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The Dual Exponentiated Weibull Model

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Abstract: A new lifetime model with various shapes of the hazard rate function for modeling uni-modal and bimodal data sets is introduced and studied along with its statistical properties. Before using the maximum likelihood method for estimating the unknown model parameters, we assessed its performance via a simulation study. The flexibility of the new model is illustrated via plots of the probability and hazard rate functions for three real data applications.

Keywords: Exponentiated Weibull, Order Statistics, Quantile Function, Moments, Moments Generating Function, Maximum Likelihood Estimation, Simulation

Introduction

Many extensions of the Weibull (W) distribution are presented in the literature. Among these models, the Exponentiated Weibull (EW) is the most popular one (Mudholkar *et al*., 1993). A Random Variable (RV) T is said to have the EW distribution if its Probability Density Function (PDF) and Cumulative Distribution Function (CDF) are given by:

$$\pi_{\alpha,\beta}(t)=\alpha\beta t^{\beta-1}\exp\left(-t^{\beta}\right)\left[1-\exp\left(-t^{\beta}\right)\right]^{\alpha-1} \quad (1)$$

and:

$$\prod{}_{\alpha,\beta}(t)=\left[1-\exp\left(-t^{\beta}\right)\right]^{\alpha}, \quad (2)$$

respectively, for $t>0$, $\alpha>0$, and $\beta>0$. For $\alpha = 1$, we have the standard one parameter Weibull model (Weibull (1951). For $\beta = 1$; we have the one parameter exponentiated exponential model (Gupta and Kundu (1999). For $\beta = 2$, we have the standard one parameter Burr type X distribution. Cordeiro *et al*. (2017) suggested a new flexible family called the Exponentiated Weibull-H (EW-H) family, the CDF of the EW-H family is given by:

$$F_{\theta,\lambda}(x)=\left\{1-\exp\left[-\left(\frac{\Pi(x)}{\overline{\Pi}(x)}\right)^{\lambda}\right]\right\}^{\theta}, \quad (3)$$

where, $\lambda>0$ and $\theta>0$ are two shape parameters, $\Pi(x)$ is the CDF of the base line model. The PDF of the EW-H class corresponding to (3) reduces to:

$$f_{\theta,\lambda}(x)=\lambda\theta\frac{\overline{\Pi}(x)^{-\lambda-1}}{\Pi(x)^{-\lambda+1}}\exp\left[-\left(\frac{\Pi(x)}{\overline{\Pi}(x)}\right)^{\lambda}\right] \times\pi(x)\left\{1-\exp\left[-\left(\frac{\Pi(x)}{\overline{\Pi}(x)}\right)^{\lambda}\right]\right\}^{\theta-1}, \quad (4)$$

where, $\pi(x)=\frac{d}{dx}\Pi(x)$ is the PDF of the base line model and $\overline{\Pi}(x)=1-\Pi(x)$. Hassan and Elgarhy (2016) introduced a similar work to Cordeiro *et al*. (2017), but we depended on Cordeiro *et al*. (2017) in this work. Using (3) and (2), we construct a model called the Dual Exponentiated Weibull (DEW) with CDF given by:

$$F_{\theta,\lambda,\alpha,\beta}(x)=\left[1-\exp\left(-\left\{\frac{\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}}{1-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}}\right\}^{\lambda}\right)\right]^{\theta} \quad (5)$$

The PDF of the DEW corresponding to (5) reduces to:

$$f_{\theta,\lambda,\alpha,\beta}(x)=\lambda\theta\alpha\beta x^{\beta-1}\exp\left(-x^{\beta}\right)\frac{\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha\lambda-1}}{\left\{1-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}\right\}^{\lambda+1}} \times\exp\left(-\left\{\frac{1-\exp\left(-x^{\beta}\right)^{\alpha}}{1-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}}\right\}^{\lambda}\right) \times\left[1-\exp\left(-\left\{\frac{\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}}{1-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}}\right\}\lambda\right)\right]^{\theta-1}. \quad (6)$$

A Random Variable (RV) X having PDF $f_{\theta,\lambda,\alpha,\beta}(x)$ (6) is denoted by X~DEW($\theta,\lambda,\alpha,\beta$). The Hazard Rate Function (HRF) can be derived from the well known relationship. The PDF and HRF plots of the DEW model are given in Fig. 1 and 2 respectively. The new additional parameters θ and λ will enable us to study the tail behavior of the new density (6) with more flexibility. Some cases of the DEW model are presented in Table 1. In Table 1, the H-model refers to the model with CDF:

$$M(x) = 1 - \exp\left[-\Pi(x)/\bar{\Pi}(x)\right]$$

which is an exponential model of the odds ratio of a continuous RV whose CDF is given by $\Pi(x)$.

Table 1: Some sub models from the DEW model

n	Reduced Model	θ	λ	α	β	CDF	Author
1	W-EW	1	λ	α	β	$1-\exp-\left(\left\{\left[1-\exp\left(-x^{\beta}\right)\right]^{-\alpha}-1\right\}^{-\lambda}\right)$	New
2	EW-W	θ	λ	1	β	$\left(1-\exp\left\{-\left[\exp\left(x^{\beta}\right)\right]-1\right\}^{-\lambda}\right)^{\theta}$	Cordeiro *et al.* (2017)
3	W-W	1	λ	1	β	$1-\exp\left\{-\left[\frac{1-\exp\left(-x^{\beta}\right)}{\exp\left(-x^{\beta}\right)}\right]^{\lambda}\right\}$	Bourguignon *et al.* (2014)
4	BrX-EW	θ	2	α	β	$\left[1-\exp\left(-\left\{\frac{\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}}{1-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}}\right\}^{2}\right)\right]^{\theta}$	Khalil *et al.* (2019)
5	EW-BrX	θ	λ	α	2	$\left[1-\exp\left(-\left\{\frac{\left[1-\exp\left(-x^{2}\right)\right]^{\alpha}}{1-\left[1-\exp\left(-x^{2}\right)\right]^{\alpha}}\right\}^{\lambda}\right)\right]^{\theta}$	New
6	BrX-BrX	θ	2	α	2	$\left[1-\exp\left(-\left\{\frac{\left[1-\exp\left(-x^{2}\right)\right]^{\alpha}}{1-\left[1-\exp\left(-x^{2}\right)\right]^{\alpha}}\right\}^{2}\right)\right]^{\theta}$	New
7	BrX-W	θ	2	1	β	$\left[1-\exp\left(-\left\{\frac{1-\exp\left(-x^{\beta}\right)}{\exp\left(-x^{\beta}\right)}\right\}^{2}\right)\right]^{\theta}$	Yousof *et al.* (2017a)
8	W-BrX	1	λ	α	2	$1-\exp\left(-\left\{\frac{\left[1-\exp\left(-x^{2}\right)\right]^{\alpha}}{1-\left[1-\exp\left(-x^{2}\right)\right]^{\alpha}}\right\}^{\lambda}\right)$	Bourguignon *et al.* (2014)
9	EE-EW	θ	1	α	β	$1-\exp\left(-\frac{\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}}{1-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}}\right)^{\theta}$	New
10	EW-EE	θ	λ	α	1	$1-\exp\left(-\left\{\frac{\left[1-\exp\left(-x\right)\right]^{\alpha}}{1-\left[1-\exp\left(-x\right)\right]^{\alpha}}\right\}^{\lambda}\right)^{\theta}$	New
11	E-W	θ	1	α	β	$\left[1-\exp-\left\{\frac{1-\exp\left(-x^{\beta}\right)}{\exp\left(-x^{\beta}\right)}\right\}\right]^{\theta}$	Mudholkar *et al.* (1993)
12	W-E	1	λ	α	1	$1-\exp\left(-\left\{\frac{\left[1-\exp\left(-x\right)\right]^{\alpha}}{1-\left[1-\exp\left(-x\right)\right]^{\alpha}}\right\}^{\lambda}\right)$	Bourguignon *et al.* (2014)

