A Fixed Point Theorem for Contraction Type Mappings in Menger Spaces

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Abstract: We proved a common fixed point theorem for a sequence of self maps satisfying a new contraction type condition in Menger spaces, results extended and generalize some known results in metric spaces and fuzzy metric spaces.

Key words: fixed point, contraction map, Menger probabilistic metric space

INTRODUCTION

There have been a number of generalizations of metric space. One such generalization is Menger space introduced in 1942 by Menger[1] who was use distribution functions instead of nonnegative real numbers as values of the metric. Schweizer and Sklar[2] studied this concept and gave some fundamental results on this space. The important development of fixed-point theory in Menger spaces was due to Sehgal[3]. The study of common fixed points of maps satisfying some contractive type condition has been at the centre of vigorous research activity. It is observed by many authors [3,4-10] that contraction condition in metric space may be translated into probabilistic metric space endowed with min norms. The purpose of this was to define and investigate a new class of self-maps satisfying a new contraction type condition in Menger spaces.

Preliminaries: We recall some definitions and known results in Menger probabilistic metric space. For more details, we refer the readers to [1-]1,4,8,11,12.

Definition 1: A triangular norm * (shorty t-norm) is a binary operation on the unit interval [0,1] such that for all a, b, c, d ∈ [0,1] the following conditions are satisfied:

(a) a * 1 = a,
(b) a * b = b * a,
(c) a * b ≤ c * d whenever a ≤ c and b ≤ d,
(d) a * (b * c) = (a * b) * c.

Some examples of t-norms are a * b = max{a+b-1,0} and a * b = min{a,b}.

Definition 2: A distribution function is a function F: [−∞,∞] → [0,1] which is left continuous on R, non-decreasing and F(−∞) = 0, F(∞) = 1. If X is a nonempty set, F: X × X → Δ is called a probabilistic distance on X and F(x,y) is usually denoted by Fxy.

Definition 3 (11): (see also [1.3,9]) The ordered pair (X,F) is called a probabilistic semimetric space (shortly PSM-space) if X is a nonempty set and F is a probabilistic metric space satisfying the following conditions: for all x, y, z ∈ X and t, s > 0,

(PM-1) Fxy(t) = H(t) ⇔ x = y,
(PM-2) Fxy = Fyx

If, in addition, the following inequality takes place:

(PM-3) Fxy(t+s) = Fxy(s) for every PSM-space (X,F), we can consider the sets of the form Uλ = \{x∈X: Fxy(t) > 1−λ \}.

The family \{Uλ\}λ>0 generates a semi uniformity denoted by UF and a topology τF called the F-topology or the strong topology. Namely, A ∈ τF iff ∀x ∈ A ∃ε > 0 and λ ∈ (0,1) such that Uλ(x) ⊂ A. UF is also generated by the family \{V_δ\}_δ>0 where V_δ := Uδ,δ

In [13], it is proved if sup_{t>0} (t * t) = 1, then UF is a uniformity, called F-uniformity, which is metrizable. The F-topology is generated by the F-uniformity and is determined by the F-convergence: x_n → x
\(\iff F_{x_n}(t) \to 1, \forall t > 0.\)

**Definition 4 (\cite{Am. J. Appl. Sci., 4 (6): 371-373, 2007}):** A sequence \(\{x_n\}\) in a Menger space \((X, F, \ast)\) is called converge to a point \(x\) in \(X\) (written as \(x_n \to x\)) if for every \(\varepsilon > 0\) and \(\lambda \in (0, 1)\), there is an integer \(n_0 = n_0(\varepsilon, \lambda)\) such that \(F_{x_n}(\varepsilon) > 1 - \lambda\) for all \(n \geq n_0\). The sequence called Cauchy if for every \(\varepsilon > 0\) and \(\lambda \in (0, 1)\), there is an integer \(n_0 = n_0(\varepsilon, \lambda)\) such that \(F_{x_n}(\varepsilon) > 1 - \lambda\) for all \(n, m \geq n_0\). A Menger space \((X, F, \ast)\) is said to be complete if every Cauchy sequence in it converges to a point of it.

**Lemma 1 (\cite{Am. J. Appl. Sci., 4 (6): 371-373, 2007}):** Let \(\{x_n\}\) be a sequence in a Menger space \((X, F, \ast)\) with continuous \(t\)-norm \(\ast\) and \(t \ast t \geq t\). If there exists a constant \(\alpha \in (0, 1)\) such that \(F_{x_n}(\alpha) \geq F_{x_{n+1}}(t)\) for all \(t > 0\) and \(n = 1, 2, \ldots\), then \(\{x_n\}\) is a Cauchy sequence in \(X\).

