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Homotopy Perturbation Method and Variational Iteration Method for Solving Zakharov-Kuznetsov Equation

[1]B. Ganjavi, [1]H. Mohammadi, [2]D.D. Ganji and [1]A. Barari
[1]Department of Civil Engineering, Mazandaran University, Babol, Iran. P.O Box 47144-484
[2]Department of Mechanical Engineering, Mazandaran University, Babol, Iran. P.O Box 47144-484

Abstract: This article investigates a nonlinear dispersive Zakharov-Kuznetsov ZK (3, 3, 3) equation. This equation governs the behavior of weakly nonlinear ion-acoustic waves in plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field. Homotopy Perturbation Method (HPM) and Variational Iteration Method (VIM) are implemented for solving the ZK equation. Homotopy results were compared with results of Adomian decomposition Method (ADM). The results reveal that the HPM and VIM are very effective, convenient and quite accurate to systems of nonlinear partial differential equations.

Keywords: Zakharov-Kuznetsov equation, Variational, Homotopy perturbation method

INTRODUCTION

The investigation of the traveling wave solution plays an important role in nonlinear science. These solutions may well describe various phenomena in nature, such as vibrations, solitons and propagation with a finite speed. The wave phenomena were observed in fluid dynamics, plasma and elastic media.
Rosenau and Hyman[1] introduced a class of partial differential equations (PDEs):

$$K(m,n): u_t + a(u^m)_x + (u^n)_{xxx} = 0, \ m > 0, 1 < n \le 3 \qquad (1)$$

which is a generalization of the korteweg-de Vries (KdV) equation. For values of m and n, the $K(m,n)$ equation has solitary waves which are compactly supported. For $m = n$ these are solitary waves or so-called compactons. Recently, Wazwaz[2] has given the new solitary patterns for the nonlinear dispersive $K(m,n)$ equations:

$$u_t - a(u^m)_x + (u^n)_{xxx} = 0, \ m, n > 1. \qquad (2)$$

The new solitary wave special solutions with compact support for the nonlinear dispersive $K(m,n)$ equations:

$$u_t + a(u^m)_x + (u^n)_{xxx} = 0, \ m, n > 1, \qquad (3)$$

are presented by Ismail and Taha[3] and Wazwaz[4] used a finite difference method and a finite element method to investigate the approximate solutions of $k(2,2)$ and $k(3,3)$ in Eq. (1).
More recently, to understand the role of nonlinear dispersion in the formation of patterns in liquid drops, Yan[5] and Zhu[6] have introduced a family of fully Boussinesq equation B(m,n):

$$u_{tt} + a(u^m)_{xx} + (u^n)_{xxxx} = 0, \ m, n > 1, \qquad (4)$$

and they have given many compactons and solitary pattern solutions, respectively.
The main goal of the present article is to investigate the ZK equation of the form (shortly called ZK(m,n,k)):

$$u_t + a(u^m)_x + b(u^n)_{xxx} + c(u^k)_{yyx} = 0, \ mnk \ne 0, \qquad (5)$$

where a, b, c are arbitrary constants and m, n, k are integers. This equation governs the behavior of weakly nonlinear ion-acoustic waves in plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field[7]. In[7], the ZK equation is solved by the sine–cosine and the hyperbolic tangent (tanh)-function methods. Recently, the numbers of solitary waves, periodic waves and kink waves of the

Corresponding Author: Behnoud Ganjavi, Department of Civil Engineering, Mazandaran University, Babol, Iran, P.O Box 47144-484 Fax: +98-32342018

modified Zakharov–Kuznetsov equation has been obtained by Zhao *et al*.[1]. In this letter, we apply the Homotopy-Perturbation Method (HPM)[9-14] and Variational Iteration Method (VIM) [15-18] to the ZK equation.

