A Priori Estimation of the Resolvent on Approximation of Born-Oppenheimer

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Abstract: In this study, we estimate the resolvent of the two bodies Schrödinger operator perturbed by a potential of Coulombian type on Hilbert space when $h$ tends to zero. Using the Feschbach method, we first distorted it and then reduced it to a diagonal matrix. We considered a case where two energy levels cross in the classical forbidden region. Under the assumption that the second energy level admits a non degenerate point well and virial conditions on the others levels, a good estimate of the resolvent were observed.

Key words: Distorsion, eigenvalues, estimation, resolvent, resonances

INTRODUCTION

The Born-Oppenheimer approximation technique\cite{1} has instigated many works one can find in bibliography the recent papers like\cite{2-5}.

It consists to study the behaviour of a many body systems, in the limit of small parameter $h$ as the particles masses (masses of nuclei) tends to infinity; (see the references therein for more information), we can describe it with a Hamiltonian of type

$$P = -h^2 \Delta_x - \Delta_y + V(x, y)$$

on $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, when $h \to 0$ and $V$ denote the interaction potentials between the nuclei of the molecule and the nuclei electrons. The idea is to replace the operator

$$Q(x) = -\Delta_x + V(x, y)$$

by the so-called electronic levels which be a family of its discrete eigenvalues: $\lambda_1(x), \lambda_2(x), \lambda_3(x), \ldots$ and to study the operators $P$ which can be approximatively given by

$$-h^2 \Delta_x + \lambda(x), \text{ on } L^2(\mathbb{R}^3)$$

Martinez and Messirdi's works, are about spectral proprieties of $P$ near the energy level $E_0$ such that $\inf \lambda \leq E_0$. Martinez in\cite{6}, studies the case where $\lambda_0(x)$ admits a nondegenerate strict minimum at some energy level $\lambda_0$, the eigenvalues of $P$ near $\lambda_0$ admits a complete asymptotic expansion in half-powers of $h$\cite{6}.

Messerdi and Martinez\cite{7} considers the case where $\lambda_0$ admits a minimum, such appears resonances for $P$. He gives an estimation of the resolvent of $O(h^{-1})$ at the neighbourhood of 0.

In this study we try to generalize this work to approximate the resolvent of $P$ where $V$ is a potential of Coulombian type at the neighbourhood of a point $x_0 \neq 0$.

In fact, we estimate the resolvent of the operator $F_\varsigma$, given by a reduction of the distorted operator $P_\varsigma$, of $P$ modified by a truncature $\varsigma$\cite{8}; and we try to have a good evaluation of the order of $O(h^{-1/2})$.

We apply the Feschbach method to study the distorted operator $P_\varsigma$ which allows us to go back to the initial problem and we put the virial conditions on $\lambda_1$ and $\lambda_3$.

Hypothesis and results

Hypothesis: Let the operator

$$P = -h^2 \Delta_x - \Delta_y + V(x, y)$$

on $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, when $h \to 0$. $V(x, y) = V(x, y_1, y_2, y_3, \ldots, y_p)$ is an interaction potential of Coulombian type

$$V(x, y) = \frac{\alpha}{|x|} + \sum_{j=1}^{p} \frac{\alpha_j}{|y_j - x|} + \sum_{j<k}^{p} \frac{\alpha_{jk}}{|y_j - y_k|}$$

where $\alpha, \alpha_j, \alpha_{jk}$ are real constants, $\alpha > 0$ ($\alpha_j$ is the charges of the nuclei).

It is well known that $P$ with domain $H^2(\mathbb{R}^3 \times \mathbb{R}^3)$ is essentially self-adjoint on $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$.
For $x \neq 0$, $Q(x) = -\Delta_x + V(x, y)$ with domain $H^2(\mathbb{R}^3)$ is essentially self-adjoint on $L^2(\mathbb{R}^3)$

**Remark 1.1:** The domain of $Q(x)$ is independent of $x$.

To describe our main results we introduce the following assumptions:

(H1) $\forall x \in \mathbb{R}^3 \setminus \{0\}$, $\sigma_{\text{disc}}(Q(x)) \geq 3$

Let $\lambda_0$ an energy level such that: $\lambda_0 \cap [-\infty, \lambda_0] \leq 3$,

denoting $\lambda_1(x), \lambda_2(x), \lambda_3(x)$ the first three eigenvalues of $Q(x)$.

