Journal of Mathematics and Statistics

Original Research Paper

On Generalized Some (*p*, *q*)-Special Polynomials

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Article history
Received: 16-02-2018
Revised: 14-04-2018
Accepted: 15-05-2018

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Abstract: In this study, we introduce a new class of generalized (*p*, *q*)-Bernoulli, (*p*, *q*)-Euler and (*p*, *q*)-Genocchi polynomials and investigate their some properties. We derive (*p*, *q*)-generalizations of some familiar formulae belonging to classical Bernoulli, Euler and Genocchi polynomials. We also obtain a (*p*, *q*)-extension of the Srivastava-Pintér addition theorem.

Keywords: (*p*, *q*)-Calculus, Generalized Bernoulli Polynomials, Generalized Euler Polynomials, Generalized Genocchi Polynomials, Generating Function, Cauchy Product

Introduction

Throughout of the paper, $\mathbb{N}$ denotes the set of the natural numbers, $\mathbb{N}_0$ denotes the set of nonnegative integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

The (*p, q*)-numbers are defined by:

$$[n]_{p,q} := \frac{p^n - q^n}{p-q},$$

Corcino (2008; Duran *et al*., 2016a; 2016b; Milovanović *et al*., 2016; Sadjang, 2013), One can readily write that $[n]_{p,q} = p^{n-1}[n]_{q/p}$, where $[n]_{q/p}$ is the *q*-number known in *q*-calculus given by $[n]_{q/p} = \frac{(q/p)^n - 1}{(q/p)-1}$. By appropriately using this obvious relation between the q-notation and its variant, the (*p*, *q*)-notation, most (if not all) of the (*p*, *q*)-results can be derived from the corresponding known q-results by merely changing the parameters and variables involved. When $p = 1$, (*p*, *q*)-numbers reduce to *q*-numbers. Note that (*p*, *q*)-numbers are symmetric: That is, $[n]_{p,q} = [n]_{q,p}$. In conjunction with the introduction of these (*p*, *q*)-numbers, the theory of (*p*, *q*)-calculus has been discussed and investigated extensively by many mathematicians and also physicists. For example, Corcino (2008) worked on the (*p*, *q*)-generalization of the binomial coefficients and also provided several useful properties parallel to those of the ordinary and *q*-binomial coefficients. Duran *et al*. (2016b) considered (*p*, *q*)-extensions of Bernoulli polynomials, Euler polynomials and Genocchi polynomials and obtained the (*p*, *q*)-analogues of familiar earlier formulas and identities. Milovanović *et al*. (2016) introduced a new generalization of Beta functions based on (*p*, *q*)-numbers and committed the integral modification of the generalized Bernstien polynomials. Sadjang (2013) developed several properties of the (*p*, *q*)-derivatives and the (*p*, *q*)-integrals and as an application, gave two (*p*, *q*)-Taylor formulas for polynomials.

The (*p*, *q*)-derivative of a function *f*, with respect to *x*, is defined by:

$$D_{p,q;x} f(x) := D_{p,q} f(x) = \frac{f(px) - (qx)}{(p-q)x} \quad (x \neq 0) \tag{1.1}$$

and $(D_{p,q}f)(0) = f'(0)$, provided that *f* is differentiable at 0.

The (*p*, *q*)-analogue of $(x + a)^n$ is given by:

$$(x\oplus a)^n_{p,q} := \begin{cases} (x+a)(px+aq) \\ \cdots\left(p^{n-2}x+aq^{n-2}\right)\left(p^{n-1}x+aq^{n-1}\right), & \text{if } n\geq 1, \\ 1, & \text{if } n=0. \end{cases}$$

The (p, q)-power basis is defined by:

$$(x\oplus a)^n_{p,q} = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} q^{\binom{n-k}{2}} x^k a^{n-k},$$

where, $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}$ and $[n]_{p,q}!$ are known as:

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!}(n\geq k)$$

and

$$[n]_{p,q}! = [n]_{p,q}[n-1]_{p,q}\cdots[2]_{p,q}[1]_{p,q}(n\in\mathbb{N}).$$

Let

$$e_{p,q}(x) = \sum_{n=0}^{\infty}\frac{p^{\binom{n}{2}}x^n}{[n]_{p,q}} \text{ and } E_{p,q}(x) = \sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}x^n}{[n]_{p,q}!}$$

denote two types of (p, q)-exponential functions. They hold:

$$e_{p,q}(x) = E_{p,q}(-x) = 1 \tag{1.2}$$

and:

$$D_{p,q}e_{p,q}(x) = e_{p,q}(px), D_{p,q}E_{p,q}(x) = E_{p,q}(qx). \tag{1.3}$$

The definite (p, q)-integral of a function f is defined by:

$$\int_0^a f(x)d_{p,q}x = (p-q)a\sum_{k=0}^{\infty}\frac{p^k}{q^{k+1}}f\left(\frac{p^k}{q^{k+1}}a\right)$$

in conjunction with (Sadjang, 2013):

$$\int_a^b f(x)d_{p,q}x = \int_0^b f(x)d_{p,q}x - \int_0^a f(x)d_{p,q}x \tag{1.4}$$

A more detailed statement of above, including (p, q)-notations, can be found in (Corcino, 2008; Duran *et al.*, 2016a; 2016b; Milovanović *et al.*, 2016; Sadjang, 2013). Special polynomials covering classical Bernoulli, Euler and Genocchi polynomials and their generalizations with many applications have been studied extensively and investigated by many mathematicians and also physicists, (Araci *et al.*, 2015; Carlitz, 1948; 1954; 1958; Chen *et al.*, 2013; Choi *et al.*, 2009; Duran *et al.*, 2016a; 2016b; He *et al.*, 2015; Kim, 2011; Kurt, 2016; Kurt and Kurt, 2016; Kurt, 2013; Luo, 2010; Luo and Srivastava, 2005; Mahmudov, 2013; Mahmudov and Keleshteri, 2014; Mahmudov, 2012; Mahmudov and Keleshteri, 2013; Milovanović *et al.*, 2016; Natalini and Bernardini, 2003; Ozden, 2010; Ozden *et al.*, 2010; Sadjang, 2013; Srivastava, 2011; Srivastava and Pintér, 2004; Srivastava and Choi, 2012; Srivastava *et al.*, 2010; Tremblay *et al.*, 2011).