METODOLOGY

Basic idea of HPM: To explain this method, let us consider the following function:

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{6}$$

With the boundary conditions of:

$$B(u, \frac{\partial u}{\partial n}) = 0, \quad r \in \Gamma, \tag{7}$$

where A , B , $f(r)$ and Γ are a general differential operator, a boundary operator, a known analytical function and the boundary of the domain Ω, respectively. Generally speaking the operator A can be divided into a linear part L and a nonlinear part $N(u)$. Eq. (6) can, therefore, be written as:

$$L(u) + N(u) - f(r) = 0 \tag{8}$$

By the homotopy technique, we construct a homotopy $v(r,p): \Omega \times [0,1] \to R$ which satisfies:

$$H(v,p) = (1-p)\left[L(v) - L(u_0)\right] + p\left[A(v) - f(r)\right] = 0, \tag{9}$$

$$p \in [0,1], r \in \Omega,$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p\left[N(v) - f(r)\right] = 0 \tag{10}$$

where $p \in [0,1]$ is an embedding parameter, while u_0 is an initial approximation of Eq. (6), which satisfies the boundary conditions. Obviously, from Eqs. (9) and (10) we will have:

$$H(v,0) = L(v) - L(u_0) = 0, \tag{11}$$

$$H(v,1) = A(v) - f(r) = 0, \tag{12}$$

The changing process of p from zero to unity is just that of $v(r,p)$ from u_0 to $u(r)$ .In topology, this is called deformation, while $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopy.

According to the HPM, we can first use the embedding parameter p as a "small parameter", and assume that the solutions of Eqs. (9) and (10) can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + ... \tag{13}$$

Setting $p = 1$ yields in the approximate solution of Eq. (6) to:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots \tag{14}$$

The combination of the perturbation method and the homotopy method is called the HPM, which eliminates the drawbacks of the traditional perturbation methods while keeping all their advantages.

The series (14) is convergent for most cases. However, the convergent rate depends on the nonlinear operator $A(v)$. Moreover, He[9] made the following suggestions: (1) The second derivative of $N(v)$ with respect to v must be small because the parameter may be relatively large, i.e. $p \to 1$. (2) The norm of $L^{-1}\frac{\partial N}{\partial v}$ must be smaller than one so that the series converges.

Basic idea of VIM: To clarify the basic ideas of VIM [15-18], we consider the following differential equation:

$$Lu + Nu = g(t), \tag{15}$$

where L is a linear operator, N is a nonlinear operator and $g(t)$ is a homogeneous term.

According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\left(Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau)\right)d\tau \tag{16}$$

where λ is a general Lagrangian multiplier which can be identified optimally via the variational theory. The subscript n indicates the n*th* approximation and u_n is considered as a restricted variation, i.e. $\delta \tilde{u}_n = 0$.

RESULTS AND DISCUSSION

Application of HPM: We consider the ZK (3, 3, 3) equation with initial value problem of:

$$u_t + (u^3)_x + 2(u^3)_{xxx} + 2(u^3)_{yyx} = 0, \qquad (17)$$
$$u(x, y, 0) = \frac{3}{2}\lambda \sinh\left[\frac{1}{6}(x+y)\right]$$

where λ is an arbitrary constant. We assume λ=1. To solve Eq. (17) by means of HPM, we consider the following process after separating the linear and nonlinear parts of the equation.

A homotopy-perturbation method can be constructed as follows:

$$H(v,p) = (1-p)\left(\frac{\partial}{\partial t}v(x,y,t) - \frac{\partial}{\partial t}u_0(x,y,t)\right) + p\begin{pmatrix}\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}v(x,y,t)^3 + 2\frac{\partial^3}{\partial x^3}v(x,y,t)^3 + \\ 2\frac{\partial^3}{\partial^2 y\,\partial x}v(x,y,t)^3\end{pmatrix} = 0 \qquad (18)$$