Table 1: Continue

13	H-EW	1	1	α	β	$1-\exp\left\{\frac{\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}}{1-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}}\right\}$	New
14	EW-H	θ	λ	1	1	$\left(1-\exp\left\{-\left[\frac{1-\exp(-x)}{\exp(-x)}\right]^{\lambda}\right\}\right)^{\theta}$	New
15	H-W	1	1	1	β	$1-\exp\left[-\frac{1-\exp\left(-x^{\beta}\right)}{\exp\left(-x^{\beta}\right)}\right]$	Bourguignon *et al*. (2014)
16	W-H	1	λ	1	1	$1-\exp\left\{-\left[\frac{1-\exp(-x)}{\exp(-x)}\right]^{\lambda}\right\}$	Bourguignon *et al*. (2014)

Linear Representation

A simple linear representation for the DEW density will be provided in this section. If $|q|<1$ and $\alpha>0$ is a real non-integer, the power series holds:

$$(1-q)^{\alpha-1}=\sum_{m=0}^{\infty}\frac{(-q)^{m}\Gamma(\alpha)}{m!\Gamma(\alpha-m)}. \tag{7}$$

Applying (7) to the last term in (6) gives:

$$\begin{aligned} f_{\theta,\lambda,\alpha,\beta}(x) &= \lambda\theta\alpha\beta x^{\beta-1} \\ &\times\frac{\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha\lambda-1}}{\left\{1-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}\right\}^{\lambda+1}} \\ &\times\sum_{i=0}^{\infty}\frac{(-1)^{i}\Gamma(\theta)}{i!\Gamma(-i+\theta)}\exp\left(-x^{\beta}\right) \\ &\underbrace{\times\exp\left(-(1+i)\left\{\frac{\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}}{1-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}}\right\}^{\lambda}\right)}_{A_i}. \end{aligned} \tag{8}$$

Expanding A_i in power series, we have:

$$A_i=\sum_{\zeta=0}^{\infty}\frac{(-1)^{\zeta}(i+1)^{\zeta}}{\zeta!}\frac{\left[1-\exp\left(-x^{\beta}\right)\right]^{\zeta\alpha\lambda}}{\left\{1-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}\right\}^{\zeta\lambda.}}$$

Inserting A_i in (8), the DEW density reduces to:

$$\begin{aligned} f_{\theta,\lambda,\alpha,\beta}(x) &= \alpha\beta x^{\beta-1}\exp\left(-x^{\beta}\right)\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha\lambda(\zeta+1)-1} \\ &\times\sum_{i,\zeta=0}^{\infty}\frac{(-1)^{\zeta+i}\theta\lambda\Gamma(\theta)(i+1)^{\zeta}}{i!\zeta!\Gamma(-i+\theta)}\left\{1-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}\right\}^{-1[1+(\zeta+1)\lambda]}, \end{aligned} \tag{9}$$

Using the generalized binomial expansion:

$$(1-q)^{-a}=\sum_{m=0}^{\infty}\frac{\Gamma(a+m)}{m!\Gamma(a)}q^{m}\,|_{(|q|<1 \text{ and } a>0)}$$

to:

$$\left\{-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}+1\right\}^{-[1+(\zeta+1)\lambda]},$$

we can write:

$$\begin{aligned} &\left\{-\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha}\right\}^{-[1+(\zeta+1)\lambda]} \\ &=\sum_{j=0}^{\infty}\frac{\Gamma\left(\lambda[\zeta+1]+j+1\right)}{j!\Gamma\left(1+\lambda(\zeta+1)\right)}\left[1-\exp\left(-x^{\beta}\right)\right]^{\alpha j}. \end{aligned} \tag{10}$$

Inserting (10) in (9), the DEW density can be expressed as an infinite linear combination of EW density function:

$$f_{\theta,\lambda,\alpha,\beta}(x)=\sum_{\zeta,j=0}^{\infty}\tau_{\zeta,j}\pi_{[(\zeta+1)\lambda+j]\alpha,\beta}(x), \tag{11}$$

where, $\pi_{[(\zeta+1)\lambda+j]\alpha,\beta}(x)$ is the EW with power parameter $[(\zeta+1)\lambda+j]\alpha$ and:

$$\tau_{\zeta,j}=\theta\lambda\sum_{i=0}^{\infty}\frac{(-1)^{\zeta+i}(i+1)^{\zeta}\Gamma\left(\lambda[\zeta+1]+j+1\right)\Gamma(\theta)}{i!\zeta!j\left\{\left[(\zeta+1)\lambda+j\right]\alpha\right\}\Gamma(-i+\theta)\Gamma\left(1+\lambda(\zeta+1)\right)}.$$

Equation (11) reveals that the new PDF of X can be expressed as a linear combination of EW densities. So, several mathematical properties of the new model can be attained by knowing those of the EW distribution. Likewise, the CDF of the DEW density can also be written as:

$$F_{\theta,\lambda,\alpha,\beta}(x)=\sum_{\zeta,j=0}^{\infty}\tau_{\zeta,j}\Pi_{[(\zeta+1)\lambda+j]\alpha,\beta(x),}$$

where, $\Pi_{[(\zeta+1)\lambda+j]\alpha,\beta(x)}$ is the CDF of the EW with power parameter $[(\zeta+1)\lambda+j]\alpha$.