(H2) we assume that the first three eigenvalues $\lambda_j$, $\forall j \in \{1, 2, 3\}$ are simple at infinity:

$\inf_{j,k \in \{1, 2, 3\}} |\lambda_j - \lambda_k| \geq \delta_1$

(H3) we suppose that $c_0 \exists \epsilon > 0$ such that $\lambda_j(x) \geq \alpha - \epsilon$

(H4) We are in the situation where $\lambda_2(x)$ admits a nondegenerate strict minimum; creating a potential well of the shape $\Gamma$

Remark 1.2: By Reed-Simon’ results, the first eigenvalue is automatically simple.

Remark 1.3: This hypothesis is still true for $\alpha, \langle \rho \rangle$

$\exists \delta_2 > 0$ such that

$\forall x \in \mathbb{R}^3 \setminus \{0\}$, $\lambda_2(x) + \delta_2 \langle \min \{\lambda_2(x), \lambda_3(x)\} \rangle$

we note by $K = \{x \in \mathbb{R}, \lambda_2(x) = \lambda_i(x)\}$

and for $\delta > 0$, we also note by:

$K_\delta = \{x \in \mathbb{R}, \text{dist}(x, K) \leq \delta\}$

Let $\delta > 0$ such that

* $K_\delta \cap K_\delta$ is simply connex
* $K_\delta \cap U = \emptyset$

* The connex composites of $\mathbb{R}^3 \setminus K_\delta$ are simply connex

(H5) Virial Conditions

It exists $d > 0$ such that for $j \in \{2, 3\}$,

The resonances of $P$ are obtained by an analytic distortion introduced by Hunziker and so they are defined as complex numbers $\rho_j \ (j = 1, ..., N)$ such that for all $\epsilon > 0$ and $\mu$ sufficiently small, $\text{Im} \mu > 0$ $\rho_j \in \sigma_{\text{disc}}(P\mu)^3$. We denote the set of the resonances of $P$ by: $\sigma(P) = \bigcup_{\mu} \sigma_{\text{disc}}(P\mu)$

Where $P\mu$ is obtained by the analytic distortion satisfying: $P\mu = U\mu P U^{-1}$. So, $P\mu$ can be extended to small enough complex values of $\mu$ as an analytic family of type.

The analytic distortion $U\mu$, for $\mu$ small enough associated to $v$ is defined on $\mathbb{R}^3 \times \mathbb{R}^3$ by

$U\mu \varphi(x, y) = \varphi(x + \mu v(x), y_1 + \mu v(y_1), ..., y_p + \mu v(y_p))$,

where $J = J(x, y) = \det(1 + \mu Dv(x))^{1/2}$

is the Jacobian of the transformation $\Psi_\mu : (x, y) \rightarrow (x + \mu v(x), y_1 + \mu v(y_1), ..., y_p + \mu v(x))$ and $v \in C^\infty(\mathbb{R}^3)$ is a vector field satisfying:

$\exists \delta > 0$, large enough such that:

$\forall v(x) = 0, \ |x| \leq \frac{2}{N}$

$\forall v(x) = x, \ |x| \geq \tau_x - \epsilon$

$\epsilon > 0$, small enough, $|t| > \frac{3 \ N}{N + \epsilon}$.

Remark 1.4: The distortion is close to the potential well.

We localise our operator near the well $v_0$ by introducing a truncate function $\zeta \in C^\infty(\mathbb{R}^3)$ satisfying:

$\zeta = 1, \ |x| \leq \frac{2}{N}$

$\zeta = 0, \ |x| \geq \frac{3}{2N}$

fixing $\alpha_0$ $v_0$, we set
\[
Q^{\pm}(x) = -U_{\mu} \Delta_{x} U_{\mu}^{-1} + \zeta(x) V_{\mu}(x,y) \alpha_{0} + (1 - \zeta(x)) \alpha_{0} \alpha_{0} \\
V_{\mu}(x,y) = (x + \mu \nu(x), y_{1} + \mu \nu(x),..., y_{p} + \mu \nu(x))
\]
We also denote:
\[
P^{\pm}_{\mu} = -h^{2}U_{\mu} \Delta_{x} U_{\mu}^{-1} + Q^{\pm}_{\mu}(x)
\]
(7)
With domain \(H^{2}(\mathbb{R}^{+})\).

