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On Hypersurfaces of Hessian Manifolds with Constant Hessian Sectional Curvature

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Abstract: In this study, we give one instrinsic inequality for Riemannian hypersurfaces in Hessian manifolds and sufficient and necessary condition for such hypersurfaces to be totally geodesic.

Keywords: Hessian Manifold, Totally Geodesic

PRELIMINARIES

Let M be a flat affine manifold with flat affine connection D. Among Riemannian metrics on M there exists an important class of Riemannian metrics compatible with the flat affine connection D. A Riemannian metric g on M is said to be Hessian metric if g is locally expressed by $g = D^2 u$ where u is a local smooth function. We call such a pair (D, g) a Hessian structure on M and a triple (M, D, g) a Hessian manifold. Geometry of Hessian manifold is deeply related to Kaehlerian geometry and affine differential geometry [1].
We use the same notations and teriminologies as [2] unless otherwise stated.
Let M be a Hessian manifold with Hessian structure (D, g). We express various geometric concepts for the Hessian structure (D, g) in terms of affine coordinate system $\{x^1, \cdots, x^{n+1}\}$ with respect to D, i.e $Ddx^i = 0$.

i. The Hessian metric;

$$g_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}$$

ii. Let γ be a tensor field of the type (1, 2) defined by

$$\gamma(X,Y) = \nabla_X Y - D_X Y$$

where ∇ is the Riemannian connection for g. Then we have

$$\gamma^i_{jk} = \Gamma^i_{jk} = \frac{1}{2} g^{ir} \frac{\partial g_{rj}}{\partial x^k}$$

$$\gamma_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k} = \frac{1}{2}\frac{\partial^3 u}{\partial x^i \partial x^j \partial x^k}$$

$$\gamma_{ijk} = \gamma_{jik} = \gamma_{kji}$$

where Γ^i_{jk} are the Christoffel's symbols of ∇.

iii. Define a tensor S of type (1, 3) by

$$S = D_\gamma$$

and call it the Hessian curvature tensor for (D, g) . Then we have

$$S^i_{jkl} = \frac{\partial \gamma^i_{jl}}{\partial x^k}$$

$$S_{ijkl} = \frac{1}{2}\frac{\partial^4 u}{\partial x^i \partial x^j \partial x^k \partial x^l} - \frac{1}{2} g^{rs} \frac{\partial^3 u}{\partial x^i \partial x^k \partial x^r} - \frac{\partial^3 u}{\partial x^j \partial x^l \partial x^s}$$

$$S_{ijkl} = S_{ilkj} = S_{kjil} = S_{jilk} = S_{klij} \ .$$

iv. The Riemannian curvature tensor for ∇ ;

$$R^i_{jkl} = \gamma^i_{rk}\gamma^r_{jl} - \gamma^i_{rl}\gamma^r_{jk} \ ,$$

$$R_{ijkl} = \frac{1}{2}\left(S_{jikl} - S_{ijkl}\right). \qquad (1.1)$$

Definition 1: For a non-zero contravariant symmetric tensor ξ_x of degree at x we set

$$h(\xi_x) = \frac{\langle \varsigma(\xi_x), \xi_x \rangle}{\langle \xi_x, \xi_x \rangle}$$

and call it the Hessian sectional curvature in the direction ξ_x .

Theorem 1: Let (M, D, g) be a Hessian manifold of dimension ≥ 2. If the Hessian sectional curvature $h(\xi_x)$ depends only x then (M, D, g) is of constant Hessian sectional curvature. (M, D, g) is of constant Hessian sectional curvature c if and only if

$$S_{ijkl} = \frac{c}{2}\left(g_{ij} g_{kl} + g_{il} g_{kj}\right) \qquad (1.2)$$

Corollary 1: If a Hessian manifold (M, D, g) is a space of constant Hessian sectional curvature c, then the Riemannian manifold (M, g) is a space of constant sectional curvature $-\frac{c}{4}$.