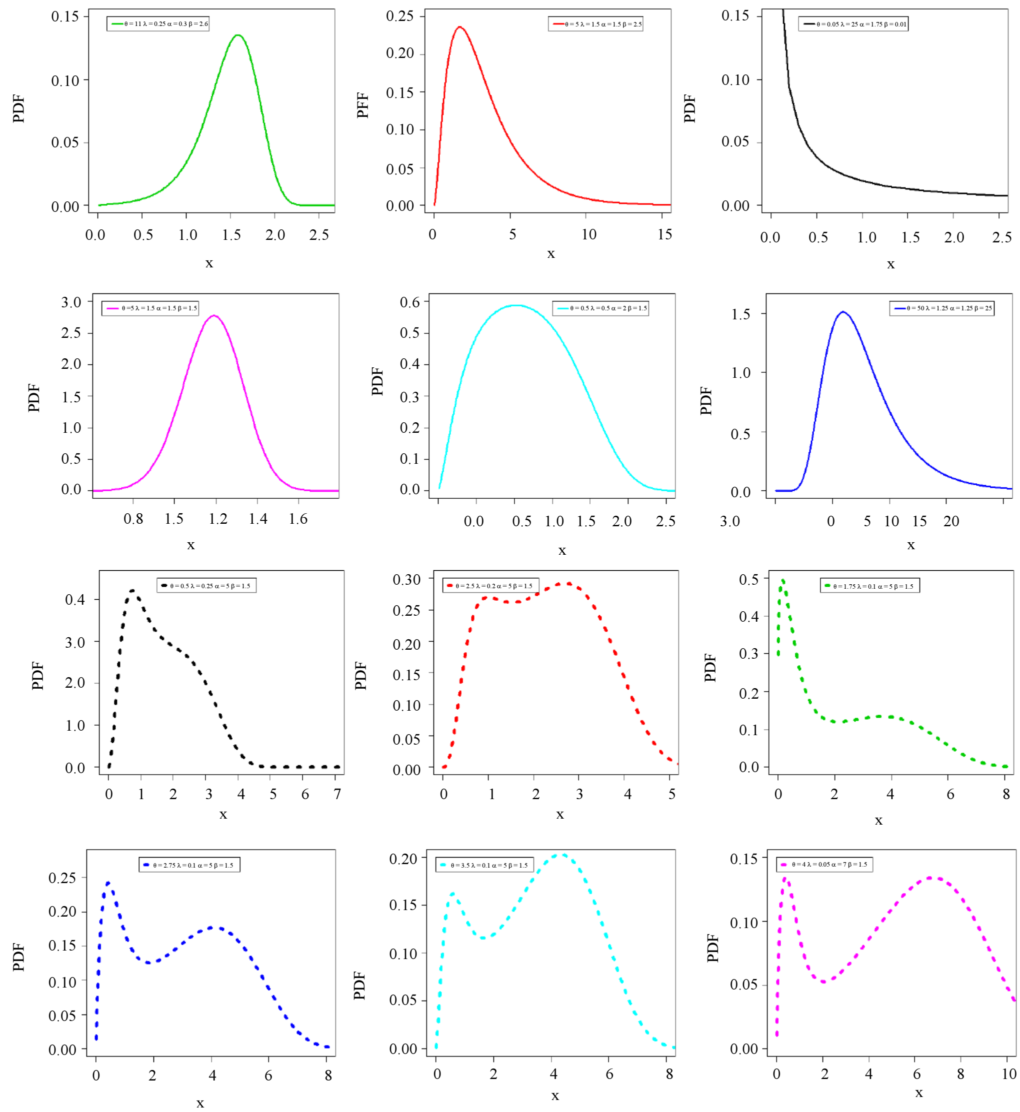


Fig. 1: Plots of the DEW PDF for selected parameter values

Mathematical Properties

Justification and Graphical Presentation

We are motivated to introduce the DEW for the following reasons:

1. The DEW model is very suitable to define special models with different kinds of hazard rates. Fig. 1 displays some plots of the PDF of the DEW for different values of θ, λ, α and β. These plots show that the DEW density can be symmetric or right-skewed or left-skewed or inverted J shape or unimodal or bimodal
2. The HRF plots of the DEW model are given in Fig. 2. The HRF of the DEW model can be increasing or increasing constant or decreasing or upside-down or bathtub, upside-down bathtub or constant or J shaped
3. The DEW model provides the best fit in modeling different types of data sets

Moments

The r^{th} moment of X, say μ'_r, follows from Equation (11) as:

$$\mu'_r = \boldsymbol{E}\left(X^r\right) = \Gamma\left(r\beta^{-1}+1\right)\sum_{\zeta,j,h=0}^{\infty} c_{\zeta,j,h}^{\{r,[(\zeta+1)\lambda+j]\alpha\}}\Big|_{(-\beta<r)}, \qquad (12)$$

where, $c_{\zeta,j,h}^{\{r,[(\zeta+1)\lambda+j]\alpha\}} = \tau_{\zeta,j}c_h^{\{r,[(\zeta+1)\lambda+j]\alpha\}}$ and:

$$c_h^{\{r,[(\zeta+1)\lambda+j]\alpha\}} = \frac{(-1)^h\left[(\zeta+1)\lambda+j\right]\alpha}{(h+1)^{(r\beta^{-1}+1)}}\binom{\left[(\zeta+1)\lambda+j\right]\alpha-1}{h},$$

when $r = 1$ in (12), we get the mean ($\boldsymbol{E}(X)$) of the new model:

$$\boldsymbol{E}(X) = \Gamma\left(\beta^{-1}+1\right)\sum_{\zeta,j,h=0}^{\infty} c_{\zeta,j,h}^{\{1,[(\zeta+1)\lambda+j]\alpha\}}|_{(-\beta<1)}.$$

Some numerical results for the $\boldsymbol{E}(X)$ and other related measures are given in Table 2.

Quantile Function and Moment Generating Function

The Quantile Function (QF) of X is defined by reversing (5). We have:

$$Q(u) = F^{-1}(u) = \left[-In-\left\{\left[-\log\left(-u^{\frac{1}{\theta}}+1\right)\right]^{-\frac{1}{\lambda}}+1\right\}^{\frac{1}{\alpha}}+1\right]^{\frac{1}{\beta}} \quad (13)$$

$,0 < u < 1.$

Next, we provide a formula for the Moment Generating Function; $M_X(t)$ of X can follow from Equation (11) as:

$$M_x(t) = \Gamma\left(r\beta^{-1}+1\right)\sum_{\zeta,j,r,h=0}^{\infty} c_{\zeta,j,h}^{\{r,[(\zeta+1)\lambda+j]\alpha\}}|_{(-\beta<r)},$$

Incomplete Moments

The s^{th} incomplete moment, say $\boldsymbol{I}_s(t)$, of X can be derived from (11) as:

$$\boldsymbol{I}_s(t) = \int_{-\infty}^{t} x^s f(x)dx = \Upsilon\left(s\beta^{-1}+1,\left(\frac{1}{t}\right)^{\beta}\right)$$
$$\sum_{\zeta,j,h=0}^{\infty} c_{\zeta,j,h}^{\{s,[(\zeta+1)\lambda+j]\alpha\}}|_{(-\beta<s)}, \quad (14)$$

where, $\Upsilon(\tau,\upsilon)$ is the incomplete gamma function:

$$\Upsilon(\tau,\upsilon) = \int_0^{\upsilon} t^{\tau-1}\exp(-t)dt = \sum_{h=0}^{\infty}\frac{(-1)^h}{h!(\tau+h)}\upsilon^{\tau+h}.$$

The first incomplete moment $I_1(t)$ is given by (14) with $s = 1$ as:

$$\boldsymbol{I}_1(t) = \Upsilon\left(1+\beta^{-1},\left(\frac{1}{t}\right)^{\beta}\right)\sum_{\zeta,j,h=0}^{\infty} c_{\zeta,j,h}^{\{1,[(\zeta+1)\lambda+j]\alpha\}}$$

Order Statistics

Let $X_1,\ldots,X_n$ be a Random Sample (RS) from the DEW distribution. The PDF of the i^{th} order statistic, say $X_{i:n}$, can be stated as:

$$f_{i:n}(x) = f_{\theta,\lambda,\alpha,\beta}(x)\sum_{j=0}^{n-1}\frac{(-1)^j}{B(i,n+1-i)}\binom{n-i}{j}\left[F_{\theta,\lambda,\alpha,\beta}(x)\right]^{j+i-1}, \quad (15)$$

where, $B(.,.)$ is the beta function. Based on Equations (5) and (6), we have:

$$f_{\theta,\lambda,\alpha,\beta}(x)\left[F_{\theta,\lambda,\alpha,\beta}(x)\right]^{j+i-1} = \sum_{\zeta,m=0}^{\infty} t_{\zeta,m}^{(j+i-1)}\pi_{[(\zeta+1)\lambda+m]\alpha,\beta}(x), \quad (16)$$

where:

$$t_{\zeta,m}^{(j+i-1)} =$$
$$\theta\lambda\sum_{l=0}^{\infty}\frac{(-1)^{l+\zeta}(1+l)^{\zeta}\,\Gamma(\lambda(\zeta+1)+1+m)\Gamma(j\theta+i\theta)}{l!\zeta!m!\Gamma(\lambda(\zeta+1)+1)\Gamma(-l+j\theta+i\theta)\left\{[(\zeta+1)\lambda+m]\alpha\right\}}.$$