The classical Bernoulli, Euler and Genocchi polynomials of order α are defined by the following Taylor series expansions at $z = 0$ (Chen *et al.*, 2013; He *et al.*, 2015; Kurt, 2013; Luo and Srivastava, 2005; Natalini and Bernardini, 2003; Ozden, 2010; Ozden *et al.*, 2010; Srivastava and Pintér, 2004; Srivastava *et al.*, 2010; Tremblay *et al.*, 2011):

$$\sum_{n=0}^{\infty}B_n^{(\alpha)}(x)\frac{z^n}{n!} = \left(\frac{z}{e^z-1}\right)^{\alpha}e^{xz} \quad (|z|<2\pi),$$

$$\sum_{n=0}^{\infty}E_n^{(\alpha)}(x)\frac{z^n}{n!} = \left(\frac{2}{e^z+1}\right)^{\alpha}e^{xz} \quad (|z|<\pi),$$

$$\sum_{n=0}^{\infty}G_n^{(\alpha)}(x)\frac{z^n}{n!} = \left(\frac{2z}{e^z-1}\right)^{\alpha}e^{xz} \quad (|z|<\pi).$$

For $m\in\mathbb{N}$ and $\alpha\in\mathbb{C}$, the generalized Bernoulli polynomials $B_n^{[m-1,\alpha]}(x)$ of order α, the generalized Euler polynomials $E_n^{[m-1,\alpha]}(x)$ of order α and the generalized Genocchi polynomials $G_n^{[m-1,\alpha]}(x)$ of order α are defined, in a suitable neighborhood of $z = 0$, by means of the following generating functions (Kurt, 2013; Mahmudov and Keleshteri, 2014; Natalini and Bernardini, 2003; Tremblay *et al.*, 2011):

$$\sum_{n=0}^{\infty}B_n^{[m-1,\alpha]}(x)\frac{z^n}{n!} = \left(\frac{z^m}{e^z-\sum_{h=0}^{m-1}\frac{z^h}{h!}}\right)^{\alpha}e^{xz}, \tag{1.5}$$

$$\sum_{n=0}^{\infty}E_n^{[m-1,\alpha]}(x)\frac{z^n}{n!} = \left(\frac{2^m}{e^z-\sum_{h=0}^{m-1}\frac{z^h}{h!}}\right)^{\alpha}e^{xz}, \tag{1.6}$$

$$\sum_{n=0}^{\infty}G_n^{[m-1,\alpha]}(x)\frac{z^n}{n!} = \left(\frac{2^m z^m}{e^z-\sum_{h=0}^{m-1}\frac{z^h}{h!}}\right)^{\alpha}e^{xz}, \tag{1.7}$$

respectively. When $x = 0$, we have $B_n^{[m-1,\alpha]}$ (0) := $B_n^{[m-1,\alpha]}$, $E_n^{[m-1,\alpha]}$ (0) := $E_n^{(\alpha)}$ and $G_n^{[m-1,\alpha]}$ (0) := $G_n^{[m-1,\alpha]}$, which are called, respectively, the generalized (*p*, *q*)-Bernoulli numbers of order α, the generalized Euler numbers of order α and the generalized (*p*, *q*)-Genocchi numbers of order α.

In the next section, we introduce a new class of generalized (*p*, *q*)-Bernoulli, (*p*, *q*)-Euler and (*p*, *q*)-Genocchi polynomials of order α. Then we investigate addition theorems, difference equations, derivative properties, integral representations, recurrence relationships for aforementioned polynomials. Also, we derive (*p*, *q*)-analogues of some known formulae belonging to usual Bernoulli, Euler and Genocchi polynomials. Further-more, we discover (*p*, *q*)-analogue of the main results given earlier by Srivastava and Pintér (2004).

Generalized (*p*, *q*)-Bernoulli, (*p*, *q*)-Euler and (*p*, *q*)-Genocchi Polynomials

We consider a new approach to higher order Bernoulli, Euler and Genocchi polynomials and numbers in the light of (*p*, *q*)-calculus. We firstly state the following Definition 1.

Definition 1

For $p, q, \alpha \in \mathbb{C}$ with $0<|q|<|p|<1$ and $m \in \mathbb{N}$, the generalized (*p*, *q*)-Bernoulli polynomials $\mathcal{B}_n^{[m-1,\alpha]}$ (*x*, *y*: *p*, *q*) of order α, the generalized (*p*, *q*)-Euler polynomials $\mathcal{E}_n^{[m-1,\alpha]}$ (*x*, *y*: *p*, *q*) of order α and the generalized (*p*, *q*)-Genocchi polynomials $\mathcal{G}_n^{[m-1,\alpha]}$ (*x*, *y*: *p*, *q*) of order α are defined, in a suitable neighborhood of $z = 0$, by the following generating functions:

$$\sum_{n=0}^{\infty} \mathcal{B}_n^{[m-1,\alpha]}(x,y:p,q)\frac{z^n}{[n]_{p,q}!} = \left(\frac{z^m}{e_{p,q}(z)-T_{m-1}^{p,q}(z)}\right)^{\alpha} e_{p,q}(xz)E_{p,q}(yz), \tag{2.1}$$

$$\sum_{n=0}^{\infty} \mathcal{E}_n^{[m-1,\alpha]}(x,y:p,q)\frac{z^n}{[n]_{p,q}!} = \left(\frac{2^m}{e_{p,q}(z)-T_{m-1}^{p,q}(z)}\right)^{\alpha} e_{p,q}(xz)E_{p,q}(yz) \tag{2.2}$$

and:

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{[m-1,\alpha]}(x,y:p,q)\frac{z^n}{[n]_{p,q}!} = \left(\frac{2^m z^m}{e_{p,q}(z)-T_{m-1}^{p,q}(z)}\right)^{\alpha} e_{p,q}(xz)E_{p,q}(yz), \tag{2.3}$$

in which $T_{m-1}^{p,q}(z) = \sum_{h=0}^{m-1} \frac{z^h}{[h]_{p,q}!}$.

Upon setting $x = 0$ and $y = 0$ in Definition 1, we then have $\mathcal{B}_n^{[m-1,\alpha]}$ (0, 0: *p*, *q*) := $\mathcal{B}_n^{[m-1,\alpha]}$ (*p*, *q*), $\mathcal{E}_n^{[m-1,\alpha]}$ (0, 0: *p*, *q*) := $\mathcal{E}_n^{(\alpha)}$ (*p*, *q*) and $\mathcal{G}_n^{[m-1,\alpha]}$ (0, 0: *p*, *q*) := $\mathcal{G}_n^{[m-1,\alpha]}$ (*p*, *q*), which are called, respectively, *n*-th generalized (*p*, *q*)-Bernoulli number of order α, *n*-th generalized (*p*, *q*)-Euler number of order α and *n*-th generalized (*p*, *q*)-Genocchi number of order α. In the case $\alpha = 1$ in Definition 1, we have:

$$\mathcal{B}_n^{[m-1,1]}(x,y:p,q) := \mathcal{B}_n^{[m-1]}(x,y:p,q),$$
$$\mathcal{E}_n^{[m-1,1]}(x,y:p,q) := \mathcal{E}_n^{[m-1]}(x,y:p,q)$$

and:

$$\mathcal{G}_n^{[m-1,1]}(x,y:p,q) := \mathcal{G}_n^{[m-1]}(x,y:p,q)$$

termed as *n*-th generalized (*p*, *q*)-Bernoulli polynomial, *n*-th generalized (*p*, *q*)-Euler polynomial and *n*-th generalized (*p*, *q*)-Genocchi polynomial.

