

Original Research Paper

A Note on “The Homotopy Category is a Homotopy Category”

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Abstract: In his paper with the title, “The Homotopy Category is a Homotopy Category”, Arne Strøm shows that the category Top of topological spaces satisfies the axioms of an abstract homotopy category in the sense of Quillen. In this study, we show by examples that Quillen’s model structure on Top fails to capture some of the subtleties of classical homotopy theory and also, we show that the whole of classical homotopy theory cannot be retrieved from the axiomatic approach of Quillen. Thus, we show that model category is an incomplete model of classical homotopy theory.

Keywords: Fibration, Cofibrations, Homotopy Category, Weak Cofibrations and Fibrations, Quillen’s Model Structure on Top

Introduction

In his paper “The Homotopy Category is a Homotopy Category” (Strøm, 1972), Arne Strøm’s in-tent is to show that the Homotopy Category $hTop$ of topological spaces is a homotopy category in the sense of Quillen. What he shows (and what he tells us he means) is that if Quillen’s fibrations, cofibrations and weak equivalences are taken to be ordinary fibrations, closed cofibrations and homotopy equivalences in $hTop$, then the objects of Quillen’s homotopy category $H_0(Top)$ has as its objects all topological spaces and as its morphisms all homotopy classes of continuous maps.

Peter May commenting on the importance of the notion of weak fibration in the sense of Dold, says that he does not know if it has a model theoretic role to play (May and Sigurdsson, 2006, Page 62). In view of the stated importance of this notion and the corresponding notion of weak cofibration we subsume his question into the larger one which asks “Does the model structure $H_0(Top)$ on Top faithfully reflect the richness of the structure of $hTop$?” Or putting this in a slightly different way, is the homotopy category $H_0(Top)$ the same as the classical homotopy category $hTop$?

We answer May’s question in the negative, showing the notion of weak fibration does not satisfy one of the fundamental axioms of model category theory. We also answer the larger question in the negative by showing that many of the classical results in $hTop$ cannot be proved within the framework of the model structure on Top . So apart from the fact that the model structure excludes the whole of weak fibration

and weak cofibration theory, the other main hurdle, which is present even when we consider the stronger notions of Hurewicz fibrations and closed Hurewicz cofibrations, has to do with duality. In particular in any model category and in particular in $H_0(Top)$, any statement involving fibrations and cofibrations that is provable from the axioms of a model category has a valid automatic dual which, moreover is automatically provable by the dual proof. We show this is not the case with respect to some of the classical results on fibrations and cofibrations in $hTop$ when we consider the corresponding duality principle known as the Eckmann-Hilton duality in classical homotopy theory.

In this study we point out two things that are missing from $hTop$ if we only look at its model structure. The first has to do with duality. It is only possible, using the model structure to replicate results for which both the result itself and the proof are self dual and that is only possible as long as the result is provable from the axioms of a model category. We exhibit results from classical homotopy theory whose duals are not true and hence cannot be proved within the framework of a model category. We show in particular that two of Strøm’s theorems, the pullback theorem and the cancellation theorem (see examples 4.4 and 4.6) are not dualizable and so cannot be proved within the model category structure on $hTop$. We also provide examples of results from classical homotopy theory where the duals are true but require separate proofs that are not self dual. We provide examples of both cases in section four. The second point we would like to mention is the exclusion of weak fibrations and

weak cofibrations from model category. We show that the category Top fails to admit a Quillen model structure with respect to weak cofibrations and weak fibrations and so consequently we would loose the full generality of the gluing and cgluing theorems which have been proved for weak fibrations and cofibrations. Because of this and some other things that we discuss in this study, there is a real sense in which the homotopy category is not a homotopy category. What we are saying is that we cannot retrieve the whole of classical homotopy theory from Quillen’s model category theory and that there are limitations to the theory.

This paper is divided into four main parts. The first part, section 2, sets forth the model category definitions and other topological definitions that are pertinent to the latter sections. Section 3 constitutes the second part and here we give an example of a weak fibration that does not have the homotopy covering property. That is, axiom MC4 of Quillen is not applicable to weak fibrations. Here is where we answer May’s question in the negative. Section 4 constitutes the third part and here we give examples from classical homotopy theory whose duals are not true in general and hence cannot be proved in a model categorical framework as well as examples of results whose duals are true but the proofs are not self dual.

The author would like to express his sincere thanks to Phil Heath for his helpful suggestions and comments pertaining to this paper.

