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On Stability and Bifurcation of Solutions of Nonlinear System of Differential Equations for AIDS Disease

El-Marouf, S.A.A.
Department of Mathematics, Faculty of Science,
Taibah University, Madinahmonwarah, Kingdom of Saudi Arabia

Abstract: Problem statement: This study aims to discuss the stability and bifurcation of a system of ordinary differential equations expressing a general nonlinear model of HIV/AIDS which has great interests from scientists and researchers on mathematics, biology, medicine and education. The existance of equilibrium points and their local stability are studied for HIV/AIDS model with two forms of the incidence rates. **Conclusion/Recommendations:** A comparison with recent published results is given. Hopf bifurcation of solutions of an epidemic model with a general nonlinear incidence rate is established. It is also proved that the system undergoes a series of Bogdanov-Takens bifurcation, i.e., saddle-node bifurcation, Hopf bifurcation and homoclinic bifurcation for suitable values of the parameters.

Key words: Epedimic models, infectious disease, HIV/AIDS model, local stability, hopf bifurcation, bogdanov-takens bifurcation

INTRODUCTION

There has been considerable interest in disease of Human Immunodeficiency Virus (HIV) infection pandemic and so it receives (Bachar and Dorfmayr, 2004; McCluskey, 2003; Elaiw, 2010; Hethcote *et al*., 1989; Hsieh and Chen, 2004; LaSalle, 1976; Moghadas and Gumel, 2003; Naresh *et al*., 2011) and references therein. There was an important report given by UN AIDS conclude that a number of more than 34 million population living with HIV by the end of the year 2003 and alos the AIDS is growing rapidly. The HIV/AIDS disease is a danger problem in poor countries. The competition between the human immunodeficiency virus and the human immune system has widely been studied (Mann and Tarantola, 1998). Mathematical and Statistical models have been proven valuable in understanding the dynamics of HIV/AIDS infection amongst a population or between interacting countries with treatment and or change of behavior (Mann and Tarantola, 1998). Recently, numerous mathematical models have been developed to describe different phenomena about this disease. The mathematical modeling has proven to be effective in improving information about the character of HIV/AIDS. Perelson and Nelson (1999) proposed an ordinary differential equations model of cell-free viral spread of immunodeficiency virus HIV in a well-mixed compartment such as bloodstream. They divided the model into the components: Uninfected healthy CD+ T-cells, latently infected CD4+T-cells and free virus. This model is very important as inspiration for many models which obtained results established the importance of mathematical techniques in HIV/AIDS researches. Hethcote *et al*. (1992). Discussed time delay in the removed class to account the period of temporarily immunity. Culshaw and Ruan (2000) studied the interaction between infection of a CD4 +T-cell and the emission of viral particles on a cellular level. In order to obtain the effect of the time delay on the stability of the endemic steady state. In their very recent study, Cai *et al*. (2009) investigated an HIV/AIDS model with treatment. They established two infective stages. Using mathematical analysis. They discussed the global analysis of the spread of the HIV disease computed by the number R_0 (basic reproduction) by which they could classify stability of equilibrium points. In this study we assume that the total population is divided into a susceptible class of size S(t) and an infectious class before the onset of AIDS. Since it is well known (Mann and Tarantola, 1998) that the infection period is very long (more than or equal ten years), it is further divided into several cases. In this study, we are going to consider the case in which the HIV/AIDS model allows for some infected people to go from the symptomatic phase to the asymptomatic phase by all sorts of treatment ways. It is very important konwing that the infected

Corresponding Author: El-Marouf, S.A.A., Department of Mathematics, Faculty of Science, Menoufia University, Egypt

individuals do not change are treated or not change their behavior no matter if they are treated or not.