Substituting (16) in Equation (15), the PDF of $X_{i:n}$ can be expressed as:

$$f_{i:n}(x) = \sum_{\zeta,m=0}^{\infty}\tau_{\zeta,m}\pi_{[(\zeta+1)\lambda+m]\alpha,\beta}(x),$$

where, $\pi_{[(\zeta+1)\lambda+m]\alpha,\beta}(x)$ is the EW density with power parameter $[(\zeta+1)\lambda+m]\alpha$ and:

$$\tau_{\zeta,m} = \sum_{j=0}^{n-i}\frac{(-1)^j t_{\zeta,m}^{(j+i-1)}}{B(i,n+1-i)}\binom{n-i}{j}.$$

The PDF of the DEW order statistics is expressed as a linear combination of EW PDF. The moments of $X_{i:n}$ are given by:

$$\boldsymbol{E}\left(X_{i:n}^s\right) = \Gamma\left(s^{\beta-1}+1\right)\sum_{\zeta,m,h=0}^{\infty}\frac{\tau_{\zeta,m}\, c_h^{\{s,[(\zeta+1)\lambda+j]\alpha\}}}{B(i,n+1-i)}|_{(-\beta<s)}. \quad (17)$$

Probability Weighted Moments (PWMs)

The $(s,r)^{th}$ PWM of the DEW distribution, say $\mu_{s;r}$, can be defined by:

$$\mu_{s,r} = \boldsymbol{E}\left[F(X)^r X^s\right] = \int_{-\infty}^{\infty} F(X)^r f(x)x^s dx.$$

From Equations (3) and (4), we can write:

$$F(X)^r f(x) = \sum_{\zeta,j=0}^{\infty} c_{\zeta,j}^{(r)}\pi_{[(\zeta+1)\lambda+j]\alpha,\beta}(x),$$

where:

$$c_{\zeta,j}^{(r)} =$$
$$\theta\lambda\sum_{i=0}^{\infty}\frac{(-1)^{\zeta+i}(i+1)^{\zeta}\,\Gamma(\theta[r+1])\Gamma(1+\lambda[\zeta+1]+j)}{i!\zeta!j!\Gamma(\lambda(\zeta+1)+1)\Gamma(r-i+\theta+1)\left\{[(\zeta+1)\lambda+j]\alpha\right\}}.$$

Then, $\mu_{s,r}$ can be expressed as:

$$\mu_{s,r} = \sum_{\zeta,j=0}^{\infty} c_{\zeta,j}^{(r)} \int_{-\infty}^{\infty} x^s \pi_{[(\zeta+1)\lambda+j]\alpha,\beta}(x)dx.$$

Finally, the $(s, r)^{th}$ PWM of X can be given by:

$$\mu_{s,r} = \Gamma\left(s\beta^{-1}+1\right) \sum_{\zeta,j=0}^{\infty} d_{\zeta,j}^{(r)} c_h^{\{s,[(\zeta+1)\lambda+j]\alpha\}} |_{(-\beta<s)}.$$

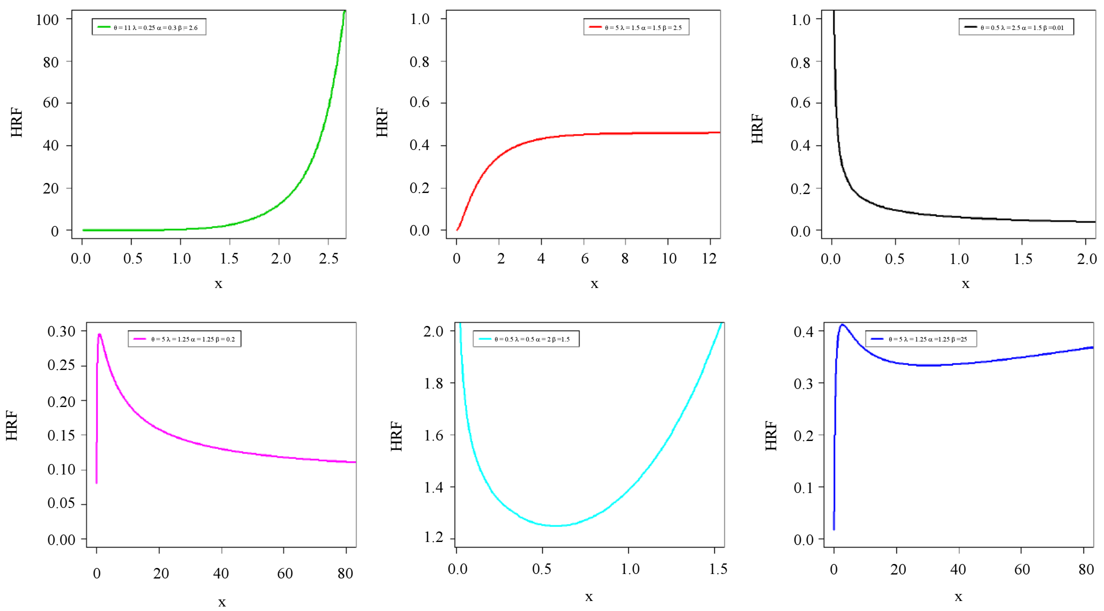
Numerical Analysis Results for Some Measures

Numerical analysis results for the $E(X)$, variance(X), skewness(X) and kurtosis(X) using (11) and well-known relationships for some selected values of parameter θ,λ,α and β are reported in Table 2. Based on Table 2 we note that:

1. The skewness of the DEW distribution is always positive and can range in the interval (1.72; 93.4)
2. The kurtosis of the DEW distribution is ranging from 7.06 to 18175.5
3. The mean of the DEW distribution increases as θ and α increases
4. The mean of the DEW distribution decreases as λ increases

Table 2: E(X), variance(X), skewness(X) and kurtosis(X) of the DEW distribution

θ	λ	α	β	E(X)	variance(X)	skewness(X)	kurtosis(X)
0.5	0.50	0.10	0.2	0.03402	0.78193	93.36140	18175.4800
1				0.06789	1.56121	66.10060	9111.3260
5				0.33414	7.70358	29.85600	1859.4610
10				0.65586	15.16610	21.36170	952.3557
50				2.90117	68.28620	10.31830	223.1026
100				5.18343	123.70200	7.83282	129.2139
1000				24.72280	622.90300	3.93418	34.20156
3	0.50	0.25	0.25	1.20275	24.64500	10.78990	204.6200
	1.50			0.00194	$4.03\times e^{-5}$	10.57980	222.7260
	1.75			0.00094	$7.48\times e^{-6}$	9.16693	167.6580
2	0.50	0.20	0.25	0.50430	8.30000	15.27990	413.5810
		0.75		6.22640	347.00000	6.95833	81.4296
		2.00		26.27700	3019.00000	4.56574	35.3904
		10.00		162.69000	4680.00000	2.75827	14.2361
		50.00		634.69000	3573.00000	1.93860	8.28824
		100.00		1031.44000	73571.00000	1.71939	7.06335
0.5	1.00	0.50	0.15	0.05110	0.38800	47.25020	4943.2100
			0.20	0.044700	0.11020	22.21120	933.9750
			0.25	0.046700	0.06310	13.74480	326.3040
			0.35	0.057610	0.04270	7.60136	91.1281
			0.40	0.064600	0.04050	6.21374	59.6693
			0.5	0.07970	0.04030	4.57647	31.9450



