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New Integer Programming Formulations of the Generalized Travelling Salesman Problem

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Abstract: The Generalized Travelling Salesman Problem, denoted by GTSP, is a variant of the classical travelling salesman problem (TSP), in which the nodes of an undirected graph are partitioned into node sets (clusters) and the salesman has to visit exactly one node from every cluster. In this paper we described six distinct formulations of the GTSP as an integer programming. Apart from the standard formulations all the new formulations that we describe are 'compact' in the sense that the number of constraints and variables is a polynomial function of the number of nodes in the problem. In order to provide compact formulations for the GTSP we used two approaches using auxiliary flow variables beyond the natural binary edge and node variables and the second one by distinguishing between global and local variables. Comparisons of the polytopes corresponding to their linear relaxations are established.

Keywords: Traveling salesman problem, generalized travelling salesman problem, integer programming, linear relaxation

INTRODUCTION

We are concerned with the generalized version of the travelling salesman problem (TSP) called the generalized travelling salesman problem (GTSP). Given an undirected graph whose nodes are partitioned into a number of subsets (clusters), the GTSP is then to find a minimum-cost Hamiltonian tour which includes *exactly* one node from each cluster. Therefore, the TSP is a special case of the GTSP where each cluster consists of exactly one node.

The GTSP has several applications to location, telecommunication problems, railway optimization, etc. More information on this problem and its applications can be found in Fischetti, Salazar and Toth[1,2], Laporte, Asef-Vaziri and Sriskandarajah[3], Laporte, Mercure and Nobert[4], Pop *et al.*[5,6]. It is worth to mention that Fischetti, Salazar and Toth[2] solved the GTSP to optimality for graphs with up to 442 nodes using a branch-and-cut algorithm and the problem was solved with various metaheuristic algorithms such as: random-key genetic algorithm[7], ant colony algorithms[5], etc.

The aim of this paper is to describe six different integer programming formulations of the GTSP and to establish relations between the polytopes corresponding to their linear relaxations.

A variant of the GTSP is the problem of finding a minimum cost Hamiltonian tour including *at least* one vertex from each cluster. This problem was introduced by introduced by Laporte and Nobert[8] and by Noon and Bean[9].

Definition and Complexity Of The Problem: Let $G = (V, E)$ be an n-node weighted undirected graph whose edges are associated with non-negative costs. We will assume w.l.o.g. that G is complete (if there is no edge between two nodes, we can add it with an infinite cost). Let $V_1, \ldots, V_m$ be a partition of V into m subsets called *clusters* (i.e., $V = V_1 \cup V_2 \cup \ldots \cup V_m$ and $V_l \cap V_k = \emptyset$ for all $l, k \in \{1, \ldots, m\}$ with $l \neq k$). We denote the cost of an edge $e = (i, j) \in E$ by c_{ij}. Let V_1 be the root cluster, and let $e = (i, j)$ be an edge with $i \in V_l$ and $j \in V_k$. If $l \neq k$, then e is called an *inter-cluster* edge; otherwise, e is called an *intra-cluster* edge.

The *generalized travelling salesman problem* (GTSP) asks for finding a minimum-cost tour H spanning a subset of nodes such that H contains exactly one nodes from each cluster V_i, $i \in \{1, ..., m\}$. The problem involved two related decisions:

- choosing a node subset $S \subseteq V$, such that $|S \cap V_k| = 1$, for all $k = 1, \dots, m$.
- finding a minimum cost Hamiltonian cycle in the subgraph of G induced by S.

We will call such a cycle a generalized Hamiltonian tour.

The GTSP and the *at least* variant of the problem are $\mathcal{NP}$-hard, as they reduce to travelling salesman problem when each cluster consists of exactly one node.

An Exact Algorithm For the GTSP: In this section, we present an algorithm that finds an exact solution to the GTSP.

Let G' be the graph obtained from G after replacing all nodes of a cluster V_i with a supernode representing V_i. For convenience, we identify V_i with the supernode representing it. We assume that G' with vertex set $\{V_1, ..., V_m\}$ is complete.

Given a sequence $(V_{k_1}, ..., V_{k_m})$ in which the clusters are visited, we want to find the best feasible Hamiltonian tour H^* (w.r.t cost minimization), visiting the clusters according to the given sequence. This can be done in polynomial time, by solving $|V_{k_1}|$ shortest path problems as we will describe below.