Substituting $v = v_0 + pv_1 + ...$ in to Eq. (18) and rearranging the resultant equation based on powers of p-terms, one has:

$$p^0 : \frac{\partial}{\partial t}v_0(x,y,t) = 0 \qquad (19)$$

$$\begin{aligned}
p^1 :\ & 12\left(\frac{\partial}{\partial x}v_0(x,y,t)\right)\left(\frac{\partial}{\partial y}v_0(x,y,y)\right)^2 + \\
& 6(v_0(x,y,t))^2\left(\frac{\partial^3}{\partial x^3}v_0(x,y,t)\right) + \\
& 6(v_0(x,y,t))^2\left(\frac{\partial^3}{\partial y^2\partial x}v_0(x,y,t)\right) + \\
& 12\left(\frac{\partial}{\partial x}v_0(x,y,t)\right)^3 + 24v_0(x,y,t) \\
& \left(\frac{\partial}{\partial y}v_0(x,y,t)\right)\left(\frac{\partial^2}{\partial y\,\partial x}v_0(x,y,t)\right) + \\
& 12v_0(x,y,t)\left(\frac{\partial^2}{\partial y^2}v_0(x,y,t)\right)\left(\frac{\partial}{\partial x}v_0(x,y,t)\right) + \\
& \left(\frac{\partial}{\partial t}v_1(x,y,t)\right) + 36v_0(x,y,t)\left(\frac{\partial}{\partial x}v_0(x,y,t)\right) \\
& \left(\frac{\partial^2}{\partial x^2}v_0(x,y,t)\right) + \\
& 3(v_0(x,y,t))^2\left(\frac{\partial}{\partial x}v_0(x,y,t)\right) = 0
\end{aligned} \qquad (20)$$

and

$$\begin{aligned}
p^2 :\ & 24v_0(x,y,t)\left(\frac{\partial}{\partial y}v_0(x,y,t)\right)\left(\frac{\partial^2}{\partial y\,\partial x}v_1(x,y,t)\right) + \\
& 36v_0(x,y,t)\left(\frac{\partial^2}{\partial x^2}v_1(x,y,t)\right) + \\
& 24v_1(x,y,t)\left(\frac{\partial}{\partial y}v_0(x,y,t)\right)\left(\frac{\partial^2}{\partial y\,\partial x}v_0(x,y,t)\right) + \\
& 36\left(\frac{\partial}{\partial x}v_0(x,y,t)\right)^2\left(\frac{\partial}{\partial x}v_1(x,y,t)\right) + \\
& 12v_0(x,y,t)v_1(x,y,t)\left(\frac{\partial^3}{\partial x^3}v_0(x,y,t)\right) + \\
& \left(\frac{\partial}{\partial t}v_2(x,y,t)\right) + 6v_0(x,y,t)v_1(x,y,t) \\
& \left(\frac{\partial}{\partial x}v_0(x,y,t)\right) + 3v_0^2(x,y,t)\left(\frac{\partial}{\partial x}v_1(x,y,t)\right) + \\
& 6v_0^2(x,y,t)\left(\frac{\partial^3}{\partial y^2\partial x}v_1(x,y,t)\right) + \\
& 12v_0(x,y,t)\left(\frac{\partial^2}{\partial y^2}v_0(x,y,t)\right)\left(\frac{\partial}{\partial x}v_1(x,y,t)\right) + \\
& 36v_0(x,y,t)\left(\frac{\partial}{\partial x}v_1(x,y,t)\right) \\
& \left(\frac{\partial^2}{\partial x^2}v_0(x,y,t)\right) + 36v_1(x,y,t)\left(\frac{\partial}{\partial x}v_0(x,y,t)\right) \\
& \left(\frac{\partial^2}{\partial x^2}v_0(x,y,t)\right) + 24v_0(x,y,t)
\end{aligned} \qquad (21)$$

