On Permutable Subgroups of n-ary Groups

Awni Fayez Al-Dababseh
Department of Mathematics and Statistics, Al-Hussein Bin Talal University, Ma'an, Jordan

Abstract: It is proved that every permutable subgroup of a finite n-ary group is subnormal.

Key words: finite n-ary group, permutable n-ary group, subnormal n-ary group

INTRODUCTION

We remind that, the system $G = < X, ( )>$ with one n-ary operation $( )$ is called n-ary group \([1,2]\), if it is associative and every one of the equations.

\((a_1, a_2, ..., a_n) = a\)

is solvable in X, where \(a_1, ..., a_n, a \in X\), \(i = 1, 2, ..., n\). Throughout this study all n-ary groups are finite. Let G be n-ary group and let H be a subgroup of G, then H is called permutable n-ary group if HT=TH for all subgroups T of G.

It is known however, that every permutable subgroup of a finite group is subnormal\([3,4]\). In this study we prove this property for n-ary groups.

PRELIMINARIES

Notation is standard\([2]\)

\(X^m_k\)-the sequence \(X_0X_1X_2...X_k\) (if \(m = k\) then \(X^m_k = X_k\)).

Definition 1: Let G be n-ary group, then \(x^{(n-i)}\) is an identity if \((x_{x^1}x^{k(n-i)}) = (X^{k(n-i)}x) = x\) for all \(x \in G\).

Definition 2: Let G be n-ary group and let \(x \in G\), then the sequence of elements \(\bar{x}\) of G is called an inverse of x if \(x\bar{x}\) is an identity.

Let \(H \subseteq G\) and \(x', y'_i\) are sequences of elements of G, where \(i + j = k(n-1)\) \([k \in N]\), then the symbol \([X'; H y'_i]\) denote all elements \((x'_i, y'_i)\) where \(h \in H\).

By analogus of binary groups n-ary subgroup H of a group G is called normal if for any \(x \in G\) and for any sequence \(\bar{x}\) we have \(xH\bar{x} = H\).

Definition 3: N-ary subgroup H of a group G is called subnormal in G if:

\(H = N_0 \subseteq N_1 \subseteq ... \subseteq N_i \subseteq G\)

where \(N_i\) is a normal in \(N_{i+1}\), \(i = 0, 1, ..., t-1\).

If H and T are subgroups of n-ary group G, then

\([H; T^{n-i}]=\set{h_1...h_i t_{n-i}}\), where \(h_i \in H\) and \(t_{i} \in T\).

Lemma 1\([2]\): Let H and T are subgroups of n-ary group G such that

\([H; T^{n-i}]=\set{T^{n-i} H}\), then \(B = [H; T]\) is a subgroup of G and \(B \supseteq H\).

Subgroup H of n-ary group G is called permutable if for any subgroup T from G we have

\(T \cap H \neq \Phi\) and \([H; T^{n-i}]=\set{T H}^{n-i}\).

Lemma 2\([2]\): Let H and T are subgroups of n-ary group G. If \(H \cap T \neq \Phi\), then

\([H; T^{n-i}]=\set{T H}^{n-i}\).

MAIN RESULTS

We are now to prove the following.

Lemma 3: If H u T are permutable subgroups of n-ary group G, then \([H; T]\) is permutable subgroup of G.

Proof: Let D any subgroup of n-ary group G, then by the definition of permutable subgroup \(H \cap D \neq \Phi\). By lemma 1 \(H \subseteq [H; D]\) and it is mean that \(H \cap D \subseteq [H; T]; D \neq \Phi\).

Now since

\([H; T^{n-i}D] = [H; T^{n-i}D^{n-i}] = [H; T^{n-i}D^{n-i}] = [H; D^{n-i}T^{n-i}]

\([H; T^{n-i}D^{n-i}] = [H; T^{n-i}D^{n-i}]

So \([H; T]\) is permutable subgroup in G.