Preliminaries

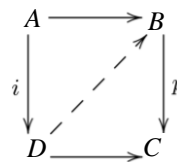
The definition below is originally due to Quillen (1967). For a gentle introduction to the subject see (Dwyer and Spaliński, 1995). Other useful sources of model categories are (Hirschhorn, 2003; Hovey 1999; May and Ponto, 2012).

Definition 2.1

A Model Category is a category with three distinguished classes of maps called weak equivalences, denoted by, fibrations and cofibrations each of which is closed under composition and contains all identity maps.

A map which is a fibration and a weak equivalence is called an acyclic fibration. A map which is a cofibration and a weak equivalence is called an acyclic cofibration. We require the following axioms:

- MC1 (Limit Axiom): Finite limits and colimits exist.
- MC2 (Two out of three axiom): If f and g are maps in M such that gf is defined and two of f , g and gf are weak equivalences, so is the third.
- MC3 (Retract Axiom): If f and g are maps in M such that f is a retract of g and g is a weak equivalence, a fibration, or a cofibration, then so is f .
- MC4 (Lifting Axiom): Given the commutative solid arrow diagram in M as shown below:

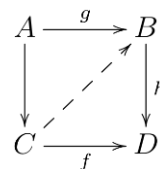


the dotted arrow exists if either i is a cofibration and p is an acyclic fibration or, i is an acyclic cofibration and p is a fibration.

- MC5 (Factorization Axiom): Any map f can be factored in two ways: $f = pi$, i is a cofibration and p is an acyclic fibration and $f = pi$, i is an acyclic cofibration and p is a fibration.

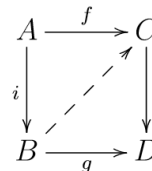
Definition 2.2: Let M be a Model category

- a. The Model category M will be called right proper if every pullback of a weak equiv-alence along a fibration is a weak equivalence. That is, if for each pullback square in M of the form:



in which h is a fibration and f is a weak equivalence, then the morphism g is a weak equivalence

- b. The Model category M will be called left proper if every pushout of a weak equivalence along a cofibration is a weak equivalence. That is, if for each pushout square in M of the form:



in which i is a cofibration and f is a weak equivalence, the morphism g is a weak equivalence.

We will also be needing the dual notions of pushouts and pullbacks in the category of topological spaces and continuous maps. We will denote the pullback of $X \xrightarrow{f} B \xleftarrow{p} E$ by $X \amalg E = \{(x,e) | f(x) = p(e)\}$ as the subspace of $X \times E$

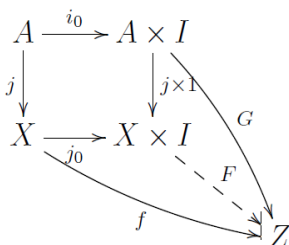
Dually, the pushout of the diagram $X \xleftarrow{f} A \xrightarrow{g} Y$ in Top, denoted by $X_f \amalg_g Y$ is the quotient set $X \amalg Y / \sim$ where \sim is the equivalence relation generated by $f(a) \sim g(a)$ for all $a \in A$. The topology on $X_f \amalg_g Y$ is

the identification topology with respect to $X \amalg Y \rightarrow X_f \amalg_g Y$.

The dual notions of cofibrations and fibrations are central in classical homotopy theory (Piccinini, 1992, Chapter 2) for definitions and properties of cofibrations and fibrations). We give the definitions below.

Definition 2.3

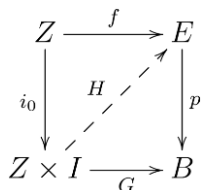
A Hurewicz cofibration(also known as h-cofibration) is a map $j: A \rightarrow X$ such that for any map $f: X \rightarrow Z$ (Z arbitrary) and any homotopy $G: A \times I \rightarrow Z$ such that $G(a, 0) = fj(a)$ for all $a \in A$, there exists a non-unique map $F: X \times I \rightarrow Z$ such that $F(j \times 1) = G$ and $F(x, 0) = f(x)$. That is, there exists a map f represented by the dotted arrow making the following diagram commutative.



We say a Hurewicz cofibration $j: A \rightarrow X$ is a closed cofibration if $j(A)$ is closed in X . However, in the category of compactly generated Hausdorff spaces, there is no need in defining “closed” cofibration since closure is automatic.