$$(\frac{\partial}{\partial y}v_1(x,y,t))(\frac{\partial^2}{\partial y\,\partial x}v_0(x,y,t))+$$
$$12v_1(x,y,t)(\frac{\partial^2}{\partial y^2}v_0(x,y,t))(\frac{\partial}{\partial x}v_0(x,y,t))+$$
$$6v_0^2(x,y,t)(\frac{\partial^3}{\partial x^3}v_1(x,y,t))+$$
$$12v_0(x,y,t)v_1(x,y,t)(\frac{\partial^3}{\partial y^2\partial x}v_0(x,y,t))+$$
$$24(\frac{\partial}{\partial x}v_0(x,y,t))(\frac{\partial}{\partial y}v_0(x,y,t))$$
$$(\frac{\partial}{\partial y}v_1(x,y,t))+12v_0(x,y,t)(\frac{\partial^2}{\partial y^2}v_1(x,y,t))$$
$$(\frac{\partial}{\partial x}v_0(x,y,t))+12(\frac{\partial}{\partial x}v_1(x,y,t))$$
$$(\frac{\partial}{\partial y}v_0(x,y,t))^2=0$$

With the following conditions:

$$v_0(x,y,0)=\frac{3}{2}\lambda\sinh\left[\frac{1}{6}(x+y)\right],\qquad v_i(x,y,0)=0,\quad i=1,2,\ldots \tag{22}$$

with the effective initial approximation for v_0 from the conditions (22) and solutions of Eqs. (19, 20 and 21) may be written as follows:

$$v_0(x,y,t)=\frac{3}{2}\sinh\left(1.6666\times10^{-1}(x+y)\right) \tag{23}$$

$$v_1(x,y,t)=4.6875\times10^{-1}t\cosh(1.6666\times10^{-1}x+1.6666\times10^{-1}y)-8.4375\times10^{-1}t\cosh(5\times10^{-1}x+5\times10^{-1}y) \tag{24}$$

$$v_2(x,y,t)=1.25\times10^{-11}t^2\times\begin{pmatrix}-145546875063\sinh(0.5x+0.5y)+\\ 16406250015\sinh(1.6666\times10^{-1}x+1.6666\times10^{-1}y)+\\ 179296875050\sinh(8.3333\times10^{-1}x+8.3333\times10^{-1}y)\end{pmatrix} \tag{25}$$

In the same manner, the rest of components were obtained using the Maple package. According to the HPM, we can conclude that

$$u(x,t)=\lim_{p\to1}v(x,t)=v_0(x,t)+v_1(x,t)+\ldots \tag{26}$$

Therefore, substituting the values of $v_0(x,t)$, $v_1(x,t)$ and $v_2(x,t)$ from Eqs. (23, 24, 25) in to Eq. (26) yields:

$$\begin{aligned}u(x,y,t)&=1.5\sinh h\left(1.6666\times10^{-1}(x+y)\right)+\\&4.6875\times10^{-1}t\cosh(1.6666\times10^{-1}x+1.6666\times10^{-1}y)-\\&8.4375\times10^{-1}t\cosh(5\times10^{-1}x+5\times10^{-1}y)+\\&1.25\times10^{-11}t^2(-145546875063\sinh(0.5x+0.5y)+\\&16406250015\sinh(1.6666\times10^{-1}x+1.6666\times10^{-1}y)+\\&179296875050\sinh(8.3333\times10^{-1}x+8.3333\times10^{-1}y))\end{aligned} \tag{27}$$

Application of VIM: To solve the Eq. (17) by means of VIM, one can construct the following correction functional:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t\lambda\begin{pmatrix}(\frac{\partial u_n(x,y,\tau)}{\partial\tau})+\left(\frac{\partial}{\partial x}u^3{}_n(x,y,\tau)\right)+\\ 2\left(\frac{\partial^3}{\partial x^3}u^3{}_n(x,y,\tau)\right)+2\left(\frac{\partial^3}{\partial y^2\partial x}u^3{}_n(x,y,\tau)\right)\end{pmatrix}d\tau \tag{28}$$