Remark 1

The order α of the Apostol type (*p*, *q*)-polynomials in Definition 1 (and also in all analogous situations occuring elsewhere in this paper) is tacitly assumed to be a nonnegative integer except possibly in those cases in which the right-hand side of the generating functions (2.1), (2.2) and (2.3) turns out to be a power series in *z*. Only in these latter cases, we can safely assume that $\alpha \in \mathbb{C}$.

Remark 2

Putting $m = 1$ in Definition 1 reduces to (*p*, *q*)-analogue of Bernoulli $\mathcal{B}_n^{(\alpha)}$ (*x*, *y*: *p*, *q*), Euler $\mathcal{E}_n^{(\alpha)}$ (*x*, *y*: *p*, *q*) and Genocchi polynomials $\mathcal{G}_n^{(\alpha)}$ (*x*, *y*: *p*, *q*) defined in (Duran *et al*., 2016b), as follows:

$$\sum_{n=0}^{\infty} \mathcal{B}_n^{(\alpha)}(x,y:p,q)\frac{z^n}{[n]_{p,q}!} = \left(\frac{z}{e_{p,q}(z)-1}\right)^{\alpha} e_{p,q}(xz)E_{p,q}(yz)$$

$$\sum_{n=0}^{\infty}\mathcal{E}_n^{(\alpha)}(x,y:p,q)\frac{z^n}{[n]_{p,q}!}=\left(\frac{2}{e_{p,q}(z)+1}\right)^{\alpha}e_{p,q}(xz)E_{p,q}(yz)$$

and:

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{(\alpha)}(x,y:p,q)\frac{z^n}{[n]_{p,q}!}=\left(\frac{2z}{e_{p,q}(z)+1}\right)^{\alpha}e_{p,q}(xz)E_{p,q}(yz).$$

We now give some special cases of Definition 1 as Corollary 3 and Corollary 4 as follows.

Remark 3

If we take p = 1 in Definition 1, we then get (Mahmudov and Keleshteri, 2014; 2013):

$$\sum_{n=0}^{\infty}\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y)\frac{z^n}{[n]_q!}=\left(\frac{z^m}{e_q(z)-\sum_{k=0}^{m-1}\frac{z^k}{[k]_q!}}\right)^{\alpha}e_q(xz)E_q(yz),$$

$$\sum_{n=0}^{\infty}\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,y)\frac{z^n}{[n]_q!}=\left(\frac{2^m}{e_q(z)+\sum_{k=0}^{m-1}\frac{z^k}{[k]_q!}}\right)^{\alpha}e_q(xz)E_q(yz),$$

$$\sum_{n=0}^{\infty}\mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,y)\frac{z^n}{[n]_q!}=\left(\frac{2^m z^m}{e_q(z)+\sum_{k=0}^{m-1}\frac{z^k}{[k]_q!}}\right)^{\alpha}e_q(xz)E_q(yz)$$

where, $\mathfrak{B}_{n,q}^{[m-1,\alpha]}(x,y)$, $\mathfrak{E}_{n,q}^{[m-1,\alpha]}(x,y)$ and $\mathfrak{G}_{n,q}^{[m-1,\alpha]}(x,y)$ are called n-th generalized q-Bernoulli polynomial of order α, n-th generalized q-Euler polynomial of order α and n-th generalized q-Genocchi polynomial of order α, respectively.

Remark 4

Taking $q\to 1$, $p = \alpha = m = 1$ and $y = 0$ in 1 gives:

$$\sum_{n=0}^{\infty}B_n(x)\frac{z^n}{n!}=\frac{z}{e^z-1}e^{xz},$$
$$\sum_{n=0}^{\infty}E_n(x)\frac{z^n}{n!}=\frac{2}{e^z+1}e^{xz}\ and\sum_{n=0}^{\infty}G_n(x)\frac{z^n}{n!}=\frac{2z}{e^z+1}e^{xz}$$

which are known as classical Bernoulli polynomials, classical Euler polynomials and Genocchi polynomials, respectively, (Kurt, 2013; Mahmudov and Keleshteri, 2014; 2013; Ozden, 2010; Tremblay *et al*., 2011).

We now discuss some properties and behaviours of the aforementioned polynomials in Definition 1. We first provide the following basic properties as Proposition 1 without proving, because they can be obtained by using Definition 1 and Cauchy product.

Proposition 1

The following relations hold true:

$$\mathcal{B}_n^{[m-1,\alpha]}(x,y:p,q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}q^{(n-k)(n-k-1)/2}\mathcal{B}_k^{[m-1,\alpha]}(x,0:p,q)y^{n-k},$$
$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{(n-k)(n-k-1)/2}\mathcal{B}_k^{[m-1,\alpha]}(0,y:p,q)x^{n-k},$$
$$\mathcal{E}_n^{[m-1,\alpha]}(x,y:p,q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}q^{(n-k)(n-k-1)/2}\mathcal{E}_k^{[m-1,\alpha]}(x,0:p,q)y^{n-k},$$
$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{(n-k)(n-k-1)/2}\mathcal{E}_k^{[m-1,\alpha]}(0,y:p,q)x^{n-k},$$
$$\mathcal{G}_n^{[m-1,\alpha]}(x,y:p,q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}q^{(n-k)(n-k-1)/2}\mathcal{G}_k^{[m-1,\alpha]}(x,0:p,q)y^{n-k},$$
$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{(n-k)(n-k-1)/2}\mathcal{G}_k^{[m-1,\alpha]}(0,y:p,q)x^{n-k}.$$

A special case of Proposition 1 is given by Corollary 1.

Corollary 1

Setting y = 1 (or x = 1) in Proposition 1 yields to:

$$\mathcal{B}_n^{[m-1,\alpha]}(x,1:p,q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}q^{(n-k)(n-k-1)/2}\mathcal{B}_k^{[m-1,\alpha]}(x,0:p,q),\quad(2.4)$$

$$\mathcal{B}_n^{[m-1,\alpha]}(1,y:p,q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{(n-k)(n-k-1)/2}\mathcal{B}_k^{[m-1,\alpha]}(0,y:p,q),\quad(2.5)$$

$$\mathcal{E}_n^{[m-1,\alpha]}(x,1:p,q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}q^{(n-k)(n-k-1)/2}\mathcal{E}_k^{[m-1,\alpha]}(x,0:p,q),\quad(2.6)$$

$$\mathcal{E}_n^{[m-1,\alpha]}(1,y:p,q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{(n-k)(n-k-1)/2}\mathcal{E}_k^{[m-1,\alpha]}(0,y:p,q),\quad(2.7)$$

$$\mathcal{G}_n^{[m-1,\alpha]}(x,1:p,q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}q^{(n-k)(n-k-1)/2}\mathcal{G}_k^{[m-1,\alpha]}(x,0:p,q),\quad(2.8)$$

$$\mathcal{G}_n^{[m-1,\alpha]}(1,y:p,q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{(n-k)(n-k-1)/2}\mathcal{G}_k^{[m-1,\alpha]}(0,y:p,q).\quad(2.9)$$

Note that Equation 2.4-2.9 are (p, q)-analogues of the following formulas (Kurt, 2013; Mahmudov and Keleshteri, 2014; 2013; Ozden, 2010; Tremblay *et al*., 2011):

$$B_n^{[m-1,\alpha]}(x+1)=\sum_{k=0}^{n}\binom{n}{k}B_k^{[m-1,\alpha]}(x),$$

$$E_n^{[m-1,\alpha]}(x+1)=\sum_{k=0}^{n}\binom{n}{k}E_k^{[m-1,\alpha]}(x)$$

and:

$$G_n^{[m-1,\alpha]}(x+1)=\sum_{k=0}^{n}\binom{n}{k}G_k^{[m-1,\alpha]}(x).$$

Now we present the addition properties of generalized (p, q)-Bernoulli, (p, q)-Euler and (p, q)-Genocchi polynomials of order α as follows.

