



Finite Element Analysis of Frictional Contact Problem During the Process of Metal Working

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Abstract. The quasistatic contact problem with dry friction, contributes to the achievement of a numerical model for the simultaneity of different kinds of viscoplastic, incompressible materials forming. Starting from a variational formulation of the problem, which extends the static case, a perturbed Lagrangean, discrete formulation within the framework of finite element method is obtained. On the contact boundary we have used a method based on contact finite elements, with three nodes, for the unilateral contact conditions with the friction law. One numerical example is presented.

Key words: Frictional contact problems, contact finite element, process of the metal working.

INTRODUCTION

The paper presents a formulation of the basic approach for finite element modeling of material forming processes with a viscoplastic, incompressible and frictional contact approximation.

Assume that the body is subject to volume forces while the input and output velocities are given and on a part of the boundary it is in a unilateral frictional contact with a rigid fixed support. Our interest is focused on the discretization by finite element of the contact area, such as the geometry of the contact area, the contact conditions and the frictional law is well approximated. The main numerical problems presented are: variational problem, evolving contact friction, modeling with finite contact element, incremental approach, meshing and remeshing of the domain, solving the non-linear set of equations using Newton-Raphson method.

The problem is also reduced to a set of optimization constrained problems. We solve this problem by using the Lagrange multipliers method. In order to illustrate the physical model the quasistatic problem is considered. The purpose is to find such velocities for the given boundary and constrained conditions that the forming processes are steady-state

MATERIALS and METHODS

Consider a Lipschitzian domain Ω on R^N , $N = 1, 2$, with boundary Γ . Let Γ_0 and Γ_1 be open and disjoint parts of Γ so that $\Gamma = \Gamma_0 \cup \Gamma_1$. We suppose that in the initial stage the body occupies the domain Ω and we restrict it to small displacement and small strains case.

Assume that the body is subjected to volume forces of density $f = (f_1, f_2)$ on Ω , while the velocities are given on Γ_0 with $m(\Gamma) > 0$ and on Γ_1 it is in unilateral contact with a rigid fixed support. We use the following notations for the normal and tangential components of the velocities and of the stress vector:

$v_n = v_i n_i$, $v_{ii} = v_i - v_n n_i$, $\sigma_n = \sigma_{ij} n_i n_j$, $\sigma_t = \sigma_{ij} n_j - \sigma_n n_i$ where $i, j = 1, 2$, $n = (n_1, n_2)$ are the outward normal unit vector on Γ and the summation convention are used for i and j .

The contact problem with friction law of elastoviscoplastic incompressible material model as the following classical formulation [1]:

Find the field of velocities $v(x) = (v_1(x), v_2(x))$, defined Ω , which satisfy the following equations and conditions):

the equilibrium equation
$$\text{div } \sigma + f = 0 \text{ on } \Omega \tag{1}$$

- the incompressibility condition
$$\text{div } v = 0 \text{ on } \Omega \tag{2}$$

- the constitutive equation
$$\sigma^D = 2\eta D_{ii} D \tag{3}$$

where $\sigma^D = \sigma - \frac{\text{tr } \sigma}{3} I$; $D_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$,

$D_{ii} = \frac{1}{2} D_{ij} D_{ij}$, $\sigma = \frac{1}{2} \sigma_{ij}^D \sigma_{ij}^D$ and $\eta > 0$ is

viscosity coefficient.

- the boundary conditions for field velocities:

$$v = v_1 \text{ on } \Gamma'_0 \quad (4)$$

$$v = v_2 \text{ on } \Gamma''_0 \quad (5)$$

- the unilateral contact conditions with friction law on $\Gamma_1 = \Gamma'_1 \cup \Gamma''_1$:

$$\begin{aligned} v_n &\leq 0 \text{ with} \\ v_n < 0 &\Rightarrow \sigma_n = 0 \\ v_n = 0 &\Rightarrow \sigma_n \leq 0 \text{ and} \\ \|\sigma_t\| < \mu \|\sigma_n\| &\Rightarrow v_t = 0 \\ \|\sigma_t\| < \mu \|\sigma_n\| &\Rightarrow \exists \lambda \geq 0 \\ \text{with } \sigma_t &= -\lambda v_t \end{aligned} \quad (6)$$

The classical formulation of this problem is as follows:

Problem P1: Find the velocities $v = (v_1, v_2)$, which satisfy (1) – (6), where f, v_1, v_2, μ and η are given.