Its stationary conditions can be obtained as follows:

$$\begin{aligned}&1-\lambda'\big|_{\tau=t}=0\\&\lambda\big|_{\tau=t}=0\\&\lambda''=0\end{aligned} \tag{29}$$

We obtain the Lagrangian multiplier:

$$\lambda = -1 \tag{30}$$

As a result, we obtain the following iteration formula:

$$u_{n+1}(x,t)=u_n(x,t)-a\int_0^t\left(\begin{array}{l}\left(\frac{\partial u_n(x,y,\tau)}{\partial\tau}\right)+\left(\frac{\partial}{\partial x}u^3{}_n(x,y,\tau)\right)+\\ 2\left(\frac{\partial^3}{\partial x^3}u^3{}_n(x,y,\tau)\right)+2\left(\frac{\partial^3}{\partial y^2\partial x}u^3{}_n(x,y,\tau)\right)\end{array}\right)d\tau \tag{31}$$

Now we start with an arbitrary initial approximation that satisfies the initial condition:

$$u_0(x,y,t)=\frac{3}{2}\lambda\sinh\left[\frac{1}{6}(x+y)\right] \tag{32}$$

Using the above variational formula (31), we have:

$$u_1(x,t)=u_0(x,t)-\int_0^t\left(\begin{array}{l}\left(\frac{\partial u_0(x,y,\tau)}{\partial\tau}\right)+\left(\frac{\partial}{\partial x}u^3{}_0(x,y,\tau)\right)+\\ 2\left(\frac{\partial^3}{\partial x^3}u^3{}_0(x,y,\tau)\right)+2\left(\frac{\partial^3}{\partial y^2\partial x}u^3{}_0(x,y,\tau)\right)\end{array}\right)d\tau \tag{33}$$

Substituting Eq. (32) in to Eq. (33) and after simplifications, we have:

$$\begin{aligned}u_1(x,y,t)&=1.5\sinh(1.6666\times10^{-1}(x+y))-\\ &3.375\cosh^3(1.6666\times10^{-1}(x+y))+\\ &3\cosh(1.6666\times10^{-1}(x+y))t\end{aligned} \tag{34}$$

In the same way, we obtain $u_2(x,t)$ as follows:

$$\begin{aligned}&u_2(x,y,t)=144.1625977t^4\times\\ &\cosh^8(1.6666\times10^{-1}(x+y))\\ &\sinh(1.6666\times10^{-1}(x+y))+3\cosh(1.6666\times\\ &10^{-1}(x+y))t+171.3867188t^4\times\\ &\cosh^4(1.6666\times10^{-1}(x+y))\times\\ &\sinh(1.6666\times10^{-1}(x+y))-32.0625t^4\times\\ &\cosh^2(1.6666\times10^{-1}(x+y))\\ &\sinh(1.6666\times10^{-1}(x+y))+35.85937501t^2\times\\ &\cosh^4(1.6666\times10^{-1}(x+y))\times\\ &\sinh(1.6666\times10^{-1}(x+y))-282.3925782t^4\times\\ &\cosh^6(1.6666\times10^{-1}(x+y))\times\\ &\sinh(1.6666\times10^{-1}(x+y))+0.75t^4\times\\ &\sinh(1.6666\times10^{-1}(x+y))+23t^3\\ &\cosh(1.6666\times10^{-1}(x+y))+4.265625t^2\times\\ &\sinh(1.6666\times10^{-1}(x+y))-\\ &128.4609375t^3\cosh^7(1.6666\times10^{-1}(x+y))+\\ &253.1953t^3\cosh^5(1.6666\times10^{-1}(x+y))-\\ &148.2187t^3\cosh^3(1.6666\times10^{-1}(x+y))-\\ &34.17187t^2\cosh^2(1.6666\times10^{-1}(x+y))\times\\ &\sinh(1.6666\times10^{-1}(x+y))-3.375t\times\\ &\cosh^3(1.6666\times10^{-1}(x+y))+\\ &1.5\sinh(1.6666\times10^{-1}(x+y))\end{aligned} \tag{34}$$

and so on. In the same way the rest of the components of the iteration formula can be obtained. Figures 1 and 2 show comparisons between the results of HPM and VIM with ADM [19]. From these comparisons, it can be clearly seen that there are very good agreement among them and the results obtained from the three methods are nearly coincident.