**Remark 1.5:** Like in \([10]\), near \(\nu\), \((P)_{\mu}\sigma\) and \((P)_{\mu}\) coincide up to exponentially small error terms. For this we will study \(P^{\pm}_{\mu}\) instead of \(P_{\mu}\).

**RESULTS**

Here we write the results of our works as following:

**Theorem 1.6:** Under assumptions (H1) to (H5) and for \(\mu \in C,|\mu|\) and \(h\) small enough, we have
\[
\left\| (F^{\pm}_{\mu} - z)^{-1} \right\| = O(h^{-1/2})
\]
where \(F^{\pm}_{\mu}\) is the Feshbach reduced operator of \(P^{\pm}_{\mu}\) verifying
\[
F^{\pm}_{\mu} = -\frac{h^{2}}{(1 + \mu)^{2}} \Delta_{x} I + M^{\pm}_{\mu} + \tilde{R}^{\pm}_{\mu}
\]
and the error \(\tilde{R}^{\pm}_{\mu}\) is satisfying:
\[
\left\| \tilde{R}^{\pm}_{\mu} \right\|_{L^2(\mathbb{R}^{3})} = O(h^{2})
\]
We need for our proof the main important theorem for the operator \(P_{2,\mu}\) which is the distorsion of the operator \(P_{2,\mu}\):
\[
P_{2,\mu} = -h^{2}U_{\mu} \Delta_{x} U_{\mu}^{-1} + \lambda_{2}(x) \alpha_{0} + \tilde{R}^{\pm}_{\mu}(x)
\]
(8)
at the neighbourhood of point \(x_{0}\) of the well such that \((\forall \varepsilon \gamma 0,\text{small enough}, \left| |x_{0}| r_{0} + \varepsilon^{*}\right|)\), the distorsion \(P_{2,\mu}\)
is in fact a dilatation of angle \(\theta\) such that \(\varepsilon^{*} = (1 + \mu)\).
We denote it by \(P_{2,0}\) \([11]\) and is defined by
\[
P_{2,0} = -h^{2} \Delta_{x} + \lambda_{2}(x) \alpha_{0}
\]
(9)
Let \(e_{j}, j = 1,...,N_{0}\) be the eigenvalues of the operator \(P_{0}\) that is bounded by \(P_{2,0}\) and is defined by
\[
P_{2,0} = -h^{2} \Delta_{x} + \lambda_{2}(x) \alpha_{0}
\]
(10)
For \(x \neq 0\), we denote also
\[
\tilde{Q}_{\mu}(x) = Q_{\mu}(x) - \frac{\alpha}{\left| x + \mu \nu(x) \right|} \quad \text{and} \quad \tilde{\lambda}_{j}(x) = \lambda_{j} - \frac{\alpha}{|k|} j \in \{1,2,3\}
\]
Let \(C(x)\) be a family of continuous closed simple loop of \(C\) enclosing \(\tilde{\lambda}_{j}(x), j \in \{1,2,3\}\) and having the rest of \(\sigma(\tilde{Q}_{\mu}(x))\) in its exterior. The gap condition (4) permits us to assume that:
\[
\min_{x \in C} \text{dist}(\gamma(x), \sigma(\tilde{Q}_{\mu}(x))) \geq \frac{\delta}{2}
\]
(11)
Using the relation (6) and (H3), we can take \(C(x)\) compact in a set of \(C\). So, we deduce from (11) the following result\([8]\).

**Lemma 2.1**

1. \(\forall j, k \in \{1,...,p\}, j \neq k, \beta \in \text{IN}^{p}\), the operators
\[
\frac{1}{|y_{j} - y_{k}|} \left( \tilde{Q}_{\mu}(x) - z \right)^{-1}, \quad \frac{1}{|y_{j} - y_{k}|} \left( \tilde{Q}_{\mu}(x) - z \right)^{-1}
\]
and \(\tilde{Q}_{\mu}(x) - z\) are uniformly bounded on \(L^{2}(\mathbb{R}^{3})\), \(x \in \text{IN}^{3}\), \(z \in C(x)\)
2. If \(\mu \in \text{small enough}\), then for \(x \in \text{IN}^{3}\), \(z \in \text{IN}^{3}\), the operator \(\left( \tilde{Q}_{\mu}(x) - z \right)^{-1}\) exists and satisfies uniformly
\[
\left( \tilde{Q}_{\mu}(x) - z \right)^{-1} = O(|\mu|).
\]
Now we define for $\mu \in C$ small enough, the spectral projector associated to $\tilde{Q}_\mu$ and the interior of $C(x)$.

$$\pi_\mu(x) = \frac{1}{2\pi} \int_{\mathbb{R}(x)} (z - \tilde{Q}_\mu(x))^{-1} \text{ and } \text{rg}\pi_\mu = 1$$

This projector permits us to construct the Grushin problem associated to the operator $P_\mu$.