2. Local Formulas: Let M' be an n-dimensional Riemannian manifold immersed in M. M' is called a hypersurface.

We choose a local field of Riemannian orthonormal frames $e_1, \cdots, e_{n+1}$ in M such that, restricted to M', $e_1, \cdots, e_n$ are tangent to M'. Let $w_1, \cdots, w_{n+1}$ be its dual frame field such that the Riemannian metric of M is given by

$$ds^2 = \sum (w_A)^2$$

Then the structure equations of M are given by

$$dw_A = -\sum w_{AB} \wedge w_B \quad w_{AB} + w_{BA} = 0 \qquad (2.1)$$

$$dw_{AB} = -\sum w_{AC} \wedge w_{CB} + \frac{1}{2} \sum K_{ABCD} w_C \wedge w_D \qquad (2.2)$$

$$K_{ABCD} = -\frac{c}{4} (\delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC}) \qquad (2.3)$$

We restrict these form to M', then

$$w_{n+1} = 0 \qquad (2.4)$$

and the Riemannian metric of M' is written as

$$ds^2 = \sum (w_i)^2.$$

Since $0 = dw_{n+1,i} = -\sum w_{n+1} \wedge w_i$ by Cartan's lemma we may write

$$w_{n+1,i} = \sum h_{ij} w_j \ , \ h_{ij} = h_{ji}. \qquad (2.5)$$

From these formulas we obtain the structure equation of M'

$$dw_i = -\sum w_{ij} \wedge w_j \ , \ w_{ij} + w_{ji} = 0, \qquad (2.6)$$

$$dw_{ij} = -\sum w_{ik} \wedge w_{kj} + \frac{1}{2} \sum R'_{ijkl} w_k \wedge w_l, \qquad (2.7)$$

$$R'_{ijkl} = \frac{c}{4} (g_{il} g_{kj} - g_{jl} g_{ik}) - (h_{ik} h_{jl} - h_{il} h_{jk}) \qquad (2.8)$$

where R'_{ijkl} are the components of the curvature tensor of M'.

We call

$$h = \sum h_{ij} w_i \otimes w_j$$

the second fundamental form of M'. The square length of h is defined by

$$S = \sum (h_{ij})^2 \qquad (2.9)$$

The mean curvature H of M' is defined by

$$H = \frac{1}{n} \sum h_{ii} \qquad (2.10)$$

At any point $x_0 \in M'$, for symmetry of (h_{ij}), we can make

$$h_{ij} = \lambda_i \delta_{ij}$$

choosing suitable orthonormal frames

$$R'_{ijkl} = \frac{1}{2} (S_{jikl} - S_{ijkl}) - \lambda_i \lambda_j (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}). \qquad (2.11)$$

In [2] , Pang, H. , Xu, S. , Dai, S. , gave one instrinsic inequality for space-like hypersurfaces in De Sitter space and sufficient and necessary condition for such hypersurfaces to be totally geodesic.

In this study we establish a Hessian version of this problem and obtain a result similar to , Pang, H. , Xu, S., Dai, S.

Theorem 2.1: Let (M, D, g) be a Hessian manifold, (M, g) be an (n+1)-dimensional Riemannian manifold of constant curvature $-\frac{c}{4}$ and M' be an n-dimensional hypersurface of M, τ' and ρ' are Ricci curvature tensor and scalar curvature of M', respectively, then

$$\left|\tau'^2\right| \geq \frac{c}{2} \rho'(1-n) - \frac{c^2}{16} n (1-n)^2.$$

Proof: *M* is a Hessian manifold then from (2.8) and (1.2) , we write

$$R'_{ijkl}=\frac{1}{2}\left[\ \frac{c}{2}\left(g_{ij}g_{kl}+g_{il}g_{kj}\right)-\frac{c}{2}\left(g_{ji}g_{kl}+g_{jl}g_{ik}\right)\ \right]-\left(h_{ik}h_{jl}-h_{il}h_{jk}\right)$$

or

$$R'_{ijkl}=\frac{c}{4}\left(g_{il}g_{kj}-g_{jl}g_{ik}\right)-\left(h_{ik}h_{jl}-h_{il}h_{jk}\right).$$