Thus the average value of contact of an person is constant for all subset of population. The disease transfer from rate is constant from stage I_1-I_2 stage. It is also supposed that infected individual in the second stage I_2 can go back by successful treatment to the first stage I_1 with rate γ. Here, we consider the treatment model with two infective stages in a general model than given by (Culshaw and Ruan, 2000; Perelson and Nelson, 1999). We consider the system Eq. 1:

$$\begin{aligned}\frac{dS}{dt} &= \alpha d - a\rho H(I_1, I_2)S - dS\\ \frac{dI_1}{dt} &= a\rho H(I_1, I_2)S - (d+k_1)I_1 + \gamma I_2\\ \frac{dI_2}{dt} &= k_1 I_1 - (d+k_2+\gamma)I_2\end{aligned} \quad (1)$$

where, $N = S + I_1 + I_2$ is the total active population size. This is a generalization of the work of Liming (Cai *et al*., 2009). Where they discussed the special case. Here S represents the number proportion of susceptible, I_1 is the number proportion of infective of first stage of treatment and I_2 is the number proportion of infective of the second stage of treatment, b is the birth rate constant, d is the natural death rate constant, α is the average number of contacts of an individual per unit of time, ρ is the probability of disease transmission per contact by an infective in I_1 and $k_1$and k_2 are the transfer rate constants $I_1 \rightarrow I_2$ and $I_1 \rightarrow$ a respectively, where A is the number of AIDS cases, α is the transfer rate constant $I_1 \rightarrow I_2$ (successful treatment) $a\rho H(I_1, I_2)$ is a nonlinear function on the average number of new infections per unite time (nonlinear incidence). This study is organized as follows. We start by discussing the existence of equilibrium points and their local stability using Routh-Hurwitz method. In next section we study the bifurcation of solutions of the system (1). The study end with a brief discussion.

Stability Properties: The total population size N (t) is variable with $N'(t) = d(\alpha - N) - k_2 I_2$. In the absence of disease, the population size N approaches carrying capacity α the differential equation for N implies that solutions of (1) starting in the positive orthant R^3_+ defined by Eq. 2:

$$\Gamma = \{(S, I_1, I_2) \in \mathbb{R}^3_+ : S + I_1 + I_2 \le \alpha\} \quad (2)$$

The following is the linearization due to the system (1) Eq. 3 and 4:

$$X' = MX, \text{where } X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, (x_1, x_2, x_3) \in \mathbb{R}^3_+ \quad (3)$$

Where:

$$M = \begin{pmatrix} a\rho H - d & -a\rho \frac{\partial H}{\partial I_1} S & -a\rho \frac{\partial H}{\partial I_2} S \\ a\rho H & a\rho \frac{\partial H}{\partial I_1} S - (d+k_1) & -a\rho \frac{\partial H}{\partial I_2} S + \gamma \\ 0 & k_1 & -(d+k_2+\gamma) \end{pmatrix} \quad (4)$$

It is clear from (1) that $\bar{P}_\circ = (\alpha, 0, 0)$ is a trivial equilibrium. The Varational matrix of (1) at $\bar{P}_\circ = (\alpha, 0, 0)$ is given by Eq. 5:

$$M_{\bar{P}_\circ} = \begin{pmatrix} -d & 0 & 0 \\ 0 & -A & \gamma \\ 0 & k_1 & -B \end{pmatrix} \quad (5)$$

where, $A = (d+k_1) > 0$ and $B = (d + k_2 + \gamma)$. The eigenvalues are $\lambda_1 = -d < 0$ and λ_2 and λ_3 are given by:

$$\lambda_{2,3} = \frac{-1}{2}(A+B) \pm \frac{1}{2}\sqrt{(A+B)^2 - 4(AB - k_1\gamma)}.$$

Since $\lambda_1 = -d < 0$, $A > 0$ and $B > 0$, hence the disease-free point $\bar{P}_\circ = (\alpha, 0, 0)$ is locally asymptotically stable.

