constituting a renewal process on $[0, \infty)$. Let $\{\tau_k\}$, $1 \leq k < \infty$, be a sequence of i.i.d. random variables with a common distribution $G(x)$ on $[0, \infty)$ with $G(0) = 0$ and a finite mean $\int_0^\infty x dG(x) = 1/\beta < \infty$. The sampling instants are defined $t_k = \sum_{i=1}^k \tau_i$.

Let $G_k(x)$ be the cumulative distribution function of $t_k$. If $G(x)$ is absolutely continuous with density $g(x)$ then $G_k(x)$ has a derivative, say $g_k(x)$, which is the probability density function of $t_k$. Define $m(t) = 2\sum_{k=1}^\infty kg_k(t)$, $t > 0$.

The quantity $m$ is often referred to in the renewal theory literature as second-order factorial density.

**Assumption 2:** The renewal-type sampling instants $\{t_k\}$ have an intensity density $g(x)$ on $[0, \infty)$ and the second-order differential $m(x)$ satisfies $m(x) \leq C(1 + x)$ on $\mathbb{R}_+$.

**Lemma 1:** Suppose $A(x)$ and $S(x)$ are bounded and satisfied with Lipschitz condition. Then, the density $f(x)$ is bounded and satisfied with Lipschitz condition.

**Proof:** By (2), we have $D = S + (A - T)^+ \leq S + A \leq C$ since $S$ and $A$ are bounded. $|f(x) - f(y)| = |S(x) - S(y) + (A(x) - T(x))^+ - (A(y) - T(y))^+| \leq |S(x) - S(y)| + |A(x) - A(y)| \leq C|x - y|$

**Assumption 3:** Suppose the joint probability density $f(x, y; \tau)$ of $(D_0, D_\tau)$ exists. There exists some constants $C$ such that it satisfies

$$
\int_0^\infty f(x, y; \tau + s) g(\tau)d\tau \leq C < \infty
$$

for all $x, y$ and $s \geq 0$.

Denote

$$
\psi(n, 1) = (\log n)^{1/2} / (nb_n)^{1/2}
$$

**Assumption 4:** For some $\ell > 0$, $(\psi(n, 1))^{-1} \sup_{x \leq x'} |f(x)| = O(1)$,
\[ \sum_{n=1}^{\infty} n(1-\int_{x\geq b_n} f(x)dx) < \infty. \]

**Assumption 5:** \( (\psi(n,1)b_n)^{-1} \sup_{x \geq b_n} K(x) = O(1). \)

**Uniform Convergence of \( f_n \):** The following lemmas are needed in the proof of Theorem. The proofs of these lemmas can be found in the Wu [17] based on assumptions.

**Lemma 2:** Suppose Assumptions 1-5 hold and \( b_n \) tends to zero slowly enough that \( nb_n / \log n \to \infty \). We have

\[ \sup_{x \leq b_n} |f_n(x) - Ef_n(x)| = O(\psi(n,1)) \quad \text{a.s. as} \quad n \to \infty. \]

**Lemma 3:** Suppose the condition of Lemma 2 holds. Then

\[ \sup_{x > 2b_n} |f_n(x) - f(x)| = O(\psi(n,1)) \quad \text{a.s. as} \quad n \to \infty. \]

**Theorem:** Suppose all assumptions hold. Further assume \( \int x |K(x)|dx < \infty \) and \( (\psi(n,1))^{-1} b_n = O(1) \). We have

\[ \sup_{x \in R} |f_n(x) - f(x)| = O(\psi(n,1)) \quad \text{a.s.} \]

**Proof:** Since \( \int K(x)dx = 1 \) and \( \int |x| |K(x)|dx < \infty \), following Roussas [15, p. 141], we have

\[ \sup_{x \in R} |Ef_n(x) - f(x)| \leq C b_n, \]

which implies \( \sup_{x \in R} |Ef_n(x) - f(x)| < C O(\psi(n,1)) \) by letting \( b_n = O(\psi(n,1)) \). From Lemmas 2 and 3, it produces

\[ \sup_{x \in R} |f_n(x) - f(x)| \leq \sup_{x \in R} |f_n(x) - Ef_n(x)| + \sup_{x \in R} |Ef_n(x) - f(x)| \]

\[ \leq \sup_{x \geq 2b_n} |f_n(x) - Ef_n(x)| + \sup_{x \geq 2b_n} |f_n(x) - Ef_n(x)| + \sup_{x \geq 2b_n} |Ef_n(x) - f(x)| \quad \text{a.s.} \]

\[ \leq O(\psi(n,1)) + \sup_{x > 2b_n} |f_n(x) - Ef_n(x)| \quad \text{a.s.} \]

\[ \leq O(\psi(n,1)) + \sup_{x > 2b_n} |f_n(x) - f(x)| + \sup_{x > 2b_n} |f(x) - Ef_n(x)| \quad \text{a.s.} \]

\[ \leq O(\psi(n,1)). \]

**Remark:** Consider an M/M/1 model where the case that \( \{ \tau_k \} \) constitutes an ordinary renewal process with \( \{ \tau_k \} \) having an exponential density function \( \theta e^{-\theta \tau} (\theta > 0) \). In this case \( f_k \) has the Gamma density function, namely

\[ \theta e^{-\theta x} / \Gamma(k), \quad \Gamma(k) = \int_0^\infty x^{k-1} e^{-x} dx. \]

From Theorem 2, it follows that \( f_n \) can achieve the uniform rate of convergence on \( R_+ \) of order \( (n^{-1} \log n)^{1/3} \).

**CONCLUSION**

In this study, we study the departure process of the GI/G/1 queue. We use kernel type estimators for the stochastic processes to prove the uniform strong consistency of the estimators and their rates of convergence. With the analysis presented, we provide a novel analytic tool for studying the departure process in a general queueing model. The stochastic processes are assumed to satisfy the strong mixing condition with the sampling instants which are random. The time scales of traffic measurement on a network are directly related to the time and space complexity of the model used to describe the departure process. We
have rigorously proved the “rate process” or the accumulated departures in a time interval may be taken to
describe the departure process. Intuitively, this result can also be extrapolated to self-similar output
processes since the strong mixing property represents an asymptotic behavior of the second-order statistics.
The generation of strong mixing of traffic sequences and its associated queueing analysis require further
study.
There are many studies about the problem of queueing system construction for various types of models.
Some of them are only suitable for independent observations or special cases. Some of them have too
strong assumptions that could not be easily reached. The weakness of the independent concept for
distributed density function of service time clearly resides in the complexity of statistical computation.
Unlike the traditional methods, estimation of a G/G/1 queue system applies the concept of strong mixing to
cooporate the realization structure change and dynamic heredity. This research liberates us from the
independent-based process and thus fewer assumptions of the system will be made.
Finally, in spite of the realistic performance for the strong mixing property, there remain some problems for
further studies. For example:
* The convergence of the proof for GI/G/1 model and the proposed assumptions have not been
  well used. This needs further investigation.
* To find an efficient procedure for the outliers as well as the intervention that make the structural
  change.
However, in order to give the popular questions, such as adaptive modeling, what if a hush point occurs, and
combined forecasting, a satisfied answer, we believe Theorem suggested in this study will be a worthwhile
approach and will stimulate more future empirical work in the GI/G/1 system.

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