At any point $x_0\in M'$, for symmetry of (h_{ij}), we can make

$$h_{ij}=\lambda_i\delta_{ij}$$

choosing suitable orthonormal frames then

$$\tau'_{ij}=\sum_k R'_{kikj}=\sum_k\left\{-\lambda_k\lambda_i\left(\delta_{kk}\delta_{ij}-\delta_{kj}\delta_{ik}\right)+\frac{c}{4}\left(\delta_{kj}\delta_{ki}-\delta_{ij}\delta_{kk}\right)\right\}$$

$$=-\sum_k\lambda_k\lambda_i\delta_{ij}+\sum_k\lambda_i\lambda_j\delta_{kj}\delta_{ik}+\frac{c}{4}\sum_k\left(\delta_{kj}\delta_{ki}-\delta_{ij}\right)$$

$$=-\lambda_i\delta_{ij}\sum_k\lambda_k+\lambda_i\lambda_j\delta_{ij}+\frac{c}{4}(1-n)\delta_{ij}$$

Then

$$|\tau'|^2=\sum_{i,j}\tau'^2_{ij}=\left\{-\lambda_i\delta_{ij}\sum_k\lambda_k+\lambda_i\lambda_j\delta_{ij}+\frac{c}{4}(1-n)\delta_{ij}\right\}^2$$

$$=\sum_{i,j}\left\{\lambda_i^2\left(\sum_k\lambda_k\right)^2\delta_{ij}+\lambda_i^2\lambda_j^2\delta_{ij}+\frac{c^2}{16}(1-n)^2\delta_{ij}\right\}$$

$$-2\lambda_i^2\lambda_j\delta_{ij}\sum_k\lambda_k-2\frac{c}{4}(1-n)\lambda_i\lambda_j\delta_{ij}$$

$$=\left(\sum_i\lambda_i^2\right)\left(\sum_k\lambda_k\right)^2+\sum_i\lambda_i^4+\frac{c^2}{16}n(1-n)^2-2\sum_i\lambda_i^3\sum_k\lambda_k$$

$$-2\frac{c}{4}(1-n)\left(\sum_i\lambda_i^2\right)+2\frac{c}{4}(1-n)\left(\sum_i\lambda_i^2\right)$$

and

$$\rho'=\sum_{i,j}R'_{ijij}=-\left(\sum_i\lambda_i\right)^2+\sum_i\lambda_i^2+\frac{c}{4}n(1-n)$$

so

$$\left(\sum_i\lambda_i\right)^2=\sum_i\lambda_i^2+\frac{c}{4}n(1-n)-\rho'$$

$$|\tau'|^2=\left(\sum_i\lambda_i\right)^2\left(\sum_k\lambda_k\right)^2+\sum_i\lambda_i^{\,4}+\frac{c^2}{16}n(1-n)^2-2\sum_i\lambda_i^3\sum_k\lambda_k$$

$$-2\frac{c}{4}(1-n)\left(\sum_i\lambda_i\right)^2+2\frac{c}{4}(1-n)\sum_i\lambda_i^2$$

$$=\left(\sum_i\lambda_i^2\right)\left(\sum_i\lambda_i^2+\frac{c}{4}n(1-n)-\rho'\right)+\sum_i\lambda_i^4+\frac{c^2}{16}n(1-n)^2-2\sum_i\lambda_i^3\sum_k\lambda_k$$

$$-2\frac{c}{4}(1-n)\left(\sum_i\lambda_i^2+\frac{c}{4}n(1-n)-\rho'\right)+2\frac{c}{4}(1-n)\sum_i\lambda_i^2$$

$$\geq\left(\sum_i\lambda_i^2\right)\left(\sum_i\lambda_i^2\right)\left(\sum_i\lambda_i^2+\frac{c}{4}n(1-n)-\rho'\right)+\sum_i\lambda_i^4$$

$$-2\left(\sum_i\lambda_i^4\right)^{\frac{1}{2}}\left(\sum_i\lambda_i^2\right)^{\frac{1}{2}}\left(\sum_i\lambda_i^2+\frac{c}{4}n(1-n)-\rho'\right)^{\frac{1}{2}}$$

$$+2\frac{c}{4}(1-n)\rho'-\frac{c^2}{16}n(1-n)^2$$

$$=\left\{\left(\sum_i\lambda_i^4\right)^{\frac{1}{2}}-\left(\sum_i\lambda_i^2\right)^{\frac{1}{2}}\left(\sum_i\lambda_i^2+\frac{c}{4}n(1-n)-\rho'\right)^{\frac{1}{2}}\right\}^2$$

$$+2\frac{c}{4}(1-n)\rho'-\frac{c^2}{16}n(1-n)^2$$

$$\geq 2\frac{c}{4}(1-n)\rho'-\frac{c^2}{16}n(1-n)^2. \qquad (2.12)$$

Using Cauchy-Schwarz inequalityTheorem 1 is proved.