It is known^[2] that a variational formulation of the Problem P1 is the following inequality:

Problem P2: Find $v = (v_1, v_2)$ such that

$$a(v, u - v) + \frac{1}{\epsilon} \int \text{div } v \text{ div}(u - v) dx + \quad (7)$$

$$+ J(v, u) - J(v, v) > (L, u - v), \forall u \in K,$$

where

$$K = \{u \in (H^1(\Omega))^N, \quad (8)$$

$$u = u^i \text{ a.i. on } \Gamma'_0, u = u^e \text{ a.i. on } \Gamma''_0,$$

$$\text{div } u = 0 \text{ on } \Omega, \text{ and } u_n \leq 0 \text{ a.e. on } \Gamma^1\},$$

$$a(v, u) = 2 \int_{\Omega} \eta(D_{ii}(v)) D_{ij}(v) D_{ij}(u) dx, \quad (9)$$

$$\forall u, v \in (H^1(\Omega))^N,$$

$$J(u, v) = \int_{\Gamma_1} \mu \|\sigma_n(v)\| \|u_t\| ds, \quad (10)$$

$$L(u) = \int_{\Omega} f_i u_i dx, \forall u \in (H^1(\Omega))^N, \quad (11)$$

where $N = 1, 2$; $f = (f_1, f_2) \in (H^1(\Omega))^N$, ϵ is the penalty parameter of the unsatisfactory of the incompressibility condition.

Finite element discretization of the problem: We consider a discrete variational problem

P2 using four nodes isoparametric finite elements on Ω , and three nodes contact element^[3,4] on Γ_1 .

The contact finite elements with 3 nodes (see Figure 1) is a bilinear isoparametric element which associates one gap with a typical slave node S is given by $g = (x_s - x_1)n$, where n denotes the normal master segment 1-2, $x_s = X_s + v_s$ defines the current position of the slave node, and $x_1 = X_1 + v_1$, $x_2 = X_2 + v_2$ define the current position of the master segment.

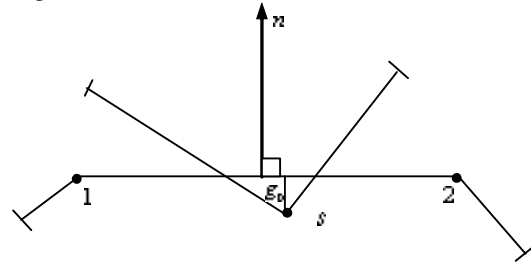


Fig. 1: Geometry of a slave node with a master segment 1-2

In all numerical applications we have derived a perturbed Lagrange formulation for the case of frictional stick and for the case of frictional slide. For the case of frictional stick the perturbed Lagrange functional for bodies in contact has the following form:

$$\begin{aligned} L(v, \Sigma_n, \Sigma_t, D_v) = & \Pi(v) + \sum_n^T G_n + \\ & + \sum_n^T G_t - \frac{1}{2\omega_n} \sum_n^T \Sigma_n - \frac{1}{2\omega_t} \sum_t^T \Sigma_t + \quad (12) \\ & + \frac{1}{2\epsilon} D_v^T D_v \end{aligned}$$

where v is the vector of nodal velocities Σ_n, Σ_t , are the vectors of normal and tangential nodal contact forces, respectively, G_n and G_t are the vectors of normal and tangential nodal gaps, and ω_n, ω_t are the normal and tangential penalty parameters respectively, $\Pi(v)$ are the total potential energy, ϵ is the penalty parameter of the unsatisfaction of the incompressibility condition and D_v is the vector obtained from the incompressibility condition.