Now we study the non-zero equilibrium point $P^* = (S^*, I_1^*, I_2^*)$ of system (1), where Eq. 6:

$$\begin{aligned} S^* &= \frac{\alpha d}{a\rho H(I_1^*, I_2^*) + d}, I_1^* = \frac{a\rho H(I_1^*, I_2^*)S^* + \gamma}{d+k_1}\\ I_2^* &= \frac{k_1 I_1^*}{d+k_2+\gamma}\end{aligned} \quad (6)$$

The Variational matrix of (1.1) at $P^* = (S^*, I_1^*, I_2^*)$ is given by Eq. 7:

$$M_{P^*} = \begin{pmatrix} a\rho H^* - d & -a\rho H_{I_1^*} S^* & -a\rho H_{I_2^*} S^* \\ a\rho H^* & a\rho H_{I_1^*} S^* - (d+k_1) & -a\rho H_{I_2^*} S^* + \gamma \\ 0 & k_1 & -(d+k_2+\gamma) \end{pmatrix} \quad (7)$$

where, $H_{I_1^*} = \frac{\partial H}{\partial H_{I_1}}|_{I_1 = I_1^*}, H_{I_2^*} = \frac{\partial H}{\partial H_{I_2}}|_{I_2 = I_2^*}$ and $H^* = H(I_1^*, I_2^*)$. We assume that $H_{I_1^*}, H_{I_2^*}$ and $H^*$are positive. Set $l_1 = a\rho H^*$, $l_2 = a\rho H_{I_1^*} S^*$ and $l_3 = a\rho H_{I_2^*} S^*$. Thus the Jacobian matrix M_{P*} becomes Eq. 8:

$$M_{P*} = \begin{pmatrix} l_1 - d & -l_2 & -l_3 \\ l_1 & l_2 - A & l_3 + \gamma \\ 0 & k_1 & -B \end{pmatrix}. \quad (8)$$

The characteristic equation of $M_{P_\circ}$ at $P_\circ$ is Eq. 9 and 10:

$$\lambda^3 + m_1\lambda^2 + m_2\lambda + m_3 = 0 \quad (9)$$

Where:

$$\begin{aligned} &m_1 = d - l_1 + A - l_2 + B, \\ &m_2 = (l_1 - d)(l_2 - A) + B(d - l_1) + \\ &(A - l_2)B - k_1(l_3 + \gamma) + l_1 l_2, \\ &m_3 = (l_1 - d)(l_2 - A)B + (l_1 - d) \\ &(l_3 + \gamma)k_1 + l_1 l_2 B + l_1 l_3 k_1 \end{aligned} \quad (10)$$

It follows from the Routh-Hurwitz criterion (El-Marouf and Alihaby, 2011; Rao, 1981) that $P^* = (S^*, I_1^*, I_2^*)$ is locally asymptotically stable if $m_1 > 0$, $m_3 > 0$ and $m_1 m_2 - m_3 > 0$, then we introduce the following result.

Theorem 2.1: Assume that the following conditions are satisfied:

$$\begin{aligned} &(A_1) A + B + d > l_1 + l_2 \\ &(A_2) l_1\left(2l_2 B + 2k_1(l_3 + \gamma)\right) > \\ &AB(l_1 - d) + d\left(l_2 B + k_1(l_3 + \gamma)\right) \\ &(A_3)(d + A + B)(x - y) + dk_1(l_3 + \gamma) \\ &> (l_1 + l_2)(x - y) + B(2l_1 l_2 + dA) + l_1 k_1(2l_3 + \gamma) \end{aligned}$$

where, $x = 2l_1 l_2 + d(A+B) + AB$ and $y = l_1(A+B) + l_2(d+B) + k_1(l_3+\gamma)$ Then the equilibrium point $P^* = (S^* I^*_1, I^*_2)$ is locally asymptotically stable.

Now we choose γ as the parameter of bifurcation for system (1.1). Let γ_c be the value of γ at which the characteristic Eq. 9 has two pure imaginary roots $\lambda_{1,2}$. From the above discussion we have the following theorem.