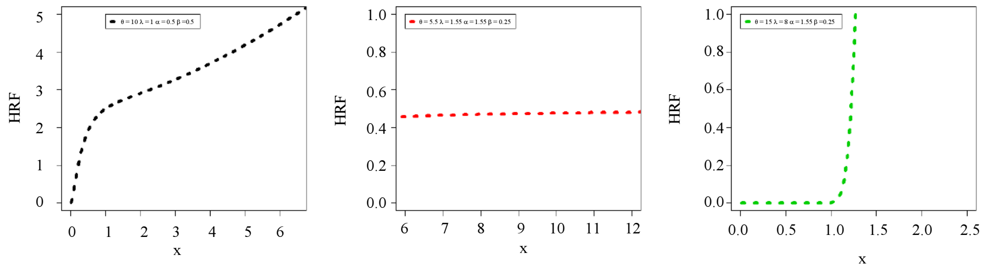


Fig. 2: Plots of the DEW HRF for selected parameter values

Estimation

Let $x_1,\ldots,x_n$ be a RS from the DEW model with parameters θ, λ ,α and β and let $\Upsilon = (\theta, \lambda, \alpha, \beta)^T$ be (4×1) parameter vector. The log-likelihood function ℓ for the DEW distribution is given by:

$$\begin{aligned}\ell &= n(\log\theta + \log\lambda + \log\alpha + \log\beta) \\ &-\sum_{i=1}^{n} p_i^{\lambda} + (\alpha\lambda - 1)\sum_{i=1}^{n}\log s_i - \sum_{i=1}^{n} x_i^{\beta} \\ &+(\beta-1)\sum_{i=1}^{n} x_i + (\theta-1)\sum_{i=1}^{n}\log q_i \\ &-(\lambda+1)\sum_{i=1}^{n}\log\left(1-s_i^{\alpha}\right),\end{aligned} \quad (18)$$

where, $s_i = 1-\exp\left(-x_i^{\beta}\right), p_i = s_i^{\alpha}/\left(1-s_i^{\alpha}\right)$ and $q_i = 1-\exp\left(-p_i^{\lambda}\right) | \left(i = 1,\ldots,n\right)$.

Simulation Results

Using (13), we simulate the DEW model via taking n = 20, 50, 200, 500 and 1000 following 3 sets of parameter values

	θ	λ	α	β
I:	3.75	2.50	4.25	0.25
II:	2.50	1.25	2.25	0.50
III:	0.50	0.75	2.50	0.50

We evaluate the ML Estimations (MLEs) of the parameters for each sample size. Then, by repeating this process N = 1000 times and calculate the Mean Squared Errors (MSEs) and the Averages of the Estimates (AEs). The numerical results in Table 3 show that both MSEs of $\hat{\theta},\hat{\lambda},\hat{\alpha}$ and $\hat{\beta}$ decline toward 0 once n increases for all initial values of θ, λ, α and β. As n increases, MSEs tend to 0 and the estimates appear to approach the true parameter values confirming the consistency property of the method of maximum likelihood estimation. Table 3 below presents the AEs, MSEs based on N = 1000 simulations of the DEW model for certain values θ, λ, α and β.

Applications

We present three applications to display the potentiality of the new PDF. To study the difference between the fits of the DEW model and other EW distributions, we will use the statistics of Anderson-Darling (A*) and Cramér-von Mises (W*). The MLE and the its Standard Errors (SEs) of the DEW model parameters are specified in Tables 4, 6 and 8. The numerical values of the A* and W* are listed in Tables 5, 7 and 9. The estimated CDF, estimated PDF, P-P plot, estimated HRF, Total Time on Test (TTT) plot and Kaplan-Meier survival plot of the three data sets of the DEW model are showed in Fig. 3, 4 and 5. We will specify some competitive models in the following three subsections, however many useful extensions of W model are available a potential competitive models, these models are presented by Nadarajah *et al*. (2013); Brito *et al*. (2017); Hamedani *et al*. (2017); Alizadeh *et al*. (2017); Merovci *et al*. (2017); Yousof *et al*. (2017b; 2018); Hamedani *et al*. (2018); Mahmoudi *et al*. (2018); Cordeiro *et al*. (2018); Korkmaz *et al*. (2019), among others. For more details about data set **I** (failure times of 84 aircraft windshield), **II** (remission times of cancer patients) and **III** (strengths of glass fibers); Lee and Wang (2003); Smith and Naylor (1987) respectively.

Modeling Failure Times

The data consist of 84 observations of failure times. Here, we assess the differences between the fits of the DEW distribution with those of other competitive models, namely: Burr-Hatke EW (BHEW), Ku- maraswamy transmuted W (KwTW), Marshall Olkin extended W (MOEW), Gamma W (GaW), W Fréchet (WFr), Poisson Topp Leone-W (PTLW), transmuted exponentiated generalized W (TEGW), Kumaraswamy W (KwW), Beta W (BW), Transmuted modified W (TMW), Modified beta W (MBW), Mcdonald W (McW), distributions, whose PDFs for x>0. The parameters of the above PDFs are all positive real numbers except for the TExGW and TMW models. Table 4 list the values of MLEs and SE for all fitted models. Table 5 reveals that the DEW distribution gives the best fit to failure times data set.

Table 3: The AEs and MSEs bsed on N = 1000 simulations

n	ϒ I	AE	MSE	ϒ II	AE	MSE	ϒ III	AE	MSE
20	θ	3.7657	0.7120	θ	2.4927	1.26480	θ	0.7112	1.1011
	λ	2.4985	0.0213	λ	1.2578	0.11280	λ	0.8224	0.5233
	α	4.2709	0.0455	α	2.2275	0.1318	α	2.7377	1.1261
	β	0.4032	0.3126	β	0.6853	0.3644	β	0.7278	0.4838
50	θ	3.7327	0.2513	θ	2.5313	0.44860	θ	0.6623	1.0187
	λ	2.4973	0.0079	λ	1.2515	0.01180	λ	0.7901	0.4101
	α	4.2515	0.0160	α	2.2273	0.01370	α	2.6925	0.9601
	β	0.3539	0.2023	β	0.6566	0.23230	β	0.6511	0.2973
200	θ	3.7319	0.0865	θ	2.4973	0.14920	θ	0.5623	0.5701
	λ	2.4918	0.0027	λ	1.2297	0.00150	λ	0.7823	0.3009
	α	4.2292	0.0056	α	2.2283	0.00490	α	2.6021	0.4445
	β	0.3145	0.1701	β	0.6138	0.12650	β	0.6030	0.1015
500	θ	3.7398	0.0255	θ	2.4991	0.04851	θ	0.5178	0.1799
	λ	2.4905	0.0008	λ	0.2292	0.00010	λ	0.7577	0.1010
	α	4.2296	0.0016	α	1.2284	0.00160	α	2.5201	0.1329
	β	0.2622	0.0492	β	0.5321	0.05611	β	0.5430	0.0522
1000	θ	3.7295	0.0122	θ	2.4969	0.02234	θ	0.4901	0.0022
	λ	2.4901	0.0002	λ	1.2298	0.00005	λ	0.7292	0.0041
	α	4.2292	0.0008	α	2.2296	0.00091	α	2.5021	0.0025
	β	0.2291	0.0007	β	0.5011	0.00320	β	0.5022	0.0021