Definition 2.4

A map $p: E \rightarrow B$ is said to be a Hurewicz fibration(h-fibration in short), if for all topological spaces Z and every map $f: Z \rightarrow E$ and homotopy $G: Z \times I \rightarrow B$ of pf , there is a homotopy $H: Z \times I \rightarrow E$ with $H(-, 0) = f$ and $pH = G$. That is, for every commutative diagram below, where $i_0(z) = (z, 0)$:



there exists a map $H: Z \times I \rightarrow E$ such that the resulting triangles commute. That is, $pH = G$ and $Hi_0 = f$.

Definition 2.5

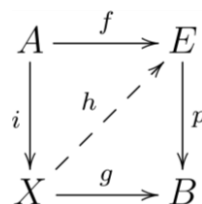
A map $f: X \rightarrow Y$ is a weak equivalence(or weak homotopy equivalence) if f induces a bijection $[K, X] \rightarrow [K, Y]$ for all CW complexes K . (Note: $[K, X]$

denotes the homotopy class of all continuous functions $K \rightarrow X$).

The notion of a weak(Dold) fibration plays an important role in the theory of fibrations. We first discuss the notion of Weak Right Lifting Property (WRLP) which is a prelude to the definition of a Dold fibration.

Definition 2.6

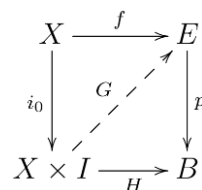
A map $p: E \rightarrow B$ has the Weak Right Lifting Property (WRLP) with respect to a map $i: A \rightarrow X$, if for every commutative square



in Top there exists a map $h: X \rightarrow E$ such that $p \circ h = g$ and $h \circ i \simeq_p f$ (i.e., there exists a homotopy H from $h \circ i$ to f such that $pH(a, t) = p h(a, 0)$ for all $(a, t) \in A \times I$. In short a fiber homotopy from $h \circ i$ to f).

Definition 2.7

A continuous function $p: E \rightarrow B$ is a weak(Dold) fibration if p has the Weak Covering Homotopy Property (WCHP) for all topological spaces X . That is, if for every commutative diagram, that is, $pf = Hi_0$:



there exists a homotopy $G: X \times I \rightarrow E$ such that $p \circ G = H$ and $G \circ i_0 \simeq_p f$ (Dieck *et al.*, 1970 for further details).

Definition 2.8

A continuous function $i: A \rightarrow X$ is called a weak cofibration if i has the Homotopy Extension Property (HEP) up to homotopy for all topological spaces Y . That is, for all continuous functions $f: X \rightarrow Y$ and all homotopies $G: A \times I \rightarrow Y$ with $G_0 = fi$, there exists a homotopy $H: X \times I \rightarrow Y$ with:

1. $H(i \times id_I) = G$ and
2. $H_0 \simeq f$ (Dieck *et al.*, 1970, 2.2) for further details

Let \mathbf{G} denote the class of weak fibrations, \mathbf{C} denote the class of closed cofibrations, \mathbf{D} the class of weak

cofibrations and \mathbf{W} the class of homotopy equivalences.

The following characterization of weak fibrations in terms of lifting maps is due to Kieboom.

Theorem 2.1

For a map $p: E \rightarrow B$, the following are equivalent:

- (i) $p \in \mathbf{G}$.
- (ii) p has the WRLP with respect all $i \in \mathbf{D} \cap \mathbf{W}$
- (iii) p has the WRLP with respect all $i \in \mathbf{C} \cap \mathbf{W}$
- (iv) p has the WRLP with respect all $i \in \mathbf{W}$

Proof: (Kieboom, 1987b) (Theorem 1)

Theorem 2.2

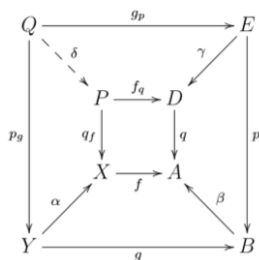
For a map $p: E \rightarrow B$, the following are equivalent:

- (i) $p \in \mathbf{G} \cap \mathbf{W}$
- (ii) p has the WRLP with respect to all $i \in \mathbf{C}$
- (iii) p has the WRLP with respect to all maps i
- (iv) p has the WRLP with respect to all $i \in \mathbf{D}$

Proof: (Kieboom, 1987b) Theorem 2.