**Problem of Grushin associated with the operator $P_\mu$:** We begin this section by the result which is (lemma 1.1 of [12] and proposition 5.1 of [7]).

**Proposition 2.2:** Assume (H1), (1.7), (1.9), (1.10) hold, then for $\mu \in C$, $z \in C$ small enough, there exist $N$ functions $\omega_{k,\mu}(x, y) \in C^0(\mathbb{R}^3, H^2(\mathbb{R}^3))$, $(k = 1, 2, 3)$, depending analytically on $\mu$ in $\mathbb{R}$, such that

i. $\langle \omega_{k,\mu}, \omega_{k,\mu} \rangle_{L^2(\mathbb{R}^3)} = \delta_{j,k}$

ii. For $|x| \geq \frac{3}{N}$, $\omega_{k,\mu}(x)$ form a basis of $\pi_\mu(x)$

iii. $\epsilon \in C^\infty\left(\left\{ |x| \leq \frac{1}{N} \right\}, H^2(\mathbb{R}^3)\right)$

iv. For $|x|$ large enough, $\omega_{k,\mu}(x)$ is an eigen function of $\tilde{Q}_\mu(x)$ associated with $\lambda_k (x + \mu\omega(x))$

We first introduce the family $\{\omega_{k,\mu}, \omega_{k,\mu}, \omega_{3,\mu}\}$ of $\pi_\mu(x)$ depending analytically on $\mu$ for $\mu$ small enough and normalized in $L^2(\mathbb{R}^3)$ by

$$\langle \omega_{k,\mu}(x), \omega_{3,\mu}(x) \rangle_{L^2(\mathbb{R}^3)} = \delta_{j,3}$$

and then we associate the two following operators

$$R^-_\mu : \bigoplus_1^3 L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$$

$$u^- = (u_1, u_2, u_3) \rightarrow R^-_\mu u^- = \sum_{k=1}^3 u_k \omega_{k,\mu}(x)$$

$$R^+_\mu = (R^-_\mu)^* : L^2(\mathbb{R}^3) \rightarrow \bigoplus_1^3 L^2(\mathbb{R}^3)$$

$$u = \langle (u, \omega_{3,\mu}) \rangle_{\mathbb{R}^3}, \langle (u, \omega_{3,\mu}) \rangle_{\mathbb{R}^3}, \langle (u, \omega_{3,\mu}) \rangle_{\mathbb{R}^3}$$

where $A^*$ denote the transposed of the operator $A$, $(\cdot, \cdot)_\mathbb{R}^3$ the inner product on $L^2(\mathbb{R}^3)$ and $(\cdot, \cdot)_\mathbb{R}^3$ is the adjoint of the operator $L^2(\mathbb{R}^3)$.

As $P_\mu^i$ and $\omega_{k,\mu}$, $k = 1, 2, 3$ have analytic extensions with $\mu$, the Grushin problem is then defined, for $z \in C$, by:

$$P_\mu^i(z) = \begin{pmatrix} P_\mu^i - z & \omega_{k,\mu} & \omega_{k,\mu} & \omega_{k,\mu} \\ \omega_{k,\mu} & 0 & 0 & 0 \\ \omega_{k,\mu} & 0 & 0 & 0 \\ \omega_{k,\mu} & 0 & 0 & 0 \end{pmatrix}$$

which sets on $H^2(\mathbb{R}^3) \oplus (\bigoplus_1^3 L^2(\mathbb{R}^3))$ to $L^2(\mathbb{R}^3) \oplus (\bigoplus_1^3 H^2(\mathbb{R}^3))$

The following proposition, gives the inverse of the operator (12) by using a result of Grushin problem. This is proved in [3, 8].