Theorem 2.2: Let (*M, D, g*) be a Hessian manifold, (*M, g)* be an (n+1)-dimensional Riemannian manifold with $-\frac{c}{4}$ constant curvature.

M' is an n- dimensional hypersurface of *M*, τ' and ρ' are Ricci curvature tensor and scalar curvature of M', respectively, then

$$\left|\tau'^2\right|=\frac{c}{2}\rho'(1-n)-\frac{c^2}{16}n(1-n)^2$$

if and only if M' is totally geodesic.

Proof: If M' is totally geodesic, i.e.

$\lambda_i=0,\quad i=1,\cdots,n$ then

$$h_{11}=\lambda(x)\quad,\quad h_{ij}=0,\qquad (i,j)\neq(1,1)$$

$$R'_{ijkl}=-\frac{c}{4}\left(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}\right)$$

and

$$\left|\tau'^2\right|=\frac{c^2}{16}n(1-n)^2\quad,\quad \rho'=\frac{c}{4}n(1-n)$$

i.e.

$$\left|\tau'^2\right| = 2\frac{c}{4}\rho'(1-n) - \frac{c^2}{16}n(1-n)^2$$

Inversely, if (2.12) becomes an equality, then all the inequality of (2.12) will become equality. From Lemma 1 in [2], there exists a constant λ such that

$$\lambda_i^2 = \lambda\lambda_i \quad , \ i = 1,\cdots,n \qquad \text{or}$$

$$\lambda\lambda_i^2 = \lambda_i \quad , \ i = 1,\cdots,n$$

For simplicity, we assume $\lambda_i = \lambda \quad i = 1,\cdots,k$ and $\lambda_j = 0$, $j = k+1,\cdots,n+1$.

If $\lambda = 0$, then M' is obviously totally geodesic.

Now, we assume $\lambda \neq 0$, so M' is not totally geodesic. Because the second inequality of (2.12) should be equality so

$$\left\{\left(\sum_i \lambda_i^4\right)^{\frac{1}{2}} - \left(\sum_i \lambda_i^2\right)^{\frac{1}{2}}\left(\sum_i \lambda_i^2 + \frac{c}{4}n(1-n) - \rho'\right)^{\frac{1}{2}}\right\}^2 = 0$$

i.e.

$$\left(k\lambda^4\right)^{\frac{1}{2}} - \left(k\lambda^2\right)^{\frac{1}{2}}\left(k\lambda^2 + \frac{c}{4}n(1-n) - \rho'\right)^{\frac{1}{2}} = 0$$

Through simple calculation we get

$$\rho' = (k-1)\lambda^2 + \frac{c}{4}n(1-n)$$

On the other hand

$$R'_{ijkl} = -\frac{c}{4}\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right)$$

so we have

$$\rho' = -\left(\sum_i \lambda_i\right)^2 + \sum_i \lambda_i^2 + \frac{c}{4}n(1-n) = \left(k-k^2\right)\lambda^2 + \frac{c}{4}n(1-n)$$

so k=1. By Lemma 2 in [2], M' is totally geodesic. This is a contradiction.

Corollary 2.1: M' is a Einstein hypersurface of M with $Ric = \frac{c}{4}(1-n)g$ (g is the Riemannian metric of M') then M' is totally geodesic.

Proof: In fact, if M' is a Einstein hypersurface of M with $Ric = \frac{c}{4}(1-n)g$, then $\rho' = \frac{c}{4}n(1-n)$ and

$$\left|\tau'^2\right| = \frac{c^2}{16}n(1-n)^2 = 2\frac{c}{4}\rho'(1-n) - \frac{c^2}{16}n(1-n)^2 .$$

The corollary follows immidiately from Theorem 2.2.

REFERENCES

1 Shima, H. and K. Yagi, 1997. Geometry of Hessian manifolds, Diff.Geometry and Its Applications, 7: 277-290.

2. Hirohiko Shima, 1995. Hessian manifolds of constant Hessian sectional curvature. J. Math. Soc. Japan, 47: 735-753.

3. Pang, H., S. Xu and S. Dai, 2002. Spacelike hypersurfaces in de Sitter space. J. Geo. and Phys., 42: 78-84.