We fix the cluster V_{k_1} as the root of the global Hamiltonian tour and in addition we duplicate the cluster V_{k_1}. The new network contains all the nodes of G plus some extra nodes v' for each $v \in V_{k_1}$. We orient all the edges away from vertices of V_{k_1} according to the global Hamiltonian tour. A directed edge $\langle V_k, V_{k+1}\rangle$ resulting from the orientation of edges of the global Hamiltonian tour defines naturally an orientation $\langle i, j\rangle$ of an edge $(i, j) \in E$, where $i \in V_k$ and $j \in V_{k+1}$, having the cost c_{ij}. Moreover, there is an arc (i, j') for each $i \in V_{k_m}$ and $j' \in V_{k_1}$ having cost $c_{ij'}$.

It is easy to observe that the best (w.r.t cost minimization) Hamiltonian tour H^* visiting the clusters in a given sequence can be found by determining all the shortest paths from each $v \in V_{k_1}$ to each $w' \in V_{k_1}$ with the property that visits exactly one node from clusters $(V_{k_2}, ..., V_{k_m})$.

The overall time complexity is then $|V_{k_1}| O(|E| + \log |V|)$, i.e. $O(|V||E| + |V|\log|V|)$ in the worst case. We can reduce the time by choosing $|V_{k_1}|$ as the cluster with minimum cardinality.

Notice that the above procedure leads to an $O((m-1)!\ (|V||E| + |V|\log|V|))$ time exact algorithm for the GTSP, obtained by trying all the $(m-1)!$ possible cluster sequences. So, we have established the following result:

Theorem 1: The above procedure provides an exact solution to the generalized travelling salesman problem in $O((m-1)!\ (|V||E| + |V|\log|V|))$ time, where m is the number of clusters in the input graph.

Clearly, the above is an exponential time algorithm unless the number of clusters m is fixed.

INTEGER PROGRAMMING FORMULATIONS

Formulations based on Hamiltonian tours properties: In order to formulate the GTSP problem as an integer program we introduce the binary variables $x_e \in \{0,1\}$, $e \in E$ and $y_i \in \{0,1\}$, $i \in V$, to indicate whether an edge e respectively a node i is contained in the Hamiltonian tour. A feasible solution to the GTSP can be seen as a cycle with m edges, connecting all the clusters and exactly one node from every cluster. Therefore the GTSP can be formulated as the following integer programming problem:

$$
\begin{array}{llll}
\min & \sum_{e \in E} c_e x_e & & \\
s.\ t. & y(V_k) = 1, & \forall\, k \in K = \overline{1,m} & (1)\\
 & x(\delta(v)) = y_v, & \forall\, v \in V & (2)\\
 & x(E(S)) \le z(S - i), & \forall\, i \in S \subset V, & \\
 & & 2 \le |S| \le n-2 & (3)\\
 & x_e \in \{0,1\}, & \forall\, e \in E & (4)\\
 & y_i \in \{0,1\}, & \forall\, i \in V. & (5)
\end{array}
$$

Here we use the notations: $x(F) = \sum_{e \in F} x_e$, $F \subseteq E$ and $y(S) = \sum_{i \in S} y_i$, $S \subseteq V$ and for $S \subseteq V$, *the cutset*, denoted by $\delta(S)$, is defined as usually:

$$\delta(S) = \{\ e = (i,j) \in E \mid i \in S,\ j \notin S\}.$$

In this formulation, the objective function clearly describes the cost of an optimal generalized tour. Constraints (1) guarantee that from every cluster we select exactly one node, constraints (2) are *degree constraints*: they specify that if a node of G is selected than it is left and entered exactly once. Constraints (3) are *subtour elimination constraints*: they prohibit the formation of subtours, i.e. tours on subsets of less than n nodes. Because of the degree constraints, subtours over one node (and hence, over $n-1$ nodes) cannot occur. Therefore, it is valid to define constraints (3) for $2 \le |S| \le n-2$ only. Finally, constraints (4) and (5) impose binary conditions on the variables.

This formulation, introduced by Fischetti *et al.*[2], is called the *generalized subtour elimination formulation* since constraints (3) eliminate all the cycles on subsets of less than n nodes.