Theorem 2.2: Assume that the assumption (17) holds, thus at $\gamma = \gamma_c$, there exists one parameter family of periodic solutions bifurcating from the equilibrium point $P^* = (S^* I^*_1, I^*_2)$ with period T, where $T \to T_\circ$ as $\gamma \to \gamma_c$ and where $T_\circ = 2\pi/\omega_\circ = 2\pi/\sqrt{m_2}$ and m_2 is shown by (10).

Proof: It is clear that Eq. 9 has at least one real root λ_3, say, we have the following analysis Eq. 11:

$$(\lambda - \lambda_3)[\lambda^2 + (\lambda_3 + m_1)\lambda + (\lambda_3^2 + m_1\lambda_3 + m_2)] = 0 \quad (11)$$

Since, by (9) Eq. 12:

$$\lambda_1 + \lambda_2 + \lambda_3 = -m_1 \quad (12)$$

also at $\gamma = \gamma_c$, we obtain Eq. 13:

$$\begin{aligned} &\lambda_3 = -m_1, \lambda_1 = \bar{\lambda}_2, \\ &\lambda_{1,2} = -\frac{1}{2}\left\{(\lambda_3 + m_1) \pm \sqrt{(\lambda_3 + m_1)^2 - 4(\lambda_3^2 + m_1\lambda_3 + m_2)}\right\} \end{aligned} \quad (13)$$

Thus, at $\gamma = \gamma_c > 0$, we can rewrite Eq. 10 as Eq. 14:

$$D_\gamma(m_1) = m_1 m_2 - m_3 \quad (14)$$

Using $m_2>0$ and $m_3>0$, at $\gamma = \gamma_c$, we get $\lambda_3 = -m_1<0$. At the critical value $\gamma = \gamma_c>0$, there is soultion of (14) which can be given by (11). Hence we obtain the equation of γ as follows Eq. 15 and 16:

$$-c_1\gamma^2 - c_2\gamma + c_3 = 0 \quad (15)$$

Where:

$$\begin{aligned} &c_1 = -(d - l_1 + d^2 + dk_1 - l_2 - k_1) \\ &c_2 = -(2d - l_1 + d^2 + dk_1 + k_2 - l_2) \\ &(d - l_1 + d^2 + dk_1 - l_2 - k_1) \\ &-(l_1 - d)(l_2 - d^2 - dk_1) - (d + k_2) \\ &(d - l_1) - (d^2 + dk_1 - l_2)(d + k_2) \\ &-k_1 l_3 + l_1 l_2 + (l_1 - d)(l_2 - d^2 - dk_1) \\ &c_3 = (2d - l_1 + d^2 + dk_1 + k_2 - l_2)[(l_1 - d) \\ &(l_2 - d^2 - dk_1)(d + k_2)(d - l_1) \\ &(d^2 + dk_1 - l_2)(d + k_2) + k_1 l_3 + l_1 l_2)] \\ &-(l_1 - d)(l_2 - d^2 - dk_1) \\ &(d + k_2) + (l_1 - d)l_3 k_1 + l_1 l_3 k_1) \end{aligned} \quad (16)$$

Conversely, we assume that $m_1>0$, $m_3>0$ and $\gamma > 0$, then we can find the solution of Eq. 15 for $\gamma_c>0$. Also we know that $m_2>0$, $\lambda_3 = -m_1 <0$ and $\lambda_{1,2}$ are conjugate imaginary. Now, we can choose both l_1, l_2 and l_3 to be sufficiently small and d, k_1 be sufficiently large. Then we have Eq. 17:

$$d(d+k_1+1) > l_1 + l_2 + k_1 \tag{17}$$

But since by (15) and (16) $c_1 > 0$ $c_2 > 0$ and $c_3 > 0$ Eq. 18:

$$D_\circ(m_1) = c_3 > 0,\ \lim D_\gamma(m_1) = \infty \tag{18}$$

Thus γ_c is uniquely determined. Now, since by (11) $\lambda_3 = -m_1 < 0$ and Eq. 19:

$$\begin{aligned} D_\gamma(m_1) &= m_1 m_2 - m_3 \\ &= (m_1+\lambda_3)(\lambda_1\lambda_2 - m_1\lambda_1) \\ \operatorname{sgn} D_\gamma(m_1) &= \operatorname{sgn}(m_1+\lambda_3) \end{aligned} \tag{19}$$