Table 4: MLEs and SEs for faliure times data set

Model			Estimates		
BHEW (θ,α,β)	814.32	20.146	0.223		
	(582.25)	(1.683)	(0.03)		
PTL-W (λ,α,b)	−5.782	4.22865	0.658		
	(1.395)	(1.167)	(0.039)		
MOEW (ϒ,β,α)	488.9	0.28	1261.96		
	(189.36)	(0.013)	(351.1)		
GaW (α,β,ϒ)	2.377	0.8481	3.53		
	(0.378)	(0.0005)	(0.665)		
DEW (θ, λ,α,β)	0.4605	26.343	1.731	0.0816	
	(0.1531)	(0.000)	(0.059)	(0.000)	
KwW (α,β,a, b)	14.433	0.2041	34.66	81.85	
	(27.09)	(0.042)	(17.53)	(52.01)	
WFr (α,β,a; b)	630.94	0.30	416.1	1.17	
	(697.9)	(0.03),	(232.36)	(0.36)	
BW (α,β,a; b)	1.36	0.2981	34.2	11.5	
	(1.002)	(0.06)	(14.8)	(6.7)	
TMW (α,β,ϒ,λ)	0.2722	1	4.6×10^{-6}	0.4685	
	(0.014)	(5.2×10^{-5})	(1.9×10^{-4})	(0.165)	
KwTW (α,β,λ a, b)	27.79	0.178	0.445	29.5	168.1
	(33.4)	(0.017)	(0.609)	(9.8)	(129.2)
M-BW (α,β,a,b,c)	10.15	0.163	57.417	19.39	2.004
	(18.7)	(0.02)	(14.063)	(10.02)	(0.66)
M-cW (α,β,a,b,c)	1.94	0.31	17.69	33.6388	16.7211
	(1.011)	(0.045)	(6.222)	(19.994)	(9.722)
TExGW (α,β,λa, b)	4.257	0.1532	0.098	5.23	1173.328
	(33.4)	(0.017)	(0.609)	(0.76)	(9.792)

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Table 5: $W^{\star}$ and $A^{\star}$ for faliure times data set

Model	$W^{\star}$	$A^{\star}$
DEW	**0.0901**	**0.6471**
BHEW	0.0917	0.8740
PTLW	0.1397	1.19390
MOEW	0.3995	4.44770
GaW	0.2553	1.94890
KwW	0.1852	1.50590
WFr	0.2537	1.95740
BW	0.4652	3.21970
TMW	0.8065	11.2050
KwTW	0.1640	1.36320
MBW	0.4717	3.26560
McW	0.1986	1.59060
TExGW	1.0079	6.23320

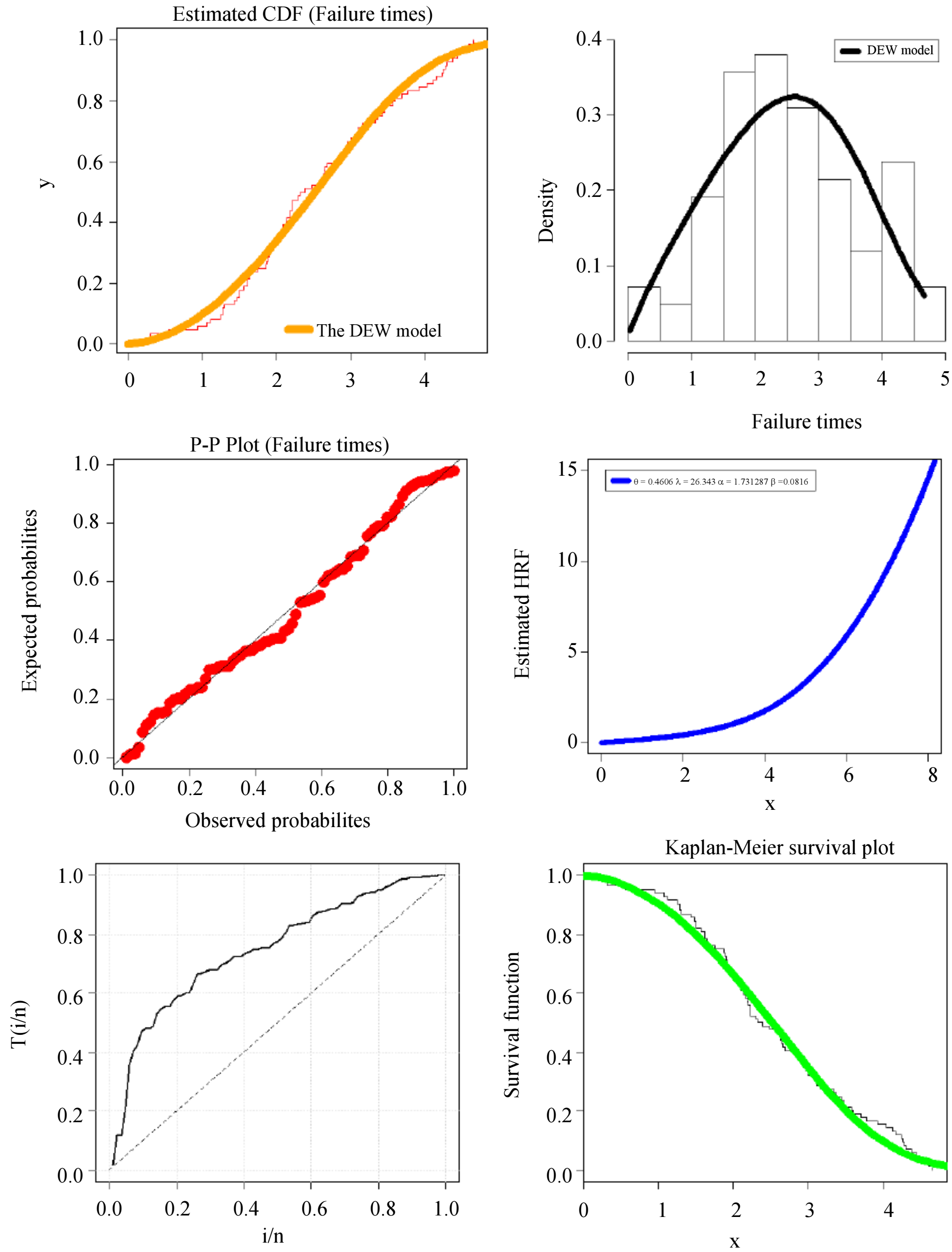

Fig. 3: CDF, PDF, P-P plot, HRF, TTT plot and Kaplan-Meier survival plot for data set I