Since in the GTSP exactly one node from each cluster must be visited, we can drop the intra-cluster edges. In view of this reduction constraints (2) are equivalent to

$x(\delta(V_k)) = 2,\ \forall\, k \in K = \{1, ..., m\}$,
and constraints (3) are equivalent to
$x(E(S)) \leq r-1, \forall\, S = \cup_{j=1}^{r} V_j$ and $2 \leq r \leq m-2$.
We may replace the subtour elimination constraints (3) by connectivity constraints, resulting *generalized cutest formulation*, as well introduced in[2]:

$$\begin{array}{lll} \min & \sum_{e\in E} c_e x_e & \\ s.\ t. & (1), (2), (4), (5) & \text{and} \\ (6) & x(\delta(S)) \geq 2(y_i + y_j - 1), & \forall S \subset V, \\ & & 2 \leq |S| \leq n-2, \\ & & i \in S,\ j \notin S. \end{array}$$

Both formulations that we have described so far have an exponential number of constraints. The formulations that we are going to consider next will have only a polynomial number of constraints but an additional number of variables.

Flow based formulations: In order to give compact formulations of the GTSP one possibility is to introduce 'auxiliary' flow variables beyond the natural binary edge and node variables.

We wish to send a flow between the nodes of the network and view an edge variable x_e as indicating whether the edge $e \in E$ is able to carry any flow or not. We consider three such flow formulations: a single commodity model, a multicommodity model and a bidirectional flow model. In each of these models, although the edges are undirected, the flow variables will be directed. That is, for each edge $(i, j) \in E$, we will have flow in the both directions i to j and j to i.

In the *single commodity model*, the source cluster V_1 sends one unit of flow to every other cluster. Let f_{ij} denote the flow on edge $e = (i, j)$ in the direction i to j. This leads to the following formulation:

$$\begin{array}{lll} \min & \sum_{e\in E} c_e x_e & \\ s.t. & y(V_k) = 1, & \forall k \in K = \{1, ..., m\} \\ & x(\delta(v)) = y_v, & \forall v \in V \\ (7) & \Delta\left(f_e\right) = \mu & \\ (8) & f_{ij}, f_{ji} \leq (m-1)x_e, & \forall\, e = (i, j) \in E \\ (9) & f_{ij}, f_{ji} \geq 0, & \forall e = (i, j) \in E \\ & x, y \in \{0, 1\}, & \end{array}$$

where $\Delta\left(f_e\right) = \sum_{e\in\delta^+(i)} f_e - \sum_{e\in\delta^-(i)} f_e$ and

$\mu = \begin{cases} (m-1)y_i & \text{for } i \in V_1 \\ -y_i & \text{for } i \in V\backslash V_1 \end{cases}$. In this model, constraints (7) restrict $m-1$ units of a single commodity flow into cluster V_1 and 1 unit of flow out of each of the other clusters. These constraints are called *mass balance* equations and imply that the network defined by any solution (x, y) must be connected. Since the constraints (1) and (2) state that the network defined by any solution one node from every cluster and satisfy the degree constraints, every feasible solution must be a generalized Hamiltonian tour. Therefore, when projected into the space of the (x, y) variables, this formulation correctly models the GTSP.

We let P_{flow} denote the projection of the feasible set of the linear programming relaxation of this model into the (x, y)-space.

A stronger relaxation is obtained by considering multicommodity flows. In this model every node set $k \in K_1 = \{2, ..., m\}$ defines a commodity. One unit of commodity k originates from V_1 and must be delivered to node set V_k. Letting f_{ij}^k be the flow of commodity k in arc (i, j) we obtain the following formulation:

$$\begin{array}{lll} \min & \sum_{e\in E} c_e x_e & \\ s.t. & y(V_k) = 1, & \forall\, k \in K = \{1, ..., m\} \\ & x(\delta(v)) = y_v, & \forall\, v \in V \\ (10) & \Delta\left(f_a^k\right) = \mu_k, & k \in K_1 \\ (11) & f_{ij}^k \leq w_{ij}, & a = (i, j) \in A,\ k \in K_1 \\ & w_{ij} + w_{ji} = x_e, & \forall\, e = (i, j) \in E \\ (12) & f_a^k \geq 0, & \forall\, a = (i, j) \in A,\ k \in K_1 \\ & x, y \in \{0, 1\}, & \end{array}$$

where $\mu_k = \begin{cases} y_i & , i \in V_1 \\ -y_i & , i \in V_k \\ 0 & , i \notin V_1 \cup V_k. \end{cases}$ and the binary

variables $w_{ij}, (i, j) \in A$ indicate whether an arc (i, j) is contained in the Hamiltonian tour and the directed

graph $D=(V,A)$ is obtained by replacing each edge $e=(i,j)\in E$ by the opposite arcs (i,j) and (j,i) in A with the same weight as the edge $(i,j)\in E$.
We let P_{mcflow} denote the projection of the feasible set of the linear programming relaxation of this model into the (x,y)-space.