Proposition 2. (Addition Properties)

Let $n\in\mathbb{N}_0$ and $\alpha,\beta\in\mathbb{C}$. We have:

$$\mathcal{B}_n^{[m-1,\alpha+\beta]}(x,y:p,q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\mathcal{B}_{n-k}^{[m-1,\alpha]}(x,0:p,q)\mathcal{B}_k^{[m-1,\beta]}(0,y:p,q),$$

$$\mathcal{E}_n^{[m-1,\alpha+\beta]}(x,y:p,q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\mathcal{E}_{n-k}^{[m-1,\alpha]}(x,0:p,q)\mathcal{E}_k^{[m-1,\beta]}(0,y:p,q),$$

$$\mathcal{G}_n^{[m-1,\alpha+\beta]}(x,y:p,q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\mathcal{G}_{n-k}^{[m-1,\alpha]}(x,0:p,q)\mathcal{G}_k^{[m-1,\beta]}(0,y:p,q).$$

Here are (p, q)-derivatives of the polynomials $\mathcal{B}_n^{[m-1,\alpha]}$ $(x, y: p, q)$, $\mathcal{E}_n^{[m-1,\alpha]}$ $(x, y: p, q)$ and $\mathcal{G}_n^{[m-1,\alpha]}$ $(x, y: p, q)$, with respect to x and y, as follows.

Proposition 3. (Derivative Properties)

We have:

$$D_{p,q;x}\mathcal{B}_n^{[m-1,\alpha]}(x,y:p,q)=[n]_{p,q}\mathcal{B}_{n-1}^{[m-1,\alpha]}(px,y:p,q),$$
$$D_{p,q;y}\mathcal{B}_n^{[m-1,\alpha]}(x,y:p,q)=[n]_{p,q}\mathcal{B}_{n-1}^{[m-1,\alpha]}(x,qy:p,q),$$
$$D_{p,q;x}\mathcal{E}_n^{[m-1,\alpha]}(x,y:p,q)=[n]_{p,q}\mathcal{E}_{n-1}^{[m-1,\alpha]}(px,y:p,q),$$
$$D_{p,q;y}\mathcal{E}_n^{[m-1,\alpha]}(x,y:p,q)=[n]_{p,q}\mathcal{E}_{n-1}^{[m-1,\alpha]}(x,qy:p,q),$$
$$D_{p,q;x}\mathcal{G}_n^{[m-1,\alpha]}(x,y:p,q)=[n]_{p,q}\mathcal{G}_{n-1}^{[m-1,\alpha]}(px,y:p,q),$$
$$D_{p,q;y}\mathcal{G}_n^{[m-1,\alpha]}(x,y:p,q)=[n]_{p,q}\mathcal{G}_{n-1}^{[m-1,\alpha]}(x,qy:p,q).$$

The (p, q)-integral representations of $\mathcal{B}_n^{[m-1,\alpha]}$ $(x, y: p, q)$, $\mathcal{E}_n^{[m-1,\alpha]}$ $(x, y: p, q)$ and $\mathcal{G}_n^{[m-1,\alpha]}$ $(x, y: p, q)$ are given by the following proposition.

Proposition 4

We have:

$$\int_a^b\mathcal{B}_n^{[m-1,\alpha]}(x,y:p,q)d_{p,q}x=\frac{\mathcal{B}_{n+1}^{[m-1,\alpha]}\left(\frac{b}{p},y:p,q\right)-\mathcal{B}_{n+1}^{[m-1,\alpha]}\left(\frac{a}{p},y:p,q\right)}{[n+1]_{p,q}},$$

$$\int_a^b\mathcal{B}_n^{[m-1,\alpha]}(x,y:p,q)d_{p,q}y=\frac{\mathcal{B}_{n+1}^{[m-1,\alpha]}\left(x,\frac{b}{q}:p,q\right)-\mathcal{B}_{n+1}^{[m-1,\alpha]}\left(x,\frac{a}{q}:p,q\right)}{[n+1]_{p,q}},$$

$$\int_a^b\mathcal{E}_n^{[m-1,\alpha]}(x,y:p,q)d_{p,q}x=\frac{\mathcal{E}_{n+1}^{[m-1,\alpha]}\left(\frac{b}{p},y:p,q\right)-\mathcal{E}_{n+1}^{[m-1,\alpha]}\left(\frac{a}{p},y:p,q\right)}{[n+1]_{p,q}},$$

$$\int_a^b\mathcal{E}_n^{[m-1,\alpha]}(x,y:p,q)d_{p,q}y=\frac{\mathcal{E}_{n+1}^{[m-1,\alpha]}\left(x,\frac{b}{q}:p,q\right)-\mathcal{E}_{n+1}^{[m-1,\alpha]}\left(x,\frac{a}{q}:p,q\right)}{[n+1]_{p,q}},$$

$$\int_a^b\mathcal{G}_n^{[m-1,\alpha]}(x,y:p,q)d_{p,q}x=\frac{\mathcal{G}_{n+1}^{[m-1,\alpha]}\left(\frac{b}{p},y:p,q\right)-\mathcal{G}_{n+1}^{[m-1,\alpha]}\left(\frac{a}{p},y:p,q\right)}{[n+1]_{p,q}},$$

$$\int_a^b\mathcal{G}_n^{[m-1,\alpha]}(x,y:p,q)d_{p,q}y=\frac{\mathcal{G}_{n+1}^{[m-1,\alpha]}\left(x,\frac{b}{q}:p,q\right)-\mathcal{G}_{n+1}^{[m-1,\alpha]}\left(x,\frac{a}{q}:p,q\right)}{[n+1]_{p,q}}.$$

Proof

Since (Sadjang, 2013):

$$\int_a^b D_{p,q}f(x)d_{p,q}x=f(b)-f(a)$$

in terms of Proposition 3 and Equation 1.3 and 1.4, we arrive at the desired result:

$$\int_a^b\mathcal{B}_{n+1}^{[m-1,\alpha]}(x,y:p,q)d_{p,q}x=\frac{1}{[n+1]_{p,q}}\int_a^b D_{p,q}\mathcal{B}_{n+1}^{[m-1,\alpha]}\left(\frac{x}{p},y:p,q\right)d_{p,q}x=\frac{\mathcal{B}_{n+1}^{[m-1,\alpha]}\left(\frac{b}{p},y:p,q\right)-\mathcal{B}_{n+1}^{[m-1,\alpha]}\left(\frac{a}{p},y:p,q\right)}{[n+1]_{p,q}}.$$

The other integral representations can be proved by utilizing similar proof technique used above.