**Proposition 2.3:** $\forall z \in C$ close enough to $\lambda_{\mu}$, $P_\mu$ is invertible and we can write its inverse:

$$P_\mu^{-1} = \begin{pmatrix} X_{\mu}^2 & X_{\mu}^3 \\ X_{\mu}^3 & X_{\mu}^4 \end{pmatrix}$$

With $X_{\mu}^2(z) = (P_\mu^i - z)^{-1} \hat{\pi}_\mu(x)$, where $P_\mu$ is the bounded inverse of the restriction of $\tilde{P}_\mu$ to $\{u \in H^2(\mathbb{R}^3), \hat{\pi}_\mu u = 0\}$.

$$X_{\mu}^2(z) = (\omega_{k,\mu} - X_{\mu}^3(z) P_\mu^i(\omega_{k,\mu}))_{k \in \mathbb{R}^3}$$

$$X_{\mu}^3(z) = \langle (1 - P_\mu^i(z) X_{\mu}^2(z), \omega_{k,\mu})_{k \in \mathbb{R}^3} \rangle$$

$$X_{\mu}^4(z) = \langle Z\delta_\mu - (P_\mu^i - P_\mu^i X_{\mu}^2(z) P_\mu^i) \omega_{k,\mu} \rangle_{L^2(\mathbb{R}^3)}$$

**Remark 2.4**

1. For $z \in C$, close enough to $\lambda_{\mu}$, we have $z \in \sigma(P_\mu^i)$ if and only if $\exists \mu, |\mu|$ small enough and $\Im \mu > 0$, such that $z \in \sigma_{\text{disc}}(X_{\mu}^4(z))$ where $X_{\mu}^4(z)$ is a pseudodifferential operator of principal symbol defined by the matrix:

$$B(x, \xi, z) = zI - (\langle \omega_{k,\mu}(x) \rangle_{k \in \mathbb{R}^3} + \omega_{k,\mu}(x))_{k \in \mathbb{R}^3}$$

and $\tilde{t}_\mu(x, \xi) = \langle \omega_{k,\mu}(x) \rangle$.

2. $z$ is a resonance of the operator $P_\mu^i$ only and only if $\exists \mu \in C$, $|\mu|$ small enough $\Im \mu > 0$, such that:

$$0 \in \sigma_{\text{disc}}(X_{\mu}^4(z)) \text{ or } 0 \in \sigma_{\text{disc}}(F_{\mu}^i(z))$$

where $F_{\mu}^i$ is the Feshbach operator $F_{\mu}^i = z - X_{\mu}^4(z)$.
Reduced Feshbach operator: To reduce the Feshbach operator in a matricial operator, we input:

$$\Phi_{\mu} = P_{\mu}^* - P_{\mu}^* X_{\mu}^*(x) P_{\mu}$$  \hspace{1cm} (13)

$$F_{\mu} = \left( \begin{array}{c} \left( \left( \Phi_{\mu} \omega_{\mu} (x) \right) (x) \right)_{1 \leq i, k \leq 3} \\ \left( \left( \Phi_{\mu} \omega_{\mu} (x) \right) (x) \right)_{1 \leq i, k \leq 3} \end{array} \right)$$  \hspace{1cm} (14)

and

$$\Phi_{\mu} (z) = \left( \begin{array}{c} \left( \left( \Phi_{\mu} \omega_{\mu} (x) \right) (x) \right)_{1 \leq i, k \leq 3} \\ \left( \left( \Phi_{\mu} \omega_{\mu} (x) \right) (x) \right)_{1 \leq i, k \leq 3} \end{array} \right)$$  \hspace{1cm} (15)

The following proposition gives us the estimation of the resolvent of the operator (15).

**Proposition 2.5:** For \( z \in C, |z| \) small enough, \( \mu \in C, |\mu| \) small enough, the operator or \( (\Phi_{\mu} (z) - z) \) is bijective for \( H^2 (\mathbb{R}^3) \) to \( L^2 (\mathbb{R}^3) \). Its inverse is extended for \( H^m \) in \( H^{m+1} \).

\( H^m = H^m (L^2 (\mathbb{R}^3), L^2 (\mathbb{R}^3)), \forall m \in Z \) and verify for \( j \in \{1, 2, 3\} \), \( h > 0 \) small enough:

$$\| (\Phi_{\mu} (z) - z) \|_{(L^p, H^{m+1})} \leq \frac{C(m)}{h^{(|\text{Im} \mu|)}}$$  \hspace{1cm} (16)

To prove this proposition, we first use a lemma in [3], to prove the following lemma:

**Lemma 2.6:** \( \forall m \in Z \), the operator \( X_{\mu} (z) \) is uniformly extensible in a bounded operator on \( H^m (L^2 (\mathbb{R}^3), L^2 (\mathbb{R}^3)) \), \( \forall m \in Z \), for \( h > 0 \), \( z \in Z \) and \( \mu \in C, |\mu| \) small enough and

$$\| X_{\mu} \|_{(L^p, H^{m+1})} = O \left( h^{-2} \right)$$

See [3] for the proof.