For the case of frictional slide the relation $\|\Sigma_t\| = \mu \|\Sigma_n\|$ must be considered, as a direct consequence of the Coulomb's friction law.

The Newton-Raphson method was applied to the discrete variational formulations that can be derived from these perturbed Lagrange functional.

In the three dimensional case^[7], a four node contact element consisting of three "master" nodes 1, 2, 3 and one "slave" node's, is employed (see Figure 2).

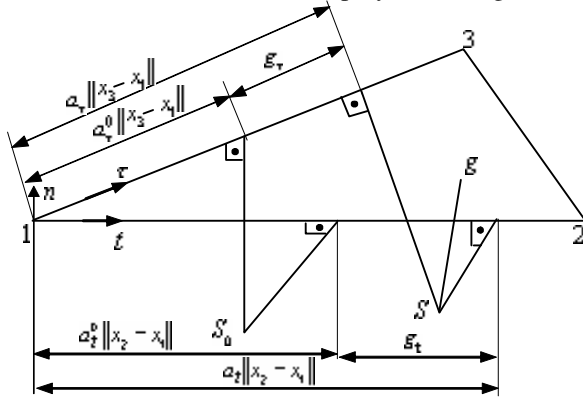


Fig. 2: Three-dimensional contact finite element

Normal vectors on a defined plane by the nodes 1, 2 and 3 and respectively the vectors, defined by directions of the nodes 1-2 and 1-3 will be:

$$n = \frac{(x_2 - x_1) \cdot (x_3 - x_1)}{\|(x_2 - x_1) \cdot (x_3 - x_1)\|}; t = \frac{x_2 - x_1}{\|x_2 - x_1\|};$$

$$\tau = \frac{x_3 - x_1}{\|x_3 - x_1\|}$$

where $x_1 = X_1 + u_1$, $x_2 = X_2 + u_2$, $x_3 = X_3 + u_3$ signify the current positions of master nodes; X_1, X_2, X_3 are reference coordinates and u_1, u_2, u_3 are current nodal displacements of points 1, 2 and 3.

In addition, we define the current "surface coordinate" as following:

$$a_t = \frac{(x_s - x_1)}{\|x_2 - x_1\|} t; a_\tau = \frac{(x_s - x_1)}{\|x_2 - x_1\|} \tau$$

where $x_s = X_s + u_s$ denotes the current position of the slave node. The normal and tangential gaps g_n, g_τ are defined as

$$g_n = (x_s - x_1)n,$$

$$g_t = (a_t - a_t^0) \|x_2 - x_1\|,$$

$$g_\tau = (a_\tau - a_\tau^0) \|x_3 - x_1\|,$$

where a_t^0 and a_τ^0 are the old surface coordinates at the last time step known.

Note that the gap g depends on the slave node s as well as on the master nodes 1, 2 and 3. Thus, the variation of the gap is obtained according to

$$g = \frac{d}{d\alpha} [g(x_s + \alpha\eta_s, x_1 + \alpha\eta_1, x_2 + \alpha\eta_2, x_3 + \alpha\eta_3)]$$

where

$$\eta(\eta_1, \eta_2, \eta_3, \eta_s) \equiv \delta u(\delta u_1, \delta u_2, \delta u_3, \delta u_s) \quad (13)$$

With respect to finite element implementations, explicit matrix expressions for the Lagrange multiplier formulation and the penalty formulation are derived as follows.