Lemma 4: Let H be a subgroup of n-ary group G. Then if for some element \(x \in G\) and for some sequence of inverse \((\bar{x})\) of x we have \([H; H_{x^{n-1}}]\) = G, where

\(H_{x^{n-1}} = xH\bar{x} \), then \(H = H_{x^{n}}\).
Proof: Let $x = (a \ b_1 \ldots \ b_{n-1})$ where $a \in H$ and $b_i \in H_i$. Let $\overline{b_i}$ be a sequence of elements from $H_i$ which are inverses for $b_i$, $i = 1, 2, \ldots, n-1$. Then $a = (ab_1 \ b_2 \ldots b_{n-1} \ b_n) = (x \overline{b_{n-1}} \ldots \overline{b_1})$.

It is clear, that $b_1 \ldots b_{n-1} x$ is the sequence of inverses for $a$. That means if $\overline{a}$ is any sequence of elements of $H$ that inverse for $a$, then $H = [aH\overline{a}] = [(x \overline{b_{n-1}} \ldots \overline{b_1}) H (b_1 \ldots b_{n-1} \overline{x})] = [xH\overline{x}] = H_1$.

Lemma 5: Let $x$ be an element of n-ary group $G$ and let $\varphi: G \to G$ a map defined by $\varphi_n(g) = xg\overline{x}$ where $g \in G$ and $\overline{x}$ is some sequence that is inverse for $x$. Then $\varphi$ is an automorphism of $G$.

Proof: For any sequence of element $g_i$ from $G$ we have $\varphi((g_1 \ldots g_n))(xg\overline{x}) = (xg_1\overline{x})(xg_2\overline{x}) \ldots (xg_n\overline{x}) = (xg_1\overline{x})(xg_2\overline{x}) \ldots (xg_n\overline{x}) = (g_1^{xg_2^{xg_3^{\ldots xg_n^{x}}}})$

So $\varphi$ is an endomorphism of n-ary group $G$.

If $g \in G$, then $\varphi((xg\overline{x})) = (x(\overline{xg}\overline{x})) = xg\overline{x} = g$. It means that $\varphi$ is an injection.

Theorem: If $H$ is a subgroup of n-ary group $G$ that is permutable with any subgroup of $G$, then $H$ is a subnormal in $G$.

Proof: We prove by induction on the order of n-ary group $G$. Let $N$ be the greatest permutable subgroup of $G$ ($N \neq G$) that contains the subgroup $H$.

We show that $N$ is a normal subgroup of $G$. Let $N$ be a normal subgroup. By the definition of normal subgroup we can find some $x \in G$ such that $xN\overline{x} \neq N$ where Error! Bookmark not defined. is some sequence that is inverse of $x$. Let $\varphi: G \to G$ defined by $\varphi_n(g) = xg\overline{x}$ for all $g \in G$. By lemma 5, $\varphi$ is an automorphism n-ary group $G$. That means $xN\overline{x}$ is a permutable subgroup of n-ary group $G u |N| = |xN\overline{x}|$.

Applying lemma 1 we have $D = \left[\begin{array}{c} x_{n-1} \\ N \ N \end{array}\right] = \left[\begin{array}{c} N \ N \end{array}\right]$ which contains $N$ subgroup n-ary group $G$, where $N_1 = xN\overline{x}$. According to lemma 2 the order of this subgroup is:

$$d = \left|\begin{array}{c} N_1 N \end{array}\right| = \left|\begin{array}{c} N_1 \ \ |N|
\left|N_1 \cap N\right|\end{array}\right|$$

Since $N \neq xN\overline{x}$ and $|N| = |xN\overline{x}|$, then $d > N$. But by Lemma 3 subgroup $D$ is permutable in $G$. That means $D = G$ and this contradict lemma 4. So $N$ is a normal subgroup of $g$. Since $|N| < |G|$ and $H$ is permutable subgroup of $N$, then by, choosing group $G$ we can conclude that $H$ is subnormal subgroup in $N$. It means $H$ is a subnormal subgroup of $G$.

REFERENCES