**Lemma 2.7:** We assume that

$$\| (P_{\mu} - z) \|_{(L^2, H^{m+1})} = O \left( \frac{1}{h^{\text{Im} \mu}} \right)$$

for \( h > 0 \), \( z \in C \) and \( \mu \in C \) small enough, where

$$P_{\mu} = -h^2 \frac{1}{(1 + \mu)^2} \Delta_x + \lambda_i (x + \mu \nu (x)) -$$

$$-h^2 \left( \Delta (\omega_{\mu} (x)) \omega_{\mu} (x) \right)_{x} -$$

$$-h^2 \left( R (x, D_x) \omega_{\mu} (x), \omega_{\mu} (x) \right)_{x}$$

\( R (x, D_x) \), is an differential operator of coefficients \( \mathcal{C}^\infty \).

**Proof of lemma 2.7:** Using (H5) we have:

$$\text{Im} \frac{1}{(1 + \mu)^2} \lambda_i (x + \mu \nu (x)) \leq -\frac{\text{Im} \mu}{C_1}$$

and we easily deduce with a simple computation that

$$\| (P_{\mu} - z) \|_{(L^2, H^{m+1})} = O \left( \frac{1}{h^{\text{Im} \mu}} \right)$$

**Proof of the proposition 2.5:** From (13) and (15), we have \( \Phi_{\mu} = \left( (P_{\mu} - P_{\mu}^* X_{\mu}^*(z) P_{\mu}^* (\omega_{\mu} (x)) (x) \right) \), then we subtitue \( P_{\mu} \) from (7) with

$$U_{\mu} \Delta_x U_{\mu}^{-1} = \frac{1}{(1 + \mu)^2} \Delta_x + R \left( x, D_x \right)$$

is a second order differential operator with \( \mathcal{C}^\infty \) coefficients in \( x \) with compact support, analytic in \( \mu \) and whose derivative of any kind compared to \( x \) are \( O \left( |\mu| \right) \) and we put

$$\Lambda_{\mu} = \frac{1}{(1 + \mu)^2} \left( \Delta_x, X_{\mu}^*(z) \omega_{\mu} (x), \omega_{\mu} (x) \right)_{x} +$$

$$+ \frac{1}{(1 + \mu)^2} \left( R \left( x, D_x \right) X_{\mu}^*(z) \omega_{\mu} (x), \omega_{\mu} (x) \right)_{x}$$

Using the fact that \( \hat{\pi}, \omega_{\mu}, \omega_{\mu}, \omega_{\mu}, \omega_{\mu}, \omega_{\mu}, \omega_{\mu}, \omega_{\mu} \) = 1, we have:

$$\Phi_{\mu} (z) = \hat{P}^\infty_{\mu} - h^2 \Lambda_{\mu} \omega_{\mu}$$

where

$$\hat{P}^\infty_{\mu} = -h^2 \frac{1}{(1 + \mu)^2} \Delta_x + \lambda_i (x + \mu \nu (x)) -$$

$$-h^2 \left( R \left( x, D_x \right) \omega_{\mu} (x), \omega_{\mu} (x) \right)_{x}$$

We have \( R \left( x, D_x \right) \) bounded, so \( \Lambda_{\mu} \) is \( O \left( h^2 \right) \) from \( H^m \) to \( H^m \) and we also see from (H5) and lemma2.6 that:

for \( h \) small enough, \( \| (P_{\mu} - z) \|_{(L^2, H^{m+1})} = O \left( \frac{1}{h^{\text{Im} \mu}} \right) \), then, we deduce

$$\| (\hat{P}^\infty_{\mu} - z) \|_{(L^2, H^{m+1})} = O \left( \frac{1}{h^{\text{Im} \mu}} \right)$$

Finally, we have:

$$\| (\Phi_{\mu} (z) - z) \|_{(L^2, H^{m+1})} = O \left( \frac{1}{h^{\text{Im} \mu}} \right)$$