**Lemma 2 (\cite{Am. J. Appl. Sci., 4 (6): 371-373, 2007}):** Let \((X, F, \ast)\) be a Menger space. If there exists a constant \(\alpha \in (0, 1)\) such that \(F_{xy}(\alpha) \geq F_{xy}(t)\) for all \(x, y \in X\) and \(t > 0\), then \(x = y\).

**Remark 1:** In a Menger space \((X, F, \ast)\), if \(t \ast t \geq t\) for all \(t \in [0, 1]\) then \(a \ast b = \min\{a, b\}\) for all \(a, b \in [0, 1]\) and it is well known that such \(t\)-norm is continuous.

**RESULTS**

**Theorem 1:** Let \(\{T_n\}_n, n = 1, 2, \ldots\) be a sequence of mappings of a complete Menger space \((X, F, \ast)\) into itself with \(t \ast t \geq t\) for all \(t \in [0, 1]\) and \(S : X \to X\) be a continuous mapping such that \(T_n(X) \subseteq S(X)\) and \(S\) is commuting with each \(T_n\). If there exists a constant \(\alpha \in (0, 1)\) such that for any two mappings \(T_i\) and \(T_j\)

\[
\begin{align*}
\min \{ & F_{xT_iT_j}(t), F_{xT_i}(t)F_{SxT_j}(t) + F_{SxT_i}(t)F_{SxT_j}(t) \} + a \cdot F_{xT_iT_j}(2at) \geq [p \cdot F_{xT_iT_j}(t) + q \cdot F_{SxT_i}(t)] \\
& F_{SxT_iT_j}(2at) \\
& \text{holds for all } x, y \in X\end{align*}
\]

\(\iff F_{Sx}(t) \to 1, \forall t > 0.\)

\[
[p \cdot F_{SxT_iT_j}(t) + q \cdot F_{SxT_i}(t)] \cdot F_{SxT_j}(2at) \quad \text{and} \quad \\
\min \{ & F_{SxT_iT_j}(2at), F_{SxT_i}(2at)F_{SxT_j}(2at) + a \cdot F_{SxT_i}(2at)F_{SxT_j}(2at) \}
\]

Thus, it follows that

\[
\min \{ & F_{SxT_iT_j}(2at), F_{SxT_i}(2at)F_{SxT_j}(2at) + a \cdot F_{SxT_i}(2at)F_{SxT_j}(2at) \}
\]

\(\iff F_{SxT_i}(2at) \geq \min \{ F_{SxT_i}(2at), F_{SxT_j}(2at) \},\)

we have

\[
F_{SxT_i}(2at) \geq [p \cdot F_{SxT_i}(t) + q \cdot F_{SxT_i}(t)] \cdot F_{SxT_j}(2at) \]

Since \(F_{SxT_i}(2at) \geq \min \{ F_{SxT_i}(2at), F_{SxT_j}(2at) \},\)

we have

\[
F_{SxT_i}(2at) \geq [p \cdot F_{SxT_i}(t) + q \cdot F_{SxT_i}(t)] \cdot F_{SxT_j}(2at) \]

Since \(p + q \ast a = 1\), we have \(F_{SxT_i}(2at) \geq F_{SxT_i}(2at)\).

By induction, \(F_{SxT_i}(2at) \geq F_{SxT_i}(2at)\) for all \(i, j \geq 1\), and we have

\[
\min \{ & F_{SxT_i}(2at), F_{SxT_j}(2at) \} + a \cdot F_{SxT_i}(2at)F_{SxT_j}(2at) \geq
\]

Using the continuity of \(S\) and taking limits on both sides, we have

\[
\min \{ & F_{SxT_i}(2at), F_{SxT_j}(2at) \} + a \cdot F_{SxT_i}(2at)F_{SxT_j}(2at) \geq
\]

Since \(F_{SxT_i}(2at) \geq \min \{ F_{SxT_i}(2at), F_{SxT_j}(2at) \},\)

we have

\[
(1 + a) \cdot F_{SxT_i}(2at) = F_{SxT_i}(2at) + a \cdot F_{SxT_i}(2at)
\]

and hence \(F_{SxT_i}(2at) \geq 1\) for all \(\alpha \in (0, 1)\) and \(t > 0\). Therefore \(Su = Tu\) for any fixed integer \(k\). Moreover, \(\min \{ F_{SxT_i}(2at), F_{SxT_i}(2at) \} \neq
\]