Proposition 1: $P_{mcflow}\subseteq P_{flow}$.
Proof. Let $(w,x,y,f)\in P_{mcflow}$, then

$$0\leq \sum_{k\in K_1} f_{ij}^k \leq |K_1|w_{ij}\leq (m-1)x_e.$$

with $f_{ij}=\sum_{k\in K_1} f_{ij}^k$ for every $e=(i,j)\in E$, we find

$$\Delta\left(f_a^k\right)=\sum_{k\in K_1}\Big(\sum_{a\in\delta^+(i)} f_a^k-\sum_{a\in\delta^-(i)} f_a^k\Big)=\mu.$$

We obtain a closely related formulation by eliminating the variables w_{ij}. The resulting formulation consists of constraints (1), (2), (4), (5), (10), (12) plus (13)

$$f_{ij}^h+f_{ij}^k\leq x_e, \forall h,k\in K_1 \text{ and } \forall e\in E.$$

We refer to this model as the *bidirectional flow formulation* of the GTSP and let P_{bdflow} denote its set of feasible solutions in (x,y)-space.

In the bidirectional flow formulation, constraints (13) which are called the *bidirectional flow inequalities*, link the flow of different commodities flowing in different directions on the edge (i,j). These constraints model the following fact: in any feasible generalized spanning tree, if we eliminate edge (i,j) and divide the nodes in two sets; any commodity whose associated node lies in the same set as the root node set does not flow on edge (i,j); any two commodities whose associated nodes both lie in the set without the root both flow on edge (i,j) in the same direction. So, whenever two commodities h and k both flow on edge (i,j), they both flow in the same direction and so one of f_{ij}^h and f_{ij}^k equals zero.
Proposition 2: $P_{mcflow}=P_{bdflow}$.
Proof. If $(w,x,y,f)\in P_{mcflow}$, using (11) we have that

$$f_{ij}^h+f_{ji}^k\leq w_{ij}+w_{ji}=x_e,$$

for all $e=(i,j)\in E$ and for all $h,k\in K_1$.

On the other hand, assume that $(x,y,f)\in P_{bdflow}$. By (13)

$$\max_h f_{ij}^h+\max_k f_{ji}^k\leq x_e,\ \ \forall e=(i,j)\in E.$$

Hence we can choose w such that $\max_h f_{ij}^h\leq w_{ij}$ and $x_e=w_{ij}+w_{ji}$ for all $e=(i,j)\in E$. For example take:

$$w_{ij}=\frac{1}{2}(x_e+\max_h f_{ij}^h-\max_k f_{ji}^k).$$

Clearly, $(w,x,y,f)\in P_{mcflow}$.

Local-global formulation: Our last model arises from distinguishing between *global* variables, i.e. variables modelling the inter-cluster (global) connections, and *local* ones, i.e. expressing wether an edge is selected between two clusters linked in the global graph G'.

Recall the construction of graph G' obtained by shrinking each cluster of G into a single node**.**

We introduce variables z_{ij} $(i,j\in\{1,...,m\})$ to describe the inter-cluster (global) connections. Hence $z_{ij}=1$ if cluster V_i is connected to cluster V_j and $z_{ij}=0$ otherwise. We assume that z represents a Hamiltonian tour. The convex hull of all these y-vectors is generally known as the Hamiltonian tour polytope on the global graph G'.

Following Miller *et al.*[10] this polytope, denoted by P_{TSP}, can be represented by the following polynomial number of constraints:

$$\sum_{\substack{j=1,\\ j\neq i}}^{m} z_{ij}=1,\quad i=\overline{1,m}$$

$$\sum_{\substack{i=1,\\ i\neq j}}^{m} z_{ij}=1,\quad j=\overline{1,m}$$

$$u_i-u_j+(m-1)z_{ij}\leq m-2,\quad \forall i,j=\overline{2,m} \text{ and } i\neq j \qquad (14)$$

$$1\leq u_i\leq m-1,\quad \forall i=\overline{2,m}, \qquad (15)$$

where the extra variables u_i represent the sequence in which city i is visited, $i\neq 1$.