**Proof of theorems**

**Proof of theorem 2.1:** Proposition3.5 permits us to reduce the Feshbach operator \( F_{\mu} \) in a matricial operator
The operators $T^j_\mu$ are defined by:

$$T^j_\mu(z)(\alpha) = (\Phi^j_\mu(\alpha z), \alpha \in L^2(\mathbb{R}))$$

hence, the spectral study of the Feshbach $F$ becomes the study of the operator $A^j_\mu$ on $L^2(\mathbb{R})$ by:

$$A^j_\mu(z)(\alpha) = (\Phi^j_\mu(z)(\alpha), \alpha \in L^2(\mathbb{R}))$$

So we establish easily that:

$$A^j_\mu = -\Delta + \lambda + \mathcal{M} \mathcal{V} \mathcal{V},$$

where $\mathcal{M}$ is a diagonal matrix outside of $0$ and it equal to:

$$M^j_\mu = \left\{ Q^j_\mu(x)(\alpha_{j,\mu}) \right\}_{j=1,2,3},$$

where $Q^j_\mu(x)(\alpha_{j,\mu})$ are the eigenvalues of $Q^j_\mu, \forall x \in IR - \{0\}$

The remainder

$$\tilde{R}^j_\mu(h, z) = O(h^2), \forall m \in Z uniformly$$

for $h \to 0$ and $z \in C$ closed to $\lambda_0$.

At the end we prove the second result. To describe it, we apply a technical of Briet Combs Duclos [13].

Let $J_1 \in C_0^\infty(\{x - x_0 \leq \delta, (\delta)0 fixed small enough and $x_0$ a point of maximum) and $J \in C^\infty(\mathbb{R}^n)$ such that:

$J_1 = 1$ near $x_0$ and $J_1 + J_2 = 1$

$J$ is an identification mapping such that:

$$J : L^2(\mathbb{R}^n) \oplus L^2(\text{supp} p_j) \to L^2(\mathbb{R}^n)$$

$$J(u \oplus w) = J_u + J w$$

It is easily proved that: $J^*J = L^2(\mathbb{R}^n)$

Now, if we note $P^\Omega_\mu$ the Dirichlet realisation of $P^j_\mu$ on $\Omega$, on $\Omega$, $x = v(x)$ and the distorsion $x + \mu v(x) = x e^0$, is an analytic dilatation (whose Dirichlet realisation is the operator $H^0$ obtained for $\zeta = 1$). We set

$$H^0_z = -h^2 e^{2\lambda} (\lambda \alpha(x)(x - x_0), (x - x_0)\mathcal{V} e^0)$$

$$H^0 = P^0 \mathcal{V} = -h^2 e^{2\lambda} (\lambda \alpha(x)$$

$$H^0 = H^0 \mathcal{V} L^2(\text{supp} p_j), \text{ with Dirichlet conditions on} \partial \text{supp} p_j$$

Remark 3.1: Since $\inf_{\text{supp} P} \text{Re} e^{2\lambda}(x e^0) \mathcal{V} 0$, $(H^0 - z)^{-1}$ is uniformly bounded for $|z|$ and $h$ small enough.

Before we prove the second result, we introduce the following lemma

Lemma 3.2: For all $p \in [0, 1]$,

$$\left\| x^p (H^0 - z)^{-1} \right\|_{L^2} = O(h^2), \text{ uniformly for} z \text{ outside of} \gamma(x)$$

$z \in [-e - x_0, C_0 h - x_0] + i[0] \text{, where} \lambda \geq 0, \text{ and} h \text{ small enough.}$

Proof of Lemma 3.2: If we put $y = x - x_0$, we can write $H^0$:

$$H^0 = \mathcal{H}^0$$

where $H^0 = -h^2 e^{2\lambda} \mathcal{V} \mathcal{V} + \mathcal{V} e^{2\lambda} \mathcal{V} e^0$, with $\mathcal{V} e^0 = \mathcal{V} (1 + (x - x_0) e^0 + \mathcal{V} e^{2\lambda} e^0)$

It is enough to show that, for $\theta = \alpha, \theta \geq 0, \text{ small enough.}$ We have from (16)

$$\left\| x^p (H^0 - z)^{-1} \right\|_{L^2} = h^{2 - \frac{p}{2}} \left\| x^p (H^0 - zh^{-1})^{-1} \right\|_{L^2}$$

and the eigenvalues of the operator $H^0$ in

$$[-\alpha, C_0 - x_0] i IR$$

We distinguish three cases for $p = 0$.