Consequently we have Eq. 20 and 21:

$$\operatorname{Re}\lambda_{1,2} = \frac{1}{2}(m_1+\lambda_3) < 0 \ \text{ for } \ \gamma > \gamma_c \tag{20}$$

$$\operatorname{Re}\lambda_{1,2} > 0 \ \text{ for } \ \gamma < \gamma_c \tag{21}$$

It follows from the above discussion that, when γ is increased though γ_c,there exists a pair of complex conjugate imaginary eigenvalues $\lambda_{1,2}$ of the Variational matrix M_{P*}. Since at $\gamma = \gamma_c$, $\lambda_3 = -m_1$, $\lambda_{1,2} = \pm i\sqrt{m_2} = \pm i\omega_\circ$ where it is clear that $\omega_\circ > 0$. Now, since $\lambda_1 = \bar{\lambda}_2$, for Eq. 22:

$$\operatorname{Re}\lambda_2 = \frac{1}{2}(\lambda_2 + \bar{\lambda}_2) = 0 \ \text{ at } \ \gamma = \gamma_c \tag{22}$$

and by above discussion we see that $\operatorname{Re}\lambda_2 > 0$ for $\gamma < \gamma_c$ and $\operatorname{Re}\lambda_2 < 0$ for $\gamma > \gamma_c$ thus Eq. 23:

$$\begin{aligned} \frac{d}{d\gamma}(\operatorname{Re}\lambda_2)|_{\gamma=\gamma_c} &= \frac{-1}{2}\frac{d}{d\gamma}(m_1+\lambda_3)|_{\gamma=\gamma_c} \\ &= \operatorname{Re}\left(\frac{d}{d\gamma}\lambda_2\right)|_{\gamma=\gamma_c} < 0. \end{aligned} \tag{23}$$

This completes the proof.

Bogdanov-Takens Bifurcations: Now we onsider the system Eq. 24:

$$\begin{aligned} \frac{dS}{dt} &= \alpha d - a\rho H(S, I_1)S - dS, \\ \frac{dI_1}{dt} &= a\rho H(S, I_1)S - (d+k_1)I_1 + \gamma I_2, \end{aligned} \tag{24}$$

With the nonlinear incidence rate of the form $a\rho H(S, I_1)$. In order to translate the interior equilibria point $P_2 = (S^*, I^*_1)$ to the origin, we set $X = S-S^*$ and $Y = I_1 - I^*_1$, then the system (24) becomes Eq. 25:

$$\begin{aligned} \frac{dX}{dt} &= a_{11}X + a_{12}Y + f_1(X,Y), \\ \frac{dY}{dt} &= a_{21}X + a_{22}Y + f_2(X,Y), \end{aligned} \tag{25}$$

where, $a_{11} = -a\rho H^* - a\rho S^* H_{S*} - d$, $a_{12} = -a\rho S^* H_{I_1^*}$, $a_{21} = a\rho H^* + a\rho S^* H_{S*}$, $a_{22} = a\rho S^* H_{I_1^*} - (d+k_1)$, and $f_1(X, Y)$ and $f_2(X, Y)$ are smooth functions in (X, Y) of order at least two. Since we are interested in codimension 2-bifurcation, we assume that:

$$(A_1) \quad 2a\rho S^* H_{I_1^*} < (d+k_1)$$

Theorem 3.1: Assume that the assumption (A_1) is satisfied. Then the equilibria point $P_2 = (S^*, I^*_1)$ of (24) is a cusp of codimension 2, i.e., it is a Bogdanov-Takens singularity.