The discrete variational equation associated with (12) take the form:

$$\delta_u \Pi(u) + \Sigma_n^T \delta_u G_n + \Sigma_t^T \delta_u G_t + \Sigma_\tau^T \delta_u G_\tau = 0 \quad (14)$$

$$\delta \Sigma_n^T \left(-\frac{1}{\omega_n} \Sigma_n + G_n \right) = 0 \quad (15)$$

$$\delta \Sigma_t^T \left(-\frac{1}{\omega_t} \Sigma_t + G_t \right) = 0 \quad (16)$$

$$\delta \Sigma_\tau^T \left(-\frac{1}{\omega_\tau} \Sigma_\tau + G_\tau \right) = 0, \quad (17)$$

where $\Pi(u) = \frac{1}{2}a(u, u) - L(u)$ is the total potential energy of the bodies in contact,

$$\delta_u G_n = (\delta_u g_n^1, \delta_u g_n^2, \dots, \delta_u g_n^s)^T,$$

$$\delta_u G_t = (\delta_u g_t^1, \delta_u g_t^2, \dots, \delta_u g_t^s)^T,$$

$$\delta_u G_\tau = (\delta_u g_\tau^1, \delta_u g_\tau^2, \dots, \delta_u g_\tau^s)^T, \quad S = \text{total number of}$$

slave nodes in contact $s = 1, 2, \dots, S$, analogous for $\delta \Sigma_n, \delta \Sigma_t, \delta \Sigma_\tau$.

The variational of a typical nodal normal gap $g_n \in G_n$ take the form:

$$\delta g_n = \sum_{j=1}^3 \frac{\partial g_n}{\partial u_s^j} \eta_s^j + \sum_{i=1}^3 \sum_{j=1}^3 \frac{\partial g_n}{\partial u_i^j} \eta_i^j, \quad \text{with the}$$

notation (13) and

$$c_n = \left(\frac{\partial g_n}{\partial u_s^1}, \frac{\partial g_n}{\partial u_s^2}, \frac{\partial g_n}{\partial u_s^3}, \frac{\partial g_n}{\partial u_1^1}, \dots, \frac{\partial g_n}{\partial u_3^3} \right),$$

$$\eta = (\eta_s^1, \eta_s^2, \eta_s^3, \eta_1^1, \dots, \eta_3^3), \text{ we obtain:}$$

$$\delta g_n = \eta^T c_n.$$

Similarly, the variation of a typical nodal tangential gap $g_t \in G_t$, $g_\tau \in G_\tau$ can be obtained according to $\delta g_t = \eta^T c_t$, $\delta g_\tau = \eta^T c_\tau$.

Moreover, the residual vector R_B and the tangent stiffness K_B associated, with the total potential energy of the contacting bodies simply read, result $\delta \Pi(u) = \eta^T R_B$ and $\delta R_B = \eta^T K_B$.

With, the convention:

$$(u^1, \dots, u^{12}) = (u_s^1, u_s^2, u_s^3, u_1^1, \dots, u_3^3) \quad (14) \text{ become:}$$

$$\eta^T \left[R_B + \sum_{s=1}^S (\sigma_n^s c_n^s + \sigma_t^s c_t^s + \sigma_\tau^s c_\tau^s) \right] = 0 \quad (18)$$

and analogously for (15)-(17) where

$$\sigma_n \in \Sigma_n, \sigma_t \in \Sigma_t, \sigma_\tau \in \Sigma_\tau.$$

To apply Newton's iteration scheme, consistent linearization of (18) and those corresponding (15)-(17), at $(u, \Sigma_u, \Sigma_t, \Sigma_\tau)$ is performed and leads

$$\left[\eta^T, \delta \Sigma_n^T, \delta \Sigma_t^T, \delta \Sigma_\tau^T \right] \begin{Bmatrix} A_1 & A_2 & A_3 & A_4 \\ A_2^T & B_2 & O & O \\ A_3^T & O & C_3 & O \\ A_4^T & O & O & D_4 \end{Bmatrix} \begin{Bmatrix} \Delta u \\ \Delta \Sigma_n \\ \Delta \Sigma_t \\ \Delta \Sigma_\tau \end{Bmatrix} = - \begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix}$$