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Taking the limits on both sides, we have

\[
\min \{ F_{uT_u}(at), F_{SuSu}(at), F_{SuSu}(at) \} + a F_{SuSu}(at) \\
F_{uT_u}(2at) \geq [p F_{uTu}(at) + q F_{uTu}(at)] F_{uT_u}(2at)
\]

and so

\[
F_{uT_u}(2at) + a F_{uT_u}(2at) \geq [p+q F_{uT_u}(at)] F_{uT_u}(2at).
\]

Thus, it follows that \( F_{uT_u}(2at) \geq 1 \) for all \( \alpha \in (0,1) \) and \( t > 0 \). Therefore \( u = Su = Tu \) for any fixed integer \( k \). Thus \( u \) is a common fixed point of \( S \) and \( T_n \) for \( n = 1,2,\ldots \)

For uniquenesses, let \( v \) be another common fixed point of \( S \) and \( T_n \) for \( n = 1,2,\ldots \). Using (3.1), we have

\[
\min \{ F_{vT_v}(at), F_{SuSu}(at), F_{SuSu}(at) \} + a F_{SuSu}(at) \\
F_{vT_v}(2at) \geq [p F_{vTv}(at) + q F_{vTv}(at)] F_{vT_v}(2at)
\]

and

\[
F_{vT_v}(2at) + a F_{vT_v}(2at) \geq [p+q F_{vTv}(at)] F_{vT_v}(2at).
\]

So \( F_{vT_v}(2at) \geq 1 \) for all \( \alpha \in (0,1) \) and \( t > 0 \). Hence, by Lemma 2, \( u = v \). This completes the proof. If we take \( a = 0 \) in the main Theorem, we have the following:

**Corollary 1:** Let \( \{ T_n \} \), \( n = 1,2,\ldots \) be a sequence of mappings of a complete Menger space \((X,F,\ast)\) into itself with \( t \ast t \geq t \) for all \( t \in [0,1] \) and \( S : X \to X \) be a continuous mapping such that \( T_n(X) \subseteq S(X) \) and \( S \) is commuting with each \( T_n \). If there exists a constant \( \alpha \in (0,1) \) such that for any two mappings \( T_i \) and \( T_j \)

\[
\min \{ F_{T_iT_j}(at), F_{ST_iT_j}(at), F_{ST_iT_j}(at) \} + a F_{ST_iT_j}(at) \\
F_{T_iT_j}(2at) \geq [p F_{T_iT_j}(at) + q F_{ST_iT_j}(at)] F_{ST_iT_j}(2at)
\]

holds for all \( x, y \in X \) and \( 0 < p, q < 1 \) such that \( p+q = 1 \), then there exists a unique common fixed point for all \( T_n \) and \( S \).

**Proof:** It is easy to verify from Theorem 1. If we take \( a = 0 \) and \( S = I_X \) (the identity map on \( X \)) in the main Theorem, we have the following:

**Corollary 2:** Let \( \{ T_n \} \), \( n = 1,2,\ldots \) be a sequence of mappings of a complete Menger space \((X,F,\ast)\) into itself with \( t \ast t \geq t \) for all \( t \in [0,1] \). If there exists a constant \( \alpha \in (0,1) \) such that for any two mappings \( T_i \) and \( T_j \)

\[
\min \{ F_{T_iT_j}(at), F_{ST_iT_j}(at), F_{ST_iT_j}(at) \} + a F_{ST_iT_j}(at) \\
F_{T_iT_j}(2at) \geq [p F_{T_iT_j}(at) + q F_{ST_iT_j}(at)] F_{ST_iT_j}(2at)
\]

holds for all \( x, y \in X \) and \( 0 < p, q < 1 \) such that \( p+q = 1 \), then for any \( x_0 \in X \) the sequence \( \{ x_n \} = \{ T_n x_n \}, n = 1,2,\ldots \) converges and its limit is the unique common fixed point for all \( T_n \).

**Proof:** Existence and uniqueness of common fixed point follows from Theorem 1. Convergence of the sequence \( \{ x_n \} \) can be proved as in Theorem 1.

**REFERENCES**