Concerns about Lack of Generality

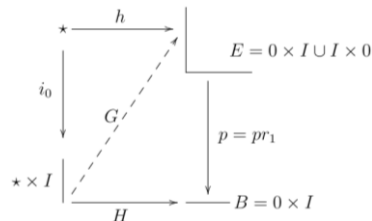
In order to illustrate the main point of this section, we recall a theorem of another paper by the author (Solomon, 2007 Theorem 1.1) see also (May and Ponto, 2012, Proposition 15.4) for an alternate proof. That is, consider the following diagram in a model category \mathcal{M} .



where, $QEYB$ and $PDXA$ are respectively the pullbacks of the front and back faces of the diagram. Suppose that p and q are fibrations and α, β, γ are weak equivalences, where A, X, B and Y are fibrant, then δ is a weak equivalence. We will show that the more general theorem where fibrations are replaced by weak fibrations cannot be proved using the model structure. The reason of course is that the lifting axiom of Quillen, ($MC4$) doesn't work for weak fibrations. A counterexample is in order.

Example(3.1)

Let $W = \{\star\}$ be a one point space and consider the following diagram:



where, $h: \{\star\} \rightarrow E$ is defined by $h(\{\star\}) = (0, 1)$, $H(\{\star\}, t) = (0, t)$ and i_0 is the canonical inclusion.

We claim that the above diagram doesn't admit a continuous lifting $G: \{\star\} \times I \rightarrow E$ such that $pG = H$ and $Gi_0 = h$.

Consider $G^{-1}(0 \times (0, 1]) \subseteq \star \times I$ where $0 \times (0, 1]$ is open in E :

$$\begin{aligned} (\star, t) \in G^{-1}(0 \times (0, 1]) &\Rightarrow G(\star, t) \in 0 \times (0, 1] \\ &\Rightarrow pG(\star, t) \in p(0 \times (0, 1]) \\ &\Rightarrow H(\star, t) \in pr_1(0 \times (0, 1]) \\ &\Rightarrow t = 0 \end{aligned}$$

Hence, $G^{-1}(0 \times (0, 1]) \subseteq (\star, 0)$ and $G(\star, 0) = Gi_0(\star) = h(\star) = (0, 1) \in 0 \times (0, 1] \Rightarrow (\star, 0) \in G^{-1}(0 \times (0, 1])$. Therefore, $G^{-1}(0 \times (0, 1]) = (\star, 0)$ is not open in $\star \times I$ and so G is not continuous.

What we have actually shown is that p is not a fibration. However, it is well known that p is a weak fibration (Piccinini, 1992, Exercise 2.2.9), (Dieck *et al.*, 1970 Example 6.2) and so axiom $MC4$, that is, Quillen's Lifting axiom is no longer true when fibration is replaced by weak fibration. In fact, if there were any model structure on Top with weak fibrations as fibrations and weak equivalences as homotopy equivalences and cofibrations as closed cofibrations, then our counterexample shows that the map $\star \rightarrow I$ would not be a cofibration. Hence, we conclude that the cogluing theorem of (Brown and Heath, 1970) which has been proved under the weaker conditions that appropriate maps are weak-fibrations and the gluing theorem for homotopy equivalences (Brown, 2006, Theorem 5.5.7 which has been generalized to weak cofibrations (Kamps, 1972, Satz 8.2) cannot be proved within the model category structure in $hTop$. Furthermore, our counterexample shows that weak fibrations and weak cofibrations do not satisfy any of the equivalent conditions of Theorem 2.1 and Theorem 2.2 given by Kieboom and so the notions of weak fibrations and weak cofibrations fail to admit a model structure on $hTop$ thus answering negatively the question raised by Peter May on the relevance of weak fibrations and cofibrations to model category theory.

Concerns About Duality

We now present some examples of well known dual theorems in classical homotopy theory which do not admit dual proofs as well as theorems in classical

homotopy theory whose duals are not true in general.

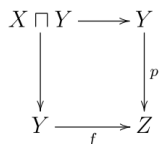
Example(4.1)

X is an H-space if and only if the canonical map $e: X \rightarrow \Omega \Sigma X$ (the adjoint of $id_{\Sigma X}$ admits a left homotopy inverse (James, 1955). Dually, Y is a co-H-space if and only if the canonical map $\pi: \Sigma \Omega Y \rightarrow Y$ (the adjoint of $id_{\Omega Y}$ admits a right homotopy inverse (Ganea, 1970). According to Roitberg (2000), no known proof of either theorem dualizes to a proof of the other.