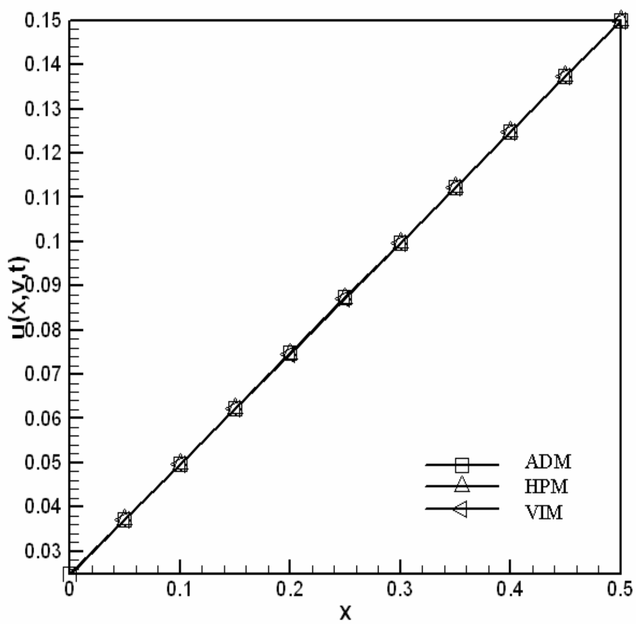

Fig.1: Comparison between the results of different solutions at $y = 0.1, t = 0.001$

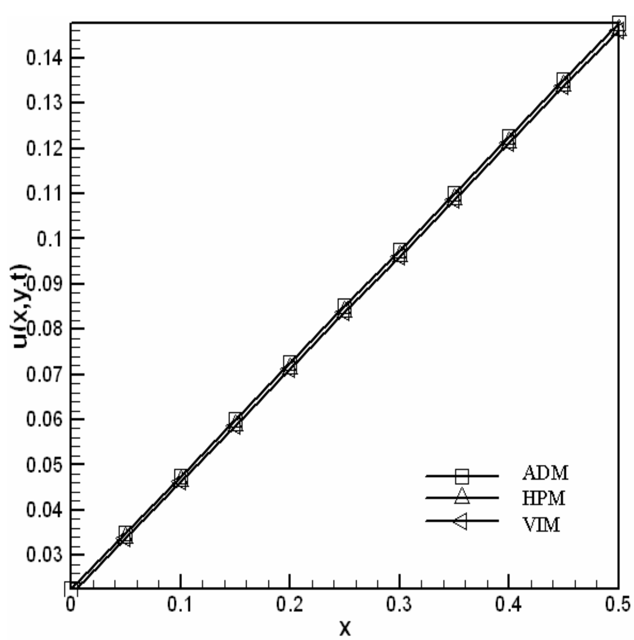

Fig.2: Comparison between the results of different solutions at $y = 0.1, t = 0.01$.

CONCLUSION

The homotopy perturbation method and variational iteration method are employed successfully to study problem of ZK (3, 3, 3). In conclusion, VIM and HPM provide highly accurate numerical solutions for nonlinear problems in comparison with other methods. They also do not require large computer memory and discretization of variable *t* neither in VIM nor in HPM. The approximations are valid not only for small parameters but also for larger ones and the initial approximation can be arbitrarily chosen with unknown constants which can be defined through different methods. As it is mentioned, these methods avoid linearization and physically unrealistic assumptions. Finally, Comparisons with Adomian's decomposition method reveals that the approximate solutions obtained by the two proposed methods converge to their exact solutions faster than those of Adomian's method. For computations and plots, Maple Package has been used.

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