We give the following recurrence relationships.

Theorem 1

We have:

$$\begin{aligned}
&\frac{[n]_{p,q}!}{[n-m]_{p,q}!}\mathcal{B}_{n-m}^{[m-1,\alpha-1]}(0,y:p,q)=\mathcal{B}_{n}^{[m-1,\alpha]}(1,y:p,q)\\
&-\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\mathcal{B}_{n-k}^{[m-1,\alpha]}(0,y:p,q)\ (n\geq m),\\
&\frac{[n]_{p,q}!}{[n-m]_{p,q}!}\mathcal{B}_{n-m}^{[m-1,\alpha-1]}(x,-1:p,q)=\mathcal{B}_{n}^{[m-1,\alpha]}(x,0:p,q)\\
&-\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\mathcal{B}_{n-k}^{[m-1,\alpha]}(x,-1:p,q)\ (n\geq m),
\end{aligned}\tag{2.10}$$

$$\begin{aligned}
&2^{m}\mathcal{E}_{n}^{[m-1,\alpha-1]}(0,y:p,q)=\mathcal{E}_{n}^{[m-1,\alpha]}(1,y:p,q)\\
&+\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\mathcal{E}_{n-k}^{[m-1,\alpha]}(0,y:p,q),\\
&2^{m}\mathcal{E}_{n}^{[m-1,\alpha-1]}(x,-1:p,q)=\mathcal{E}_{n}^{[m-1,\alpha]}(x,0:p,q)\\
&+\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\mathcal{E}_{n-k}^{[m-1,\alpha]}(x,-1:p,q),
\end{aligned}\tag{2.11}$$

$$\begin{aligned}
&\frac{2^{m}[n]_{p,q}!}{[n-m]_{p,q}!}\mathcal{G}_{n-m}^{[m-1,\alpha-1]}(0,y:p,q)=\mathcal{G}_{n}^{[m-1,\alpha]}(1,y:p,q)\\
&+\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\mathcal{G}_{n-k}^{[m-1,\alpha]}(0,y:p,q)\ (n\geq m),\\
&\frac{2^{m}[n]_{p,q}!}{[n-m]_{p,q}!}\mathcal{G}_{n-m}^{[m-1,\alpha-1]}(x,-1:p,q)=\mathcal{G}_{n}^{[m-1,\alpha]}(x,0:p,q)\\
&+\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\mathcal{G}_{n-k}^{[m-1,\alpha]}(x,-1:p,q)\ (n\geq m),
\end{aligned}\tag{2.12}$$

Proof

By inspiring the proof technique in (Mahmudov and Keleshteri, 2013) and utilizing the following relation:

$$\begin{aligned}
&T_{m-1}^{p,q}(z)\left(\frac{z^{m}}{e_{p,q}(z)-T_{m-1}^{p,q}(z)}\right)^{\alpha}E_{p,q}(yz)\\
&=\sum_{n=0}^{m-1}\frac{z^{n}}{[n]_{p,q}!}\sum_{n=0}^{\infty}\mathcal{B}_{n}^{[m-1,\alpha]}(0,y:p,q)\frac{z^{n}}{[n]_{p,q}!}\\
&=\sum_{n=0}^{\infty}\mathcal{B}_{n}^{[m-1,\alpha]}(0,y:p,q)\left(\frac{z^{n}}{[n]_{p,q}!}+\frac{z^{n+1}}{[n]_{p,q}!}+\frac{z^{n+2}}{[n]_{p,q}![2]_{p,q}!}\cdots+\frac{z^{n+m-1}}{[n]_{p,q}![m-1]_{p,q}!}\right)\\
&=\sum_{n=0}^{\infty}\mathcal{B}_{n}^{[m-1,\alpha]}(0,y:p,q)\frac{z^{n}}{[n]_{p,q}!}+\sum_{n=0}^{\infty}[n]_{p,q}\mathcal{B}_{n}^{[m-1,\alpha]}(0,y:p,q)\frac{z^{n}}{[n]_{p,q}!}\\
&+\cdots+\sum_{n=0}^{\infty}\frac{[n]_{p,q}\cdots[n-m+2]_{p,q}}{[m-1]_{p,q}!}\mathcal{B}_{n-m+1}^{[m-1,\alpha]}(0,y:p,q)\frac{z^{n}}{[n]_{p,q}!}\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\mathcal{B}_{n-k}^{[m-1,\alpha]}(0,y:p,q)\frac{z^{n}}{[n]_{p,q}!},
\end{aligned}$$

we then get:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\mathcal{B}_{n}^{[m-1,\alpha]}(1,y:p,q)-\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\mathcal{B}_{n-k}^{[m-1,\alpha]}(0,y:p,q)\right)\frac{z^{n}}{[n]_{p,q}!}\\
&=\left(\frac{z^{m}}{e_{p,q}(z)-T_{m-1}^{p,q}(z)}\right)^{\alpha}e_{p,q}(z)E_{p,q}(yz)\\
&-T_{m-1}^{p,q}(z)\left(\frac{z^{m}}{e_{p,q}(z)-T_{m-1}^{p,q}(z)}\right)^{\alpha}E_{p,q}(yz)\\
&=\left(\frac{z^{m}}{e_{p,q}(z)-T_{m-1}^{p,q}(z)}\right)^{\alpha}E_{p,q}(yz)\left(e_{p,q}(z)-T_{m-1}^{p,q}(z)\right)\\
&=z^{m}\left(\frac{z^{m}}{e_{p,q}(z)-T_{m-1}^{p,q}(z)}\right)^{\alpha-1}E_{p,q}(yz)\\
&=\sum_{n=0}^{\infty}\mathcal{B}_{n}^{[m-1,\alpha-1]}(0,y:p,q)\frac{z^{n+m}}{[n]_{p,q}!}.
\end{aligned}$$

Checking against the coefficients of z^n, then we have the Equation 2.10. The others in this theorem can be similarly proved.

By the Equation 2.5 and 2.10, Equation 2.7 and 2.11, Equation 2.9 and 2.12, we acquire the following formulas.