where

$$A_1 = K_B + \sum_{s=1}^S (k_n^s + k_t^s + k_\tau^s), A_2 = \sum_{s=1}^S c_n^s, A_3 = \sum_{s=1}^S c_t^s,$$

$$A_4 = \sum_{s=1}^S c_\tau^s, B_2 = -\frac{1}{\omega_n} I, C_3 = -\frac{1}{\omega_t} I, D_4 = -\frac{1}{\omega_\tau} I,$$

$$R_1 = R_B + \sum_{s=1}^S (\sigma_n^s c_n^s + \sigma_t^s c_t^s + \sigma_\tau^s c_\tau^s)$$

$$R_2 = -\frac{1}{\omega_n} \Sigma_n + G_n, R_3 = -\frac{1}{\omega_t} \Sigma_t + G_t,$$

$$R_4 = -\frac{1}{\omega_\tau} \Sigma_\tau + G_\tau,$$

O is the matrix zero, and

$$(k_n^s)_{ji} = \frac{\partial c_n^{si}}{\partial u_j} = \frac{\partial^2 g_n^s}{\partial u_i \partial u_j}, (k_t^s)_{ji} = \frac{\partial c_t^{si}}{\partial u_j} = \frac{\partial^2 g_t^s}{\partial u_i \partial u_j},$$

$$(k_\tau^s)_{ji} = \frac{\partial c_\tau^{si}}{\partial u_j} = \frac{\partial^2 g_\tau^s}{\partial u_i \partial u_j}.$$

Finally after the discrete formulation within the framework FEM, a standard assembly procedure can be used to add the contact contributions of each contact element to the global tangent stiffness and residual matrix and thus we obtain:

$$K U = R, \quad (19)$$

$$\text{where } K = K_B + \sum_{s=1}^S K_C^s, R = - \left(R_B + \sum_{s=1}^S R_C^s \right),$$

K_B , R_B are mechanical global tangent stiffness matrix and residual vector, K_C^s , R_C^s are mechanical contact contributions of contact nod s , $U = (\Delta u, \Delta \Sigma_n, \Delta \Sigma_t, \Delta \Sigma_\tau)^T$, S is the total number of the slave nodes. And for $\omega_n = \omega_t = \omega_\tau = \omega$, and $\sigma_n = \omega g_n$, $\sigma_t = \omega g_t$, $\sigma_\tau = \omega g_\tau$ result

$$K_C = \sum_{s=1}^S \omega \left(g_n^s k_n^s + g_t^s k_t^s + g_\tau^s k_\tau^s + c_n^{sT} c_n^s + c_t^{sT} c_t^s + c_\tau^{sT} c_\tau^s \right) \\ R_C = \sum_{s=1}^S \omega \left(g_n^{sT} c_n^s + g_t^{sT} c_t^s + g_\tau^{sT} c_\tau^s \right). \quad (20)$$

For the case of frictional slide we used the relation $|\Sigma_{\tan}| = \mu |\Sigma_n|$, where μ is the coefficient of friction and Σ_{\tan} is the result force of the Σ_t and Σ_τ , forces in the tangent plane of the contact surface.

Note with β the angle between the sides $x_2 - x_1$ and $x_3 - x_1$; we obtain $\cos \beta = t\tau$ and

$$|\lambda_{\tan}| = \mu \sqrt{g_t^2 + g_\tau^2 + 2\mathcal{E} |g_t| |g_\tau| \cos \beta} \text{ where}$$