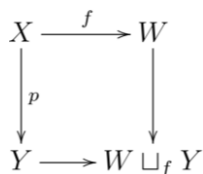
Example(4.2)

If $p: Y \rightarrow Z$ is a fibration and $f: X \rightarrow Z$ is a weak homotopy equivalence [Definition 2.5], then $X \amalg Y \rightarrow Y$ is also a weak homotopy equivalence. (Note: $X \amalg Y$ is the pullback of the diagram $X \xrightarrow{f} Z \xleftarrow{p} Y$) (Munson and Volić, 2015, Proposition 2.1.23).

That is, in the following pullback diagram.



$X \amalg Y \rightarrow Y$ is a weak homotopy equivalence. Dually, If $p: X \rightarrow Y$ is a cofibration and $f: X \rightarrow W$ is a weak homotopy equivalence, then $Y \rightarrow \amalg_f Y$ is also a weak homotopy equivalence. (Note: $W \amalg_f Y$ is the pushout of the diagram $Y \xrightarrow{p} X \xleftarrow{f} W$) (Munson and Volić, 2015, Proposition 2.2.23] That is, in the following pushout diagram.



$Y \rightarrow W \amalg_f Y$ is a weak homotopy equivalence. As explained by Munsen and Volić, (2015, Proposition 2.3.19) neither of the proofs dualizes.

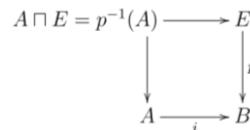
Example(4.3)

Let $i: A \rightarrow X$ be a cofibration, then i is an injection (Strøm, 1966, Theorem 1) or (Cockroft and Jarvis, 1964). Dually, if $p: E \rightarrow B$ is a fibration, then it is not necessarily true that p is a surjection. A trivial example is the unique map $p: \phi \rightarrow B$ for any set B .

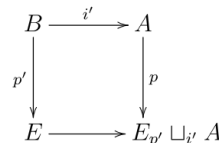
Example(4.4)

Strom’s pullback theorem (Strøm,1968, Theorem 12; Booth, 1974, Corollary 3) states that if $i: A \rightarrow B$ is a closed cofibration and $p: E \rightarrow B$ is a Hurewicz fibration,

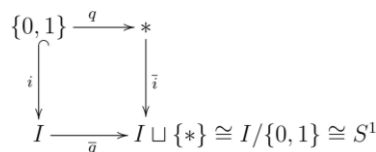
then $p^{-1}(A) \rightarrow E$ is a cofibration. That is, in the following pullback diagram:



$A \amalg E = p^{-1}(A) \rightarrow E$ is a cofibration. For the dual, we consider the following commutative diagram:



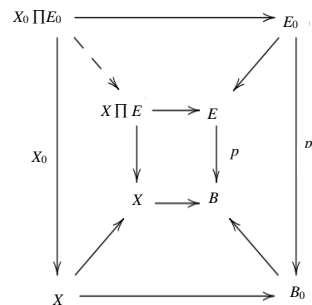
where, i' is a fibration, p' is a cofibration and $E_{p'} \amalg_{i'} A$ is the pushout of the diagram $E \xleftarrow{p'} B \xrightarrow{i'} A$. The conclusion of the dual statement would then require that the map $E \rightarrow E_{p'} \amalg_{i'} A$ is a fibration. As a counterexample we consider the pushout of the diagram $I \hookrightarrow \{0, 1\} \rightarrow *$. That is, the following is a commutative diagram by construction.



where, $q: \{0, 1\} \rightarrow *$ is a fibration and $i: \{0, 1\} \rightarrow I$ is a cofibration. The quotient set $I / \{0, 1\} \cong S^1$ via the map $[t] \mapsto e^{2\pi it}$. However, \bar{q} is not a fibration since the fibers are not of the same homotopy type.

Example(4.5)

(Kieboom’s Pullback Theorem for Cofibrations) (Kieboom, 1987a): Consider the following commutative diagram in Top.

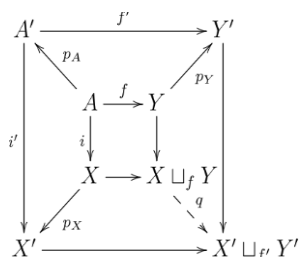


in which:

- (a) The inclusions $B_0 \hookrightarrow B$, $E_0 \hookrightarrow E$ and $X_0 \hookrightarrow X$ are closed cofibrations
- (b) $p: E \rightarrow B$ and $p_0: E_0 \rightarrow B_0$ are fibrations

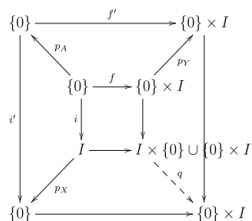
then the inclusion $X_0 \sqcup E_0 \hookrightarrow X \sqcup E$ is also a closed cofibration. (Note: $X \sqcup E$ denotes the pullback of $X \rightarrow B \leftarrow E$).