Proof: Using the assumption (A_1) the value of the determinant of the matrix Eq. 26:

$$M_{P_2=(S^*,I_1^*)} = \begin{pmatrix} a\rho H^* - a\rho S^* H_{S^*} & -a\rho S^* H_{I_1^*} \\ a\rho H^* + a\rho S^* H_{S^*} & a\rho S^* H_{I_1^*} - (d+k_1) \end{pmatrix} \tag{26}$$

is zero. Also, by the assumption (A_1) the matrix M has two zero eigenvalues. By the same transformation, we can write system (24) in the form Eq. 27:

$$\begin{aligned} \frac{dX}{dt} &= -(a\rho S^* H_{I_1^*})Y - \frac{1}{2}\left[a\rho S^* H_{S^*S^*} + a\rho H_{S^*}\right] \\ &X^2 + f_3(X,Y), \\ \frac{dY}{dt} &= \frac{1}{2}\left[a\rho S^* H_{S^*S^*} + 2a\rho H_{S^*}\right]X^2 + f_4(X,Y) \end{aligned} \tag{27}$$

where, $H_{S^*S^*} = \dfrac{\partial^2 H(S^*, I_1^*)}{\partial S^2}$, $f_3(X, Y)$ and $f_4(X, Y)$ are C^∞ functions at least of third order. After that, we discuss the normal form for the system (27) in the two dimensional and appling the center manifold theorem. Making the following affine transformation:

$$x = X \ \text{ and } \ y = -(a\rho S^* H_{I_1^*})Y$$

system (27) can be written as Eq. 28:

$$\frac{dx}{dt} = y - \frac{1}{2}\left[a\rho S^* H_{s^*s^*} + 2a\rho H_{s^*}\right]x^2 + f_5(x,y),$$
$$\frac{dy}{dt} = -\frac{1}{2}a\rho S^* H_{I_1^*}\left[a\rho S^* H_{s^*s^*} + 2a\rho H_{s^*}\right]x^2 + f_6(x,y) \quad (28)$$

where, $f_5(X, Y)$ and $f_6(X, Y)$ are C^∞ functions at least of order three. In order to find the canonical normal form of the cusp, we take:

$$X = x \text{ and } Y = y - \frac{1}{2}\left[a\rho S^* H_{s^*s^*} + 2a\rho H_{s^*}\right]x^2 + f_5(x,y)$$

then the system (28) becomes Eq. 29:

$$\frac{dX}{dt} = Y,$$
$$\frac{dY}{dt} = -\frac{1}{2}a\rho S^* H_{I_1^*}\left[a\rho S^* H_{s^*s^*} + 2a\rho H_{s^*}\right]X^2 - \left[a\rho S^* H_{s^*s^*} + 2a\rho H_{s^*}\right]XY + f_7(X,Y) \quad (29)$$

where, $f_7(X, Y)$ is C^∞ functions at least of order three. Now, since $(a\rho S^* H_{S*S*} + 2a\rho H_{S*}) > 0$, hence $P_2 = (S^*, I^*_1)$ is a cusp of codimension 2. This completes the proof.

The above result indicates that Eq. 24 can satisfies the Bogdanov-Takens bifurcation with a small perturbation if the bifurcation parameters are chosen by suitable method. For convenience, we denote:

$$\beta^\circ = (\alpha_1, a_1, d_1, \gamma_1, k_{11}) \text{ and } \beta = (\alpha, a, d, \gamma, k_1).$$