$\mathcal{E} = \text{sgn}(g_t g_\tau)$. As a direct consequence of Coulomb's friction law, it results $\mu \omega |g_n| = \omega r$,

where $r = \sqrt{g_t^2 + g_\tau^2 + 2\mathcal{E} |g_t| |g_\tau| \cos \beta}$ therefore

$$\lambda_t = \lambda \tan \frac{g_t}{r} \omega g_n = -\mu \text{sgn}(g_t) \frac{g_t}{r} \omega g_n = -\mu \frac{|g_t|}{r} \omega g_n,$$

$$\lambda_\tau = -\mu \frac{|g_\tau|}{r} \omega g_n.$$

If we write $d_t = \frac{|g_t|}{r}$, $d_\tau = \frac{|g_\tau|}{r}$, $b_t = \frac{\partial d_t}{\partial u}$, $b_\tau = \frac{\partial d_\tau}{\partial u}$, from linearized kinematics (i.e., by neglecting non-linear terms k_t and k_τ), we obtain:

$$K_C = \sum_{s=1}^S (SL_1^s + SL_2^s), \text{ with}$$

$$SL_1^S = \omega \left(g_n^s k_n^s - \mu g_n^s d_t^s k_t^s - \mu g_n^s d_\tau^s k_\tau^s + c_n^{sT} c_n^s - \mu d_t^s c_n^{sT} c_t^s \right)$$

$$SL_2^S = \omega \left(-\mu d_t^s c_n^{sT} c_\tau^s - \mu g_n^s b_t^{sT} c_t^s - \mu g_n^s b_\tau^{sT} c_\tau^s \right) \text{ and}$$

$$R_c = \sum_{s=1}^S \omega \left(\mu g_n^s d_t^{sT} c_t^s + \mu g_n^s d_\tau^{sT} c_\tau^s - g_n^{sT} c_n^s \right).$$

The Newton-Raphson method was applied to the discrete variational formulation that can be derived from these perturbed Lagrangian functional. The scheme of solving the linearized problem is following:

- (i) initialisation set the iterative count $k = 0$, $u^{(0)} = 0$;
- (ii) compute the system stiffness and residual excluding contact;
- (iii) compute contact stiffness and residual for each finite element s , $s = 1, \dots, NC$

- a) compute the normal gap $g_n^{(s)(k)}$;
- b) check for contact finite element status:
 - IF $g_n^{(s)(k)} > TOL$ then out of contact
 - ELSE in contact. Check for frictional stick or slip; compute $\sum_n^{(k)}$, $\sum_t^{(k)}$, $\sum_\tau^{(k)}$ and

IF $\left| \text{SQRT} \left(\left(\sum_t^{(k)} \right)^2 + \left(\sum_\tau^{(k)} \right)^2 \right) \right| \leq \mu \left| \sum_n^{(k)} \right|$ then frictional stick; ELSE frictional slip; ENDIF

- ENDIF
- c) compute the total stiffness and residual;
- (iv) solve the total system to obtain displacement increment $\Delta u^{(k)}$;
- (v) check for convergence:

IF $\left| \Delta u^{(k)} - \Delta u^{(k+1)} \right| < TOL1$ then converge and exit.

- ELSE go to step (vi);
- (vi) update the displacement field $u^{(k+1)} = u^{(k)} + \Delta u^{(k+1)}$;
- (vii) set $k = k + 1$ and go to back to (ii).

Several examples have been taken from other works to be compared and the results obtained by us agreed to the ones^[7].

RESULTS and DISCUSSION

Based on these matrices, a standard assembly procedure can be used to add the contact contributions of each contact node to the global tangent stiffness and residual.

We consider^[5,6] a simplified model with the geometry and loading given in Figure 3, a discretization with 32 nodes and 20 finite elements with four nodes isoparametric and for contact boundary, three nodes contact element^[4]. Data of the problem:

- $h_i = 38 \text{ mm}$;
 - $h_0 = 9,5 \text{ mm}$;
 - $\mu = 0,001$ on Γ_1' and $\mu = 0,1$ on Γ_1'' ;
 - $\varepsilon = 10^{-7} \eta_m$, with η_m in average value of η , and $\eta_0 = 0,278 \text{ Mpas}$;
 - $\beta = 70^0$, $v_i = 0,19 \text{ m/s}$, where we have used index i for input data and index o for output data.
- Obtained values for v_1 and $\|v_2\|$, where $v = (v_1, v_2)$, are represented in Figure 4, respective Figure 5.