1/ If $z \in \left[-Ch - x_0, C_0 h - x_0\right] + i\left[-Ch - x_0, C_0 h - x_0\right]$ we deduce for all $C$, $(H^0 - zh^{-1})^{-1}$ is bounded on $L^2$ uniformly for $z$ outside the $\gamma_j$, so (17) is verified.

2/ If $z \in \left[-e - x_0, C_0 h - x_0\right] + i\left[-e - x_0, C_0 h - x_0\right]$ then for $u \in C^\infty_0(\mathbb{R}^n)$:
\[ e^{2\theta}H_{0}^{0} = -\Delta y + \frac{1}{2} \langle \lambda ''(x_{0})y, y \rangle e^{i0} + \]
\[ h^{-1}(z + i(1 + (x - x_{0}))e^{i0} + \frac{1}{2}(x - x_{0})^{2} e^{i0}) \]
and
\[ \text{Im} \langle e^{2\theta}(H_{0}^{0} - zh)^{-1}u, u \rangle = \frac{1}{2} \sin 2\alpha \langle \lambda ''(x_{0})y, y \rangle u, u \rangle - \]
\[ h^{-1}(z\sin 2\alpha + \text{Im} z\cos 2\alpha + h^{-1}(y\sin 3\alpha + z\cos 4\alpha) \parallel u \parallel^{2} \]
We take particularly \( \alpha \) small enough and \( C \) large enough such that: \( C\cos 2\alpha \cdot C \sin 2\alpha \)
At least we obtained
\[ \parallel e^{2\theta}(H_{0}^{0} - zh)^{-1}u, u \parallel \geq h^{-1} \langle x_{0} \sin 2\alpha + y\sin 3\alpha \rangle \parallel u \parallel^{2} \]
so the result is also verified. It remain the case:
3) If \( z \in [e - x_{0}, -C\sin x_{0}] + i[-C - x_{0}, C\cos x_{0}] : \]
\[ \text{Re} \langle e^{2\theta}(H_{0}^{0} - zh)^{-1}u, u \rangle \]
\[ \geq h^{-1} \langle \text{Re} z\cos 4\alpha - \text{Im} z\sin 2\alpha + y\sin 3\alpha \rangle \]
we deduce the estimation when \( C\cos 2\alpha \cdot \alpha \) small enough and \( C \) large enough such that \( \cos 4\alpha \cdot \sin 2\alpha \)
Now we consider the case when \( p \neq 0 , \)
\[ e^{2\theta}(H_{0}^{0} - zh)^{-1} = -\Delta + \frac{1}{2} e^{i0} \langle \lambda ''(x_{0})y, y \rangle \]
and
\[ zh^{-1} e^{2\theta} + h^{-1} e^{2\theta} \Sigma(\epsilon) \]
\[ -\Delta + \frac{1}{2} e^{i0} \langle \lambda ''(x_{0})y, y \rangle - zh^{-1} e^{2\theta} + h^{-1} e^{2\theta} \Sigma(\epsilon) \]
\[ \geq \frac{1}{2} \cos 4\alpha \langle \lambda ''(x_{0})y, y \rangle u \parallel u \parallel^{2} \geq \frac{1}{C} \parallel u \parallel^{2} \]
if we put \( u = (H_{0}^{0} - zh^{-1})^{-1} v \) the result is deduced from an a priori standard estimation.

**Proof of theorem 1.2:** We put \( H_{0}^{d} = H_{0}^{0} \oplus H_{0}^{0} \) and \( \Pi = H_{0}^{0}J - JH_{0}^{0} \), for \( z \) outside the spectrum of \( H_{0}^{0} \), with a simple calculation we obtain:
\[ (H_{0}^{d} - z)^{-1} = J(H_{0}^{0} - z)^{-1}J'(1 + \Pi(H_{0}^{d} - z)^{-1}J')^{-1} \] (18)
Using the lemma3.2 (with \( p = 2 \)) and the lemma3.1 of Briet Combs Duclos[13], we can easily prove that: \( \exists \beta \langle 1 \]
such that
\[ \parallel \Pi(H_{0}^{0} - z)^{-1}J' \parallel \leq \beta \] (19)
Using the lemma3.2 and (19), we obtain from (18)
\[ \parallel (H_{0}^{d} - z)^{-1} \parallel \leq C \parallel (H_{0}^{0} - z)^{-1} \parallel , \]
finally the result is obtained from lemma3.2 and remark3.1

**REFERENCES**