Note that:

$$\begin{aligned}
&\mathcal{B}_{n}^{[m-1,0]}(x,y:p,q)=\mathcal{E}_{n}^{[m-1,0]}(x,y:p,q)\\
&=\mathcal{G}_{n}^{[m-1,0]}(x,y:p,q)=(x\oplus y)_{p,q}^{n}.
\end{aligned}\tag{2.13}$$

From (2.13) and Corollary 1, we have:

$$y^{n-m}=\frac{[n-m]_{p,q}!}{q^{\binom{n-m}{2}}[n]_{p,q}!}\left(\begin{aligned}&\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}p^{\binom{n-m}{2}}\mathcal{B}_{k}^{[m-1]}(0,y:p,q)\\&-\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\mathcal{B}_{n-k}^{[m-1]}(0,y:p,q)\end{aligned}\right)\ (n\geq m)$$

$$y^{n}=\frac{2^{-m}}{q^{\binom{n}{2}}}\left(\begin{aligned}&\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}p^{\binom{n-k}{2}}\mathcal{E}_{k}^{[m-1]}(0,y:p,q)\\&+\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\mathcal{E}_{n-k}^{[m-1]}(0,y:p,q)\end{aligned}\right)$$

and:

$$y^{n-m}=\frac{2^{-m}[n-m]_{p,q}!}{q^{\binom{n-m}{2}}[n]_{p,q}!}\left(\begin{aligned}&\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}p^{\binom{n-k}{2}}\mathcal{G}_{k}^{[m-1]}(0,y:p,q)\\&+\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\mathcal{G}_{n-k}^{[m-1]}(0,y:p,q)\end{aligned}\right)\ (n\geq m).$$

From Theorem 1, we get the following Corollary 2.

Corollary 2

The following equalities:

$$\mathcal{B}_n^{[m-1,\alpha]}(1,y:p,q)-\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\mathcal{B}_{n-k}^{[m-1,\alpha]}(1,y:p,q)$$

$$=[n]_{p,q}\sum_{k=0}^{n-1}\begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q}\mathcal{B}_k^{[m-1,\alpha]}(0,y:p,q)\mathcal{B}_{n-1-k}^{[0,-1]}(p,q),$$

$$\mathcal{E}_n^{[m-1,\alpha]}(1,y:p,q)+\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\mathcal{E}_{n-k}^{[m-1,\alpha]}(0,y:p,q)$$

$$=2\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\mathcal{E}_k^{[m-1,\alpha-1]}(0,y:p,q)\mathcal{E}_{n-k}^{[0,-1]}(p,q),$$

$$\mathcal{G}_n^{[m-1,\alpha]}(1,y:p,q)+\sum_{k=0}^{\min(n,m-1)}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}\mathcal{G}_{n-k}^{[m-1,\alpha]}(0,y:p,q)$$

$$=2[n]_{p,q}\sum_{k=0}^{n-1}\begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q}\mathcal{G}_k^{[m-1,\alpha]}(0,y:p,q)\mathcal{G}_{n-1-k}^{[0,-1]}(p,q)$$

hold true.

We here present some new connections among the polynomials $\mathcal{B}_n^{[m-1,\alpha]}$ (x, y: p, q), $\mathcal{E}_n^{[m-1,\alpha]}$ (x, y: p, q) and $\mathcal{G}_n^{[m-1,\alpha]}$ (x, y: p, q) and our main results as given by subsequent theorems.

Theorem 2

For $n\in\mathbb{N}_0$ and x, y, $\alpha\in\mathbb{C}$, the following relationships are valid:

$$\mathcal{B}_n^{[m-1,\alpha]}(x,y:p,q)=\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_{p,q}\mathcal{B}_{n-l}^{[m-1,\alpha]}(0,y:p,q)\sum_{k=0}^{l}\begin{bmatrix} l \\ k \end{bmatrix}_{p,q}\mathcal{E}_k^{[m-1,\alpha]}(x,0:p,q)\mathcal{E}_{l-k}^{[m-1,\alpha]}(p,q)$$

$$=\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_{p,q}\mathcal{B}_{n-l}^{[m-1,\alpha]}(0,y:p,q)\sum_{k=0}^{l}\begin{bmatrix} l \\ k \end{bmatrix}_{p,q}\mathcal{G}_k^{[m-1,\alpha]}(x,0:p,q)\mathcal{G}_{l-k}^{[m-1,-\alpha]}(p,q)$$

$$\mathcal{E}_n^{[m-1,\alpha]}(x,y:p,q)=\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_{p,q}\mathcal{E}_{n-l}^{[m-1,\alpha]}(0,y:p,q)\sum_{k=0}^{l}\begin{bmatrix} l \\ k \end{bmatrix}_{p,q}\mathcal{B}_k^{[m-1,\alpha]}(x,0:p,q)\mathcal{B}_{l-k}^{[m-1,\alpha]}(p,q)$$

$$=\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_{p,q}\mathcal{E}_{n-l}^{[m-1,\alpha]}(0,y:p,q)\sum_{k=0}^{l}\begin{bmatrix} l \\ k \end{bmatrix}_{p,q}\mathcal{G}_k^{[m-1,\alpha]}(x,0:p,q)\mathcal{G}_{l-k}^{[m-1,-\alpha]}(p,q)$$

$$\mathcal{G}_n^{[m-1,\alpha]}(x,y:p,q)=\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_{p,q}\mathcal{G}_{n-l}^{[m-1,\alpha]}(0,y:p,q)\sum_{k=0}^{l}\begin{bmatrix} l \\ k \end{bmatrix}_{p,q}\mathcal{B}_k^{[m-1,\alpha]}(x,0:p,q)\mathcal{B}_{l-k}^{[m-1,\alpha]}(p,q)$$

$$=\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_{p,q}\mathcal{G}_{n-l}^{[m-1,\alpha]}(0,y:p,q)\sum_{k=0}^{l}\begin{bmatrix} l \\ k \end{bmatrix}_{p,q}\mathcal{E}_k^{[m-1,\alpha]}(x,0:p,q)\mathcal{E}_{l-k}^{[m-1,-\alpha]}(p,q).$$

Proof

From Definition 1 and using Cauchy product, we get:

$$\sum_{n=0}^{\infty}\mathcal{B}_{n-l}^{[m-1,\alpha]}(x,y:p,q)\frac{z^n}{[n]_{p,q}!}$$

$$=\left(\frac{z^m}{e_{p,q}(z)-T_{m-1}^{p,q}(z)}\right)^{\alpha}e_{p,q}(xz)E_{p,q}(yz)\left(\frac{2^m}{e_{p,q}(z)+T_{m-1}^{p,q}(z)}\right)^{\alpha}e_{p,q}(xz)\left(\frac{2^m}{e_{p,q}(z)+T_{m-1}^{p,q}(z)}\right)^{-\alpha}$$

$$=\sum_{n=0}^{\infty}\mathcal{B}_n^{[m-1,\alpha]}(0,y:p,q)\frac{z^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}\mathcal{E}_n^{[m-1,\alpha]}(x,0:p,q)\frac{z^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}\mathcal{E}_n^{[m-1,-\alpha]}(p,q)\frac{z^n}{[n]_{p,q}!}$$

$$=\sum_{n=0}^{\infty}\mathcal{B}_n^{[m-1,\alpha]}(0,y:p,q)\frac{z^n}{[n]_{p,q}!}\left(\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_{p,q}\mathcal{E}_l^{[m-1,\alpha]}(x,0:p,q)\mathcal{E}_{n-l}^{[m-1,-\alpha]}(p,q)\right)\frac{z^n}{[n]_{p,q}!}\right)$$

$$=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_{p,q}\mathcal{B}_{n-l}^{[m-1,\alpha]}(0,y:p,q)\sum_{k=0}^{l}\begin{bmatrix} l \\ k \end{bmatrix}_{p,q}\mathcal{E}_k^{[m-1,\alpha]}(x,0:p,q)\mathcal{E}_{l-k}^{[m-1,-\alpha]}(p,q)\right)\frac{z^n}{[n]_{p,q}!}.$$

DOI: 10.3844/jmssp.2018.129.140

Checking against the coefficients of $z^n/[n]_{p,q}!$, then we have desired result in the first equation. The others in this theorem can be proved in a like manner.