By reversing the arrows and interchanging cofibrations and fibrations we can consider the following situation:



where, $p_A: A \rightarrow A'$, $p_X: X \rightarrow X'$ and $p_Y: Y \rightarrow Y'$ are fibrations and $i: A \rightarrow X$ and $i': A' \rightarrow X'$ are cofibrations. The expected conclusion is then that $q: X \sqcup_f Y \rightarrow X' \sqcup_{f'} Y'$ is a fibration which is not true by the following example given by Kieboom (1987a, example).

Let $A = A' = X' = 0$, $X = I$, $Y = Y' = 0 \times I$. Let $f = f': 0 \rightarrow (0, 0)$ and let $p_y = 1_Y$. The adjunction spaces $X \sqcup_f Y$ and $X' \sqcup_{f'} Y'$ can be identified with the subspaces $I \times \{0\} \cup \{0\} \times I$ and $\{0\} \times I$ of $I \times I$ respectively. The map $q: X \sqcup_f Y \rightarrow X' \sqcup_{f'} Y'$ is defined by $q(s, 0) = (0, 0)$ and $q(0, t) = (0, t)$ for all $s, t \in I$:



As shown in Example 3.1, the induced map $q: I \times 0 \cup 0 \times I \rightarrow \{0\} \times I$ is not a fibration and so the theorem cannot be dualized.

Example(4.6)

(A. Strøm's Cancellation Theorem). Strøm (1972, lemma 5), A. Strøm proved the following theorem: (See diagram below):



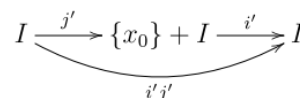
If $i': B \rightarrow A$ and $i: A \rightarrow X$ are maps such that i and ij are cofibrations, then j is also a cofibration. The dual will

then be(see diagram below):



If $i'j'$ is a fibration and $j': X \rightarrow A$ is a fibration, then $i': A \rightarrow B$ is a fibration. Once again, we will provide a counterexample to show that the conclusion is false.

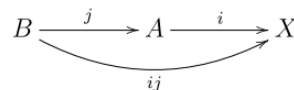
Consider the following diagram:



where, $\{x_0\}$ is a one point space, $\{x_0\} + I$ denotes the topological sum and $j': I \rightarrow \{x_0\} + I$ is defined by $j'(t) = t$ for all $t \in I$ and $i': \{x_0\} + I \rightarrow I$ is defined by $i'(x_0) = 0$ and $i'(t) = t$ for $t \in I$. Now $i'j' = 1_I: I \rightarrow I$ and $j': I \rightarrow \{x_0\} + I$ are fibrations, whereas $i': \{x_0\} + I \rightarrow I$ is not a fibration see (Dieck *et al.*, 1970, Example 5.17).

Remark

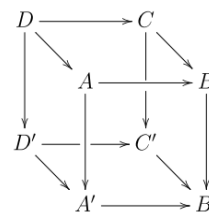
If in the diagram:



we assume ij is a cofibration and i is an acyclic cofibration where all spaces are assumed to be fibrant, then it is easy to prove that j is a cofibration in the model categorical sense. We leave the proof as an easy exercise to the reader. In (Riehl, 2008, Page 7) Emily Riehl remarks that if ij is a cofibration and i is split monic, then j is a cofibration.

Example(4.7)

Consider the cube diagram in which all faces are homotopy commutative squares:



If the four vertical faces are homotopy pullback squares and the bottom square is a homotopy pushout square, then the top square is a homotopy pushout square. This theorem has been proved by Mather (1976, Theorem 25). Doeraene (1998, page 22) has shown that the dual of this result is false.

Finally we would like to point out that in a specific model category, a statement with a non-model categorical proof may hold without its dual statement holding. That is, a model category can be left proper (the class of weak equivalences is closed under cobase change along cofibrations) but not right proper (the class of weak equivalences is closed under base change along fibrations) or vice versa (see definition 2.2). The paper by Rezk (2002, Examples 2.10-2.12) gives examples of model categories that are right proper but not left proper and remarks that failure of left properness is a "generic" property of certain examples of model categories. In this study, we have focused on the general theory of model categories which is self dual rather than the examples of a model category which might fail to be self dual. This was after all what Strøm was working with.

Ethics

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and no ethical issues involved.

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