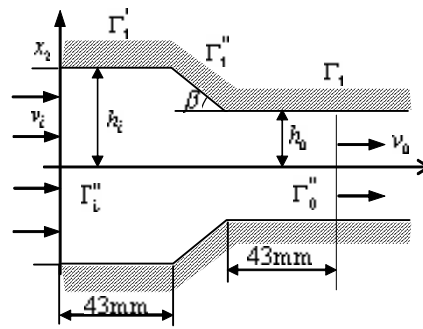


Fig. 3: Geometry of domain Ω

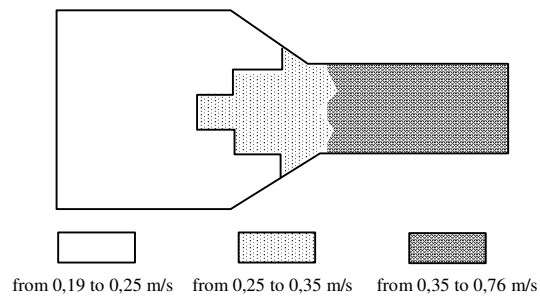


Fig. 4: Values of the v_1

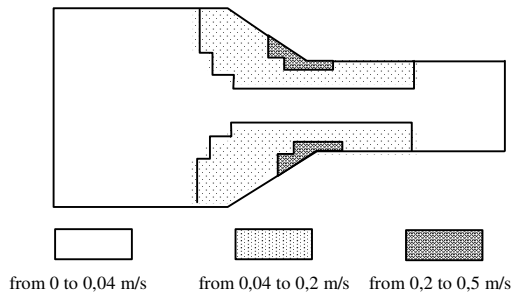


Fig. 5: Values of the $\|v_2\|$

CONCLUSION

This study investigates the simulation model of different kinds of viscoplastic, incompressible materials forming. Starting with the initial value $\eta_0 = 0,278$ MPas for viscosity coefficient, a fast convergence of the sequence (7) in the discretization variant had been observed. Only three iterations are enough to obtain a relative error less than one percent. The problem can be extended in the three-dimensional case and for crack analysis^[8].

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REFERENCES

1. Teodosiu, C., Soós, E. and Rosu, T., 1990. A finite element model of the hot working of axisymmetric products, Rev. Roum., Sci., Tech., Mec. Appl., III, 35: 225-229
2. Klarbring, A., Mikelic, A., Shillor, M., 1988. Frictional Contact Problems with Normal Compliance, Int. J. Eng. Sci. 26 : 811-832
3. Ju, J. W. and Taylor, R. L., 1988. A Perturbed Lagrangean Formulation for the Finite Element Solution of Nonlinear Contact Problems, J. de Méc. Théor. et App., Special Issue, Supplement no. 1 to vol. 7:1-14.
4. Wriggers, P. and Simo, J. C., 1985. A Note on Tangent Stiffness for Frictional Contact Problems, Comm. in App. Num. Meth., 1: 199-203.
5. Cocu, M., 1991. Nonlinear boundary value problems in the thermo-viscoplasticity of the metals (in Romanian), Ph Thesis, Bucharest University.
6. Rous, M., Chabrand, P. and Lebon, F., 1988. Numerical Methods for Frictional Contact Problems and Applications, J. de Méc. Théor. et App., Special Issue, Supplement no. 1 to vol. 7 : 111-126
7. Pop, N., 2004. Numerical Simulations for the 3D Frictional Contact Problems, Inginerare, Revista Falcultad de Ingenieria de la Pontificia Universidad Catolica de Valparaiso, CHILE, 17: 33-38
8. Labibzadeh, M., Sadrnejad, S.A., 2007. Crack Analysis of Concrete Arch Dams Using Micro-Planes Damage Based Constitutive Relations, American Journal of Applied Sciences, 4: 197-202.