The constraints (14) ensure that the solution contains no subtour on a set of nodes S with $|S|\leq m-1$ and hence, no subtour involving less than m nodes. Constraints (15) ensure that the u_i variables are uniquely defined for any feasible tour.

If the vector y describes a Hamiltonian tour on the global graph G', the corresponding best (w.r.t. cost minimization) *generalized Hamiltonian tour* $(x,y)\in\{0,1\}^{|E|\times|V|}$ can be obtained either by determining all the shortest paths from each $v \in V_{k_1}$ to each $w' \in V_{k_1}$ with the property that visits exactly one node from clusters $(V_{k_2}, ..., V_{k_m})$, or by solving the following 0-1 programming problem:

$$\begin{array}{lll} \min & \sum_{e \in E} c_e x_e & \\ s.t. & y(V_k) = 1, & \forall k \in K = \{1,...,m\} \\ & x(V_l, V_r) = z_{lr}, & \forall l, r \in K = \{1,...,m\}, \\ & & l \neq r \\ & x(i, V_r) \leq y_i, & \forall r \in K, \forall i \in V \setminus V_r \\ & x_e, z_i \in \{0,1\}, & \forall e = (i,j) \in E, \forall\, i \in V, \end{array}$$

where $x(V_l, V_r) = \sum_{\substack{i \in V_l, \\ j \in V_r}} x_{ij}$ and $x(i, V_r) = \sum_{j \in V_r} x_{ij}$.

For given z, we denote the feasible set of the linear programming relaxation of this program by $P_{local}(z)$.

Pop *et al*[11]. proved that if z is the 0-1 incidence vector of a spanning tree of the contracted graph then the polyhedron $P_{local}(y)$ is integral. But as we are going to show in the next example a similar result doesn't hold when z is the incidence vector of the Hamiltonian tour, namely if z is the 0-1 incidence vector of a Hamiltonian tour on the contracted graph then $P_{local}(z)$ may not be integral.

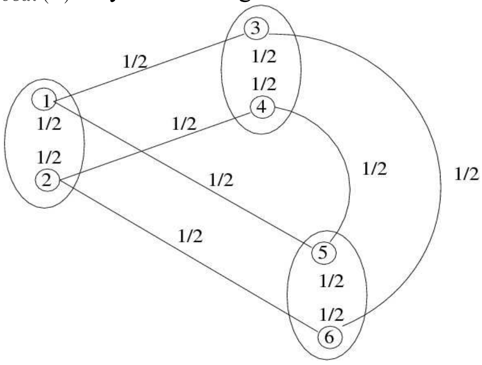

Fig. 1: Example showing that $P_{local}(z)$ may have fractional extreme points if z is the is the 0-1 incidence vector of a Hamiltonian tour on the contracted graph

If the lines drawn in the Figure 1 (i.e., $\{1,3\}$, $\{2,4\}$ etc.) have cost 1 and all the other lines (i.e., $\{1,4\}$, $\{2,3\}$ etc.) have cost $M \gg 1$, then $y \equiv \frac{1}{2}$ and $x \equiv \frac{1}{2}$ on the drawn lines is an optimal solution of $P_{local}(z)$, showing that the polyhedron $P_{local}(z)$ is not integral.

The observations presented so far lead to our final formulation, called the *local-global formulation* of the GTSP as an 0-1 mixed integer programming problem:

$$\begin{array}{llll} (P) & \min & \sum_{e \in E} c_e x_e & \\ & s.t. & z \in P_{TSP} & \\ & & (x,y) \in P_{local}(z) & \\ & & z_{lr} \in \{0,1\}, & 1 \leq l, r \leq m. \end{array}$$

This new formulation of the GTSP was obtained by incorporating the constraints characterizing P_{TSP}, with $z \in \{0,1\}$, into $P_{local}(z)$.

CONCLUSION

In this paper we present six distinct formulations of the Generalized Travelling Salesman Problem as an integer programming. Apart from the standard formulations all the new formulations that we describe are compact in the sense that the number of constraints and variables is a polynomial function of the number of nodes in the problem. Comparisons of the polytopes corresponding to their linear relaxations are established.

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