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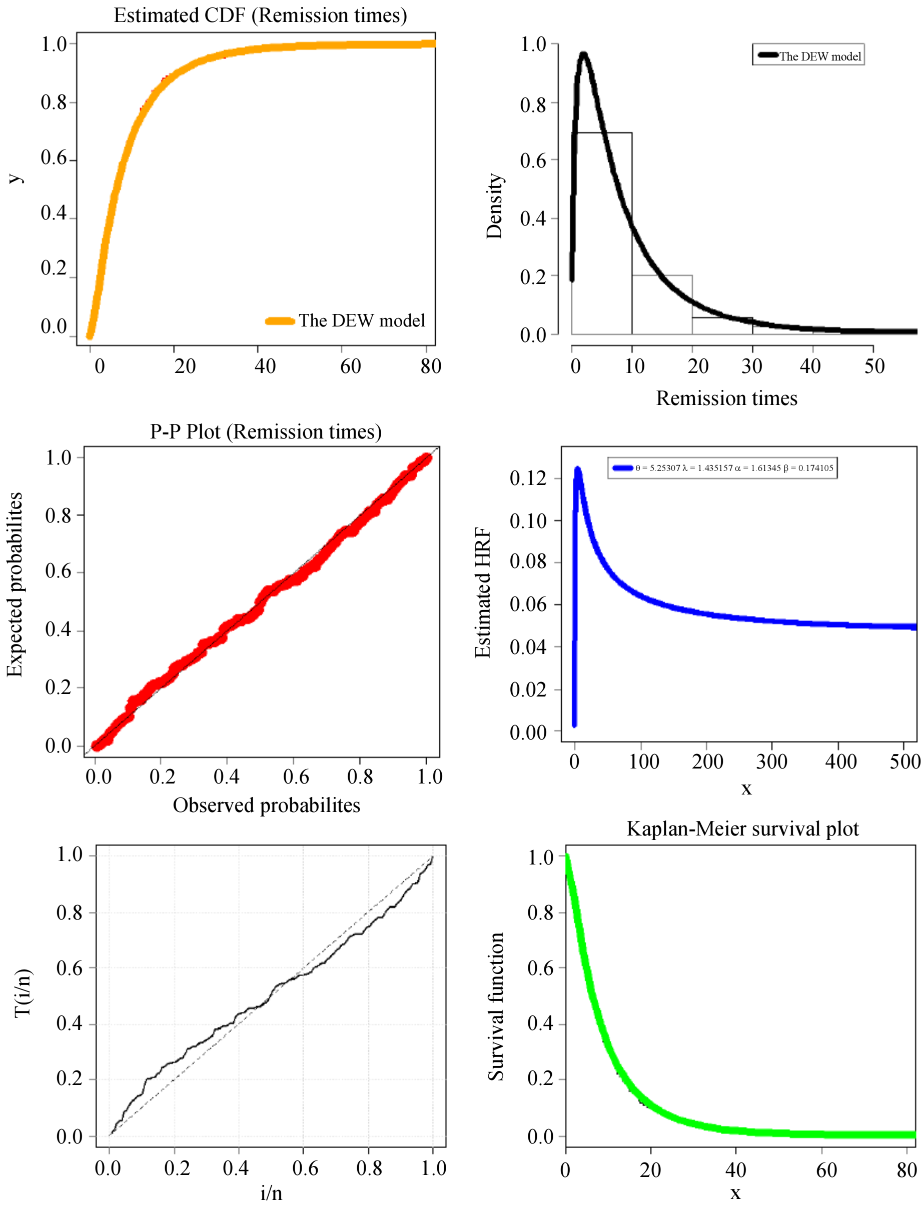
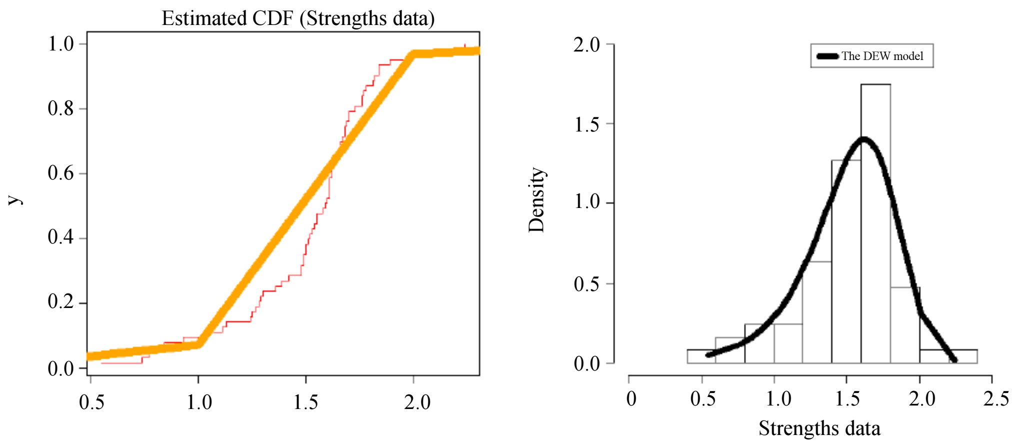


Fig. 4: CDF, PDF, P-P plot, HRF, TTT plot and Kaplan-Meier survival plot for data set II



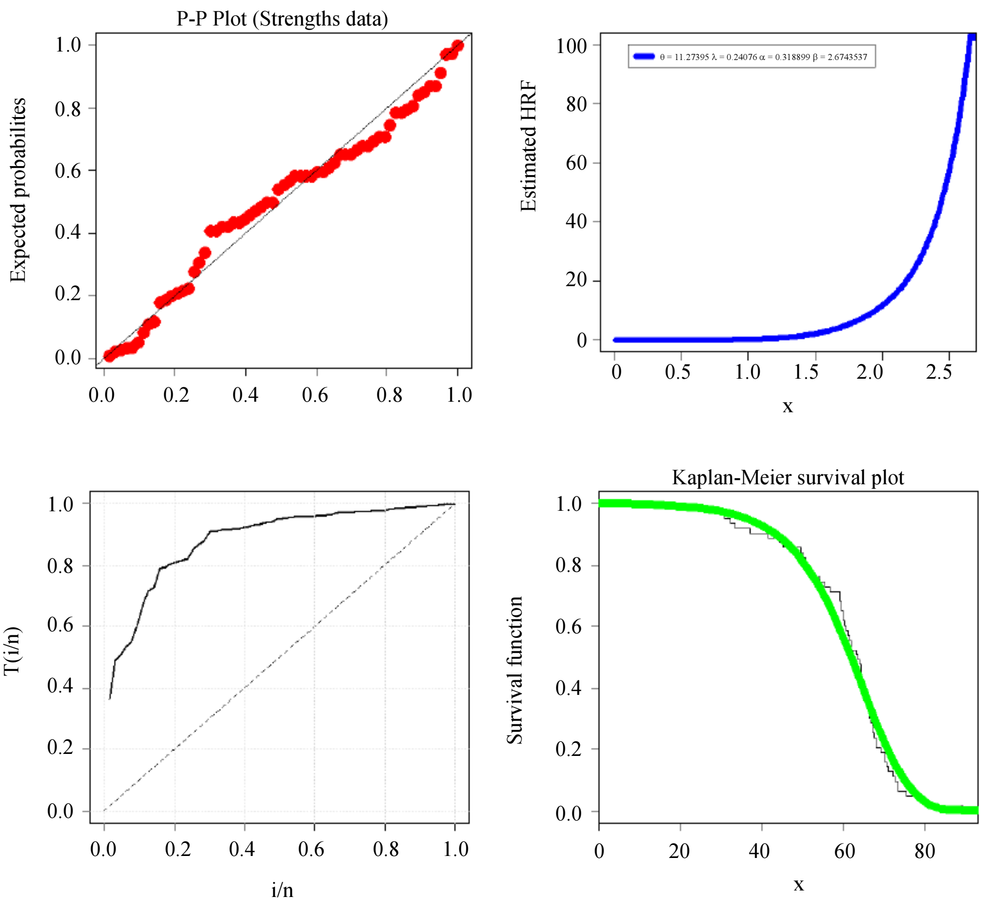


Fig. 5: CDF, PDF, P-P plot, HRF, TTT plot and Kaplan-Meier survival plot for data set III

Modeling Remission Times

The second data set denotes the remission times (in months) of a random sample of 128 bladder cancer patients. We intend to compare the fits of the DEW distribution with those of other models, namely: the WW, the Odd WW (OWW), the Gamma Exponentiated Exponential (GaEE) distributions, whose PDFs for $x>0$. Table 6 list the results of MLEs and SE for all fitted models. It is clear from Table 7 that the DEW distribution provides the best fit to remission times data set.

Modeling the Strengths

This data consists of 63 observations of the strengths of 1.5 cm glass fibers, originally obtained by workers at the UK National Physical Laboratory. For this data set, we compare the fits of the DEW distribution with some competitive models like BHEW, EW, TW and OLLW. Table 8 list the values of MLEs and SE for all models. This data have also been analyzed in a paper.