Then the system (2) in a small neighborhood of (S^*, I^*_1) can be written as Eq. 30:

$$\frac{dx}{dt} = y + W_1(x,y;\beta),$$
$$\frac{dy}{dt} = -\frac{1}{2}a\rho S^* H_{I_1^*}\left[a\rho S^* H_{s^*s^*} + 2a\rho H_{s^*}\right]x^2 + \left[a\rho S^* H_{s^*s^*} + 2a\rho H_{s^*}\right]xy + W_2(x,y;\beta) \quad (30)$$

where, W_1 and W_2 are C^∞ functions, $W_1(x, y; \beta^\circ) = 0$, $W_2(x, y; \beta) = f_7(X,Y)$, x and y belong to small neighborhood of (0,0) and β is a small neighborhood of β°. Next we obtain versal unfolding depending on the original parameters in Eq. 4. By this method,we will compute the approximating bifurcations curves. As the bifurcation parameters, we can choose the parameters a, d and ρ. Assume that the following assumption holds:

$$(A_2) \quad H_{I_1^* I_1^*} = 0$$

Assume that a, d and ρ satisfy (A_1) and (A_2). Let $a = a_\circ + \mu_1$ and $d = d_\circ + \mu_2$ and $\rho = \rho_\circ + \mu_3$. Substituting $X = S - S^*$ and $Y = I - I^*_1$ into (24) and using Taylor expansion, we obtain Eq. 31:

$$\frac{dx}{dt} = a_\circ + a_1x + a_2y + a_3x^2 + a_4xy + p_1(x,y),$$
$$\frac{dy}{dt} = b_0 + b_1x + b_2y + b_3x^2 + b_4xy + p_2(x,y) \quad (31)$$

Where:

$$a_\circ = \alpha d - a\rho S^* H^* - dS^*, \quad a_1 = -(a\rho H^* + 2a\rho S^* H_{s^*} + d)$$
$$a_2 = -a\rho S^* H_{I_1^*} \quad a_3 = \frac{1}{2}\left[-2a\rho H_{s^*} - a\rho S^* H_{s^*s^*}\right]$$
$$a_4 = -\left[a\rho H_{I_1^*} + a\rho S^* H_{s^* I_1^*}\right], b_0 = a\rho S^* H^* - (d + k_1 - \gamma)I_1^*$$
$$b_1 = a\rho S^* H_{s^*} + a\rho H^*, b_2 = a\rho H_{I_1^*} - (d + k_1),$$
$$b_3 = \frac{1}{2}\left[a\rho S^* H_{s^*s^*} + 2a\rho H_{s^*}\right], b_4 = a\rho H_{s^* I_1^*} + a\rho S^* H_{I_1^*}$$

and $p_1(x, y)$ and $p_2(x, y)$, are C^∞ functions of (x, y) up to the third order. Using the change of variables:

$$X = x \text{ and } Y = a_\circ + a_1x + a_2y + a_3x^2 + a_4xy + p_1(x,y)$$

and rewrite X, Y as x, y respectively, system (31) becomes Eq. 32:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = a_5 + a_6x + a_7y + a_8x^2 + a_9y^2 + a_{10}xy + p_3(x,y,\mu) \quad (32)$$

Where:

$$a_5 = a_\circ a_1 + a_2b_0,$$
$$a_6 = a_1^2 + a_2b_1 + 2a_0a_3 + a_4b_0,$$
$$a_7 = a_1a_2 + a_2b_2 + a_\circ a_4,$$
$$a_8 = a_1a_3 + a_2b_3 + 2a_1a_3 + a_4b_1,$$
$$a_9 = a_2a_4, \quad a_{10} = a_1a_4 + a_2b_4 + 2a_2a_3 + a_4b_2$$

and p_3 is a continous function. Let $X = x + \frac{a_7}{a_{10}}$ and rewriting X as x,we obtain Eq. 33:

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = b_5 + b_6 x + a_8 x^2 + a_9 y^2 + a_{10} xy + p_4(x, y, \mu) \qquad (33)$$

where $b_5 = \left(a_5 - \frac{a_6 a_7}{a_{10}} + \frac{a_8 a_7^2}{a_{10}^2}\right)$, $b_6 = \left(a_6 - \frac{2a_7 a_8}{a_{10}}\right)$ and $p_4(x, y, \mu)$ is a continous function in the variables x, y and the parameter μ. Now, introduce the new time τ by $dt = (1-a_1x)\, d\tau$ and rewrite τ as t, we have Eq. 34:

$$\frac{dx}{dt} = (1 - a_1 x)y, \quad \frac{dy}{dt} = (1 - a_1 x)(b_5 + b_6 x + a_8 x^2 + a_9 y^2 + a_{10} xy + p_4(x, y, \mu)), \qquad (34)$$

Set X = x and $Y = (1-a_1x)\, y$ and rename X, Y as x, y respectively, we obtain Eq. 35:

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = b_5 + C_1 x + C_2 x^2 + a_{10} xy + p_5(x, y, \mu), \qquad (35)$$

where, $C_1 = b_6-2a_9b_5$, $C_2 = a_8-2a_9b_6$ and $p_5(x, y, \mu)$ is a smooth function of in xy-plane and μ at least of third order. Setting the change of variables $X = \frac{a_{10}^2}{C_2} x$, $Y = \frac{a_{10}^3}{C_2^2} y$, $\tau = \frac{C_2}{a_{10}} t$. Also, we denote them again by x, y and t, respectively, we get Eq. 36:

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = \xi_1 + \xi_2 x + x^2 + xy + p_6(x, y, \mu), \qquad (36)$$

where $\xi_1 = \frac{b_5 a_{10}^4}{C_2^3}, \xi_2 = \frac{a_{10}^2 C_1}{C_2^2}$ and $p_6(x, y, \mu)$ is a continous function in the variables x, y and the parameter μ. As in (Bogdanov, 1981a; 1981b), we will get the following bifurcation curves.

Theorem 3.2: Suppose (A_1) and (A_2) hold. Then system (24) satisfies the following bifurcation curves:

- The saddle-node bifurcation curve $SN = \{(\xi_1, \xi_2) : \xi_1 = \frac{1}{4}\xi_2^2\}$
- The Hopf bifurcation curve $H = \{(\xi_1, \xi_2) : \xi_1 = 0, \xi_2 < 0\}$
- The homoclinic bifurcation curve $HL = \left\{(\xi_1, \xi_2) : \xi_1 = \frac{-6}{25}\xi_2^2 + 0(\| \xi \|^2)\right\}$

CONCLUSION

In this study, we considered a general HIV/AIDS system with treatment model. The incidence rates $\rho H(I_1, I_2)$ and $\rho H(S, I_1)$ are of nonlinear form. The local asymptotic stability of the disease-free equilibrium points $\bar{P}_\circ = (\alpha, 0, 0)$ and $P^* = (S^*, I^*_1, I^*_2)$ for systems (1) and (2), respectively are established. The obtained results in here are consistent with those obtained by (Perelson and Nelson, 1999). We proved that The disease-free solution $\bar{P}_\circ$ = (a, 0, 0) is locally asymptotically stable in the interior of the feasible region and the disease always dies out. Also we showed that the non-trivial equilibrium point $P^* = (S^*, I^*_1, I^*_2)$ exists and is locally asymptotically stable in the considered region. In Theorem 3.1 we proved that if the two conditions (A_1) and (A_2) hold, then the equilibrium point $P_2 = (S^*, I^*_1)$ of system (24) is a cusp of codimension 2, i,e., it is a Boggdanov-Takens singularity. Also we have proved that if the additional condition (A_2) holds, then the system (24) exhibits Boggdanov-Takens bifurcation, that is, there are three types of bifurcations, saddle-node bifurcation, Hopf bifurcation and homoclinic bifurcation. Our results that obtained througout this study are considered as improvement and partial generalization for those obtained by (Anderson, 1988; Busenberg and Driessche, 1990; Pedro and Tchuenche, 2010; Wang and Li, 2006; Xu, 2011; Yang and Xia, 2010).

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