We give some relations between the new and old (p, q)-polynomials as follows.

Theorem 3

For $n \in \mathbb{N}_0$ and $x,y,\ \alpha \in \mathbb{C}$, the following relations holds true:

$$\mathcal{B}_n^{[m-1,\alpha]}(x,y:p,q)$$
$$=\frac{1}{[n+1]_{p,q}}\sum_{u=0}^{n+1}\begin{bmatrix} n+1 \\ u \end{bmatrix}_{p,q}\mathcal{B}_{n-u+1}(0,ly:p,q)l^{u-n}$$
$$\times\left(\sum_{s=0}^{u}\begin{bmatrix} u \\ s \end{bmatrix}_{p,q}\mathcal{B}_s^{[m-1,\alpha]}(x,0:p,q)l^{s-u}p^{\binom{u-s}{2}}-\mathcal{B}_u^{[m-1,\alpha]}(x,0:p,q)\right),$$
$$\mathcal{B}_n^{[m-1,\alpha]}(x,y:p,q)$$
$$=\frac{1}{[n+1]_{p,q}}\sum_{u=0}^{n+1}\begin{bmatrix} n+1 \\ u \end{bmatrix}_{p,q}\mathcal{B}_{n-u+1}(lx,0:p,q)l^{u-n}$$
$$\times\left(\sum_{s=0}^{u}\begin{bmatrix} u \\ s \end{bmatrix}_{p,q}\mathcal{B}_s^{[m-1,\alpha]}(0,y:p,q)l^{s-u}p^{\binom{u-s}{2}}-\mathcal{B}_n^{[m-1,\alpha]}(0,y:p,q)\right),$$
$$\mathcal{E}_n^{[m-1,\alpha]}(x,y:p,q)=\frac{1}{[2]_{p,q}}\sum_{u=0}^{n}\begin{bmatrix} n \\ u \end{bmatrix}_{p,q}\mathcal{E}_{n-u}(0,ly:p,q)l^{u-n}$$
$$\times\left(\sum_{s=0}^{u}\begin{bmatrix} u \\ s \end{bmatrix}_{p,q}\mathcal{E}_s^{[m-1,\alpha]}(x,0:p,q)l^{s-u}p^{\binom{u-s}{2}}-\mathcal{E}_n^{[m-1,\alpha]}(x,0:p,q)\right),$$
$$\mathcal{E}_n^{[m-1,\alpha]}(x,y:p,q)=\frac{1}{[2]_{p,q}}\sum_{u=0}^{n}\begin{bmatrix} n \\ u \end{bmatrix}_{p,q}\mathcal{E}_{n-u}(lx,0:p,q)l^{u-n}$$
$$\times\left(\sum_{s=0}^{u}\begin{bmatrix} u \\ s \end{bmatrix}_{p,q}\mathcal{E}_s^{[m-1,\alpha]}(0,y:p,q)l^{s-u}p^{\binom{u-s}{2}}-\mathcal{E}_n^{[m-1,\alpha]}(0,y:p,q)\right),$$
$$\mathcal{G}_n^{[m-1,\alpha]}(x,y:p,q)$$
$$=\frac{1}{[2]_{p,q}[n+1]_{p,q}}\sum_{u=0}^{n+1}\begin{bmatrix} n+1 \\ u \end{bmatrix}_{p,q}\mathcal{G}_{n-u+1}(0,ly:p,q)l^{u-n}$$
$$\times\left(\sum_{s=0}^{u}\begin{bmatrix} u \\ s \end{bmatrix}_{p,q}\mathcal{G}_s^{[m-1,\alpha]}(x,0:p,q)l^{s-u}p^{\binom{u-s}{2}}-\mathcal{G}_n^{[m-1,\alpha]}(x,0:p,q)\right), \tag{2.14}$$
$$\mathcal{G}_n^{[m-1,\alpha]}(x,y:p,q)$$
$$=\frac{1}{[2]_{p,q}[n+1]_{p,q}}\sum_{u=0}^{n+1}\begin{bmatrix} n+1 \\ u \end{bmatrix}_{p,q}\mathcal{G}_{n-u+1}(lx,0:p,q)l^{u-n}$$
$$\times\left(\sum_{s=0}^{u}\begin{bmatrix} u \\ s \end{bmatrix}_{p,q}\mathcal{G}_s^{[m-1,\alpha]}(0,y:p,q)l^{s-u}p^{\binom{u-s}{2}}-\mathcal{G}_n^{[m-1,\alpha]}(0,y:p,q)\right)$$

where $\mathcal{B}_n(x, y: p, q)$, $\mathcal{E}_n(x, y: p, q)$ and $\mathcal{G}_n(x, y: p, q)$ are defined in (Duran *et al.*, 2016b), called (p, q)-Bernoulli polynomials, (p, q)-Euler polynomials and (p, q)-Genocchi polynomials, respectively.

Proof

Indeed:

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{[m-1,\alpha]}(x,y:p,q)\frac{z^n}{[n]_{p,q}!}$$
$$=\left(\frac{2^m z^m}{e_{p,q}(z)+T_{m-1}^{p,q}(z)}\right)^{\alpha}e_{p,q}(xz)\frac{e_{p,q}\left(\frac{z}{l}\right)+1}{\frac{z}{l}[2]_{p,q}}E_{p,q}\left(ly\frac{z}{l}\right)\frac{\frac{z}{l}[2]_{p,q}}{e_{p,q}\left(\frac{z}{l}\right)+1}$$
$$=\frac{l}{[2]_{p,q}z}\left[\sum_{n=0}^{\infty}\mathcal{G}_n^{[m-1,\alpha]}(x,0:p,q)\frac{z^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}p^{\binom{n}{2}}l^{-n}\frac{z^n}{[n]_{p,q}!}\right.$$
$$\left.+\sum_{n=0}^{\infty}\mathcal{G}_n^{[m-1,\alpha]}(x,0:p,q)\frac{z^n}{[n]_{p,q}!}\right]\sum_{n=0}^{\infty}\mathcal{G}_n(0,ly:p,q)\frac{1}{l^n}\frac{z^n}{[n]_{p,q}!}$$
$$=\frac{l}{[2]_{p,q}}\sum_{n=0}^{\infty}\sum_{u=0}^{n}\begin{bmatrix} n \\ u \end{bmatrix}_{p,q}$$
$$\left\{\sum_{s=0}^{u}\begin{bmatrix} u \\ s \end{bmatrix}_{p,q}\mathcal{G}_s^{[m-1,\alpha]}(x,0:p,q)l^{s-u}p^{\binom{u-s}{2}}+\mathcal{G}_u^{[m-1,\alpha]}(x,0:p,q)\right\}$$
$$\times\mathcal{G}_{n-u}(0,ly:p,q)l^{u-n}\frac{z^{n-1}}{[n]_{p,q}!}.$$

Comparing the coefficients of $z^n/[n]_{p,q}!$, we get desired result for (2.14). The others in this theorem can be proved in a like manner.