Table 6: MLEs and SEs for remission times data set

Model	Estimates			
WW (θ,α,β)	2.6594	0.6933	0.0270	
	(0.7129)	(0.1707)	(0.019)	
OWW (θ,α,β)	11.158	0.0881	0.457	
	(4.545)	(0.036)	(0.077)	
GaEE (θ,α,β)	2.114	2.601	0.008	
	(1.33)	(0.56)	(0.005)	
DEW ($\theta,\lambda,\alpha,\beta$)	5.2531	1.4352	1.6135	0.174
	(22.47)	(7.586)	(2.369)	(0.61)

Table 7: $W^\star$ and $A^\star$ for remission times data set

Model	$W^\star$	$A^\star$
DEW	0.0462	0.3074
WW	0.1427	0.7811
OWW	0.4494	2.4764
GaEE	0.3150	1.7208

Table 8: MLEs and SEs for the strengths data set

Model	Estimates			
BHEW (θ, α, β)	908.64	21.127	0.5	
	(577.31)	(1.61)	(0.06)	
EW (θ, α,)	0.671	7.285,	1.718	
	(0.249)	(1.707)	(0.09)	
TW (θ, α, β)	−0.501	5.15,	0.646	
	(0.27)	(0.67)	(0.024)	
OLLW (θ, α, β)	0.9439	6.0256	0.6159	
	(0.2689)	(1.348)	(0.016)	
DEW (θ, λ, α, β)	11.27	0.241	0.32	2.6
	(13.8)	(0.11)	(1.047)	(0.4)

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Table 9: $W^\star$ and A for the strengths data set

Model	$W^\star$	$A^\star$
DEW	0.141	0.082
BHEW	0.316	1.73
EW	0.636	3.484
TW	1.036	0.169
OLLW	1.236	0.219

Concluding Remarks

A new lifetime Exponentiated weibull model called the Dual Exponentiated Weibull Model (DEW) with various shapes of the HRF for modeling unimodal and bimodal data sets is introduced and studied along with its statistical properties. Before using the maximum likelihood method for estimating the unknown model parameters, we assessed its performance via a simulation study. The flexibility of the new model is illustrated via plots of the probability and hazard rate functions and three real data applications. The DEW model is very attractive to define special models with different types of hazard rates and its density can be symmetric or right-skewed or left-skewed or reversed **J** shape or unimodal or bimodal. The HRF of the DEW model can be increasing or increasing constant or decreasing or upside-down or bathtub or upside-down bathtub or constant or **J** shaped. The DEW model provides the best fit in modeling different types of data sets as illustrated using three real data sets.

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Author Contributions

The author wrote and developed the paper.

Ethics

The author declares that there is no conflict of interests regarding the publication of this article.

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Appendix A:

```
CDF_DEW<- function(par,x){
theta=par[1]
lameda=par[2]
alpha=par[3]
beta=par[4]
g=alpha*beta*x^(beta-1)*exp(-(x)^beta)*((1-exp(-(x)^beta))^(alpha-1))
G=(1-exp(-(x)^beta))^(alpha)
g.D=((lameda*theta*g*(G^(lameda-1)))/((1-G)^(lameda+1)))*exp(-(G/(1-G))^lameda)*(1-exp(-
(G/(1-G))^lameda))^(theta-1)
G.D=(1-exp(-(G/(1-G))^lameda))^theta
return(G.D)
}
PDF-DEW <- function(par,x){
theta=par[1]
lameda=par[2]
alpha=par[3]
beta=par[4]
g=alpha*beta*x^(beta-1)*exp(-(x)^beta)*((1-exp(-(x)^beta))^(alpha-1))
G=(1-exp(-(x)^beta))^(alpha)
g.D=((lameda*theta*g*(G^(lameda-1)))/((1-G)^(lameda+1)))*exp(-(G/(1-G))^lameda)*(1-exp(-
(G/(1-G))^lameda))^(theta-1)
G.D=(1-exp(-(G/(1-G))^lameda))^theta
return(g.D)
}
n = 20;
#theta = 3:75; lameda = 2.50; alpha = 4.25; beta=0.25
theta =2.50; lameda = 1.25; alpha = 2.25; beta=0:50
theta =0.50; lameda = 0:75; alpha = 2:50; beta=0:50
a=f(u)
x = f(a)
.t = goodness..t(PDF=PDF-DEW, CDF=CDF-DEW,
starts = c(1,1,1,1), data=x,
method="", domain=c(0,Inf), mle=NULL)
.t$mle
#————————————————————————
#theta = 3:75; lameda = 2.50; alpha = 4.25; beta=0.25
theta =2.50; lameda = 1.25; alpha = 2.25; beta=0:50
theta =0.50; lameda = 0:75; alpha = 2:50; beta=0:50
para=c(theta,lameda, alpha, beta)
#M=10^2
M = 1000
pa=matrix(para=M,nc=3,byrow=T)
NN=seq(50,1000,by=50)
bias=MSE=matrix(NA,nr=length(NN),nc=4)
row.names(bias)=row.names(MSE)=NN
for(i in 1:length(NN)){
    N = NN[i]
    cat("i=",i," n=",N,.nn.)
    ml = matrix(NA=M, nc=4, byrow=T)
```

```
        j = 1
        while(j <= M){
  a=f(u)
  x = f(a)
  .t = goodness..t(PDF=PDF-DEW, CDF=CDF-DEW,
  starts = c(1,1,1,1),
  data=x, method="N", domain=c(0,Inf), mle=NULL)
            if(.t$Convergence == 0) {
                ml[j,] = .t$mle
                j = j + 1
            }
        }
   # for(k in 1:length(ml[,1])){
   # if(ml[k,1] > 1.5*theta) ml[k,1] = NA
   # if(ml[k,1] > 1.5*lameda) ml[k,1] = NA
   # if(ml[k,1] > 1.5*alpha) ml[k,1] = NA
   # if(ml[k,1] > 1.5*beta) ml[k,1] = NA
   # }
        bias[i,]=apply((ml-pa), 2, FUN=mean, na.rm=TRUE)
        MSE[i,]=apply((ml-pa)^2, 2, FUN=mean, na.rm=TRUE)
   }
   bias; MSE
   ml
   write.table(data.frame(bias),"E://bias.txt")
   write.table(data.frame(MSE),"E://MSE.txt")
   MSE=read.table("E://MSE.txt")
   bias=read.table("E://bias.txt")
   MSE.theta = MSE[,1]; MSE.a = MSE[,2]; MSE.b = MSE[,3]
   bias.theta = bias[,1]; bias.a = bias[,2]; bias.b = bias[,3]
```

Appendix B:

```
   Application I, II and III
   x = (Data)
   hist(x)
   #==================================================
   CDF_DEW <- function(,){
   theta=par[1]
   lameda=par[2]
   alpha=par[3]
   beta=par[4]
   g=alpha*beta*x^(beta-1)*exp(-(x)^beta)*((1-exp(-(x)^beta))^(alpha-1))
   G=(1-exp(-(x)^beta))^(alpha)
   g.D=((lameda*theta*g*(G^(lameda-1)))/((1-G)^(lameda+1)))*exp(-(G/(1-G))^lameda)*(1-exp(-
(G/(1-G))^lameda))^(theta-1)
   G.D=(1-exp(-(G/(1-G))^lameda))^theta
   return(G.D)
   }
   PDF_DEW <- function(par,x){
   theta=par[1]
   lameda=par[2]
   alpha=par[3]
   beta=par[4]
   g=alpha*beta*x^(beta-1)*exp(-(x)^beta)*((1-exp(-(x)^beta))^(alpha-1))
   G=(1-exp(-(x)^beta))^(alpha)
   g.D=((lameda*theta*g*(G^(lameda-1)))/((1-G)^(lameda+1)))*exp(-(G/(1-G))^lameda)*(1-exp(-
(G/(1-G))^lameda))^(theta-1)
   G.D=(1-exp(-(G/(1-G))^lameda))^theta
   return(g.D)
   }
   goodness.t(PDF=PDF-DEW, CDF=CDF-DEW, starts = c(1,1,1,1), data=x,
   method="", domain=c(0,Inf), mle=NULL)
   # ==============================================
```