Here we present new correlations including the polynomials $\mathcal{B}_n^{[m-1,\alpha]}(x, y: p, q)$, $\mathcal{E}_n^{[m-1,\alpha]}(x, y: p, q)$ and $\mathcal{G}_n^{[m-1,\alpha]}(x, y: p, q)$ by the following theorem.

Theorem 4

For $n \in \mathbb{N}_0$, $m \in \mathbb{N}$ and $x,\ y,\ \alpha \in \mathbb{C}$, the following correlations hold true:

$$\mathcal{B}_n^{[m-1,\alpha]}(x,y:p,q)=\frac{1}{[2]_{p,q}}\sum_{u=0}^{n}\begin{bmatrix} n \\ u \end{bmatrix}_{p,q}\mathcal{E}_{n-u}(0,ly:p,q)l^{u-n}$$
$$\times\left(\sum_{s=0}^{u}\begin{bmatrix} u \\ s \end{bmatrix}_{p,q}\mathcal{B}_s^{[m-1,\alpha]}(x,0:p,q)l^{s-u}p^{\binom{u-s}{2}}+\mathcal{B}_u^{[m-1,\alpha]}(x,0:p,q)\right),$$
$$\mathcal{B}_n^{[m-1,\alpha]}(x,y:p,q)=\frac{1}{[2]_{p,q}[n+1]_{p,q}}\sum_{u=0}^{n+1}\begin{bmatrix} n+1 \\ u \end{bmatrix}_{p,q}\mathcal{G}_{n-u+1}(0,ly:p,q)l^{u-n}$$
$$\times\left(\sum_{s=0}^{u}\begin{bmatrix} u \\ s \end{bmatrix}_{p,q}\mathcal{B}_s^{[m-1,\alpha]}(x,0:p,q)l^{s-u}p^{\binom{u-s}{2}}+\mathcal{B}_u^{[m-1,\alpha]}(x,0:p,q)\right),$$
$$\mathcal{E}_n^{[m-1,\alpha]}(x,y:p,q)=\frac{1}{[n+1]_{p,q}}\sum_{u=0}^{n+1}\begin{bmatrix} n+1 \\ u \end{bmatrix}_{p,q}\mathcal{B}_{n-u+1}(0,ly:p,q)l^{u-n}$$
$$\times\left(\sum_{s=0}^{u}\begin{bmatrix} u \\ s \end{bmatrix}_{p,q}\mathcal{E}_s^{[m-1,\alpha]}(x,0:p,q)l^{s-u}p^{\binom{u-s}{2}}+\mathcal{E}_u^{[m-1,\alpha]}(x,0:p,q)\right),$$

$$\mathcal{E}_n^{[m-1,\alpha]}(x,y:p,q)$$
$$=\frac{1}{[2]_{p,q}[n+1]_{p,q}}\sum_{u=0}^{n+1}\begin{bmatrix} n+1 \\ u \end{bmatrix}_{p,q}\mathcal{G}_{n-u+1}(0,ly:p,q)l^{u-n}$$
$$\times\left(\sum_{s=0}^{u}\begin{bmatrix} u \\ s \end{bmatrix}_{p,q}\mathcal{E}_s^{[m-1,\alpha]}(x,0:p,q)l^{s-u}p^{\binom{u-s}{2}}+\mathcal{E}_u^{[m-1,\alpha]}(x,0:p,q)\right),$$
$$\mathcal{G}_n^{[m-1,\alpha]}(x,y:p,q)$$
$$=\frac{1}{[n+1]_{p,q}}\sum_{u=0}^{n+1}\begin{bmatrix} n+1 \\ u \end{bmatrix}_{p,q}\mathcal{B}_{n-u+1}(0,ly:p,q)l^{u-n} \tag{2.15}$$
$$\times\left(\sum_{s=0}^{u}\begin{bmatrix} u \\ s \end{bmatrix}_{p,q}\mathcal{G}_s^{[m-1,\alpha]}(x,0:p,q)l^{s-u}p^{\binom{u-s}{2}}+\mathcal{G}_u^{[m-1,\alpha]}(x,0:p,q)\right),$$
$$\mathcal{G}_n^{[m-1,\alpha]}(x,y:p,q)=\frac{1}{[2]_{p,q}}\sum_{u=0}^{n}\begin{bmatrix} n \\ u \end{bmatrix}_{p,q}\mathcal{E}_{n-u}(0,ly:p,q)l^{u-n}$$
$$\times\left(\sum_{s=0}^{u}\begin{bmatrix} u \\ s \end{bmatrix}_{p,q}\mathcal{G}_s^{[m-1,\alpha]}(x,0:p,q)l^{s-u}p^{\binom{u-s}{2}}+\mathcal{G}_u^{[m-1,\alpha]}(x,0:p,q)\right).$$

Proof

We consider:

$$\sum_{n=0}^{\infty}\mathcal{B}_n^{[m-1,\alpha]}(x,y:p,q)\frac{z^n}{[n]_{p,q}!}$$
$$=\left(\frac{z^m}{e_{p,q}(z)-T_{m-1}^{p,q}(z)}\right)^{\alpha}$$
$$e_{p,q}(xz)\frac{e_{p,q}\left(\frac{z}{l}\right)+1}{[2]_{p,q}\frac{z}{l}}E_{p,q}\left(yl\frac{z}{l}\right)\frac{\frac{z}{l}[2]_{p,q}}{e_{p,q}\left(\frac{z}{l}\right)+1}$$
$$=\frac{l}{[2]_{p,q}z}\left[\sum_{n=0}^{\infty}\mathcal{B}_n^{[m-1,\alpha]}(x,0:p,q)\frac{z^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}p^{\binom{n}{2}}l^{-n}\frac{z^n}{[n]_{p,q}!}\right.$$
$$\left.+\sum_{n=0}^{\infty}\mathcal{B}_n^{[m-1,\alpha]}(x,0:p,q)\frac{z^n}{[n]_{p,q}!}\right]\sum_{n=0}^{\infty}\mathcal{G}_n(0,ly:p,q)\frac{1}{l^n}\frac{z^n}{[n]_{p,q}!}$$
$$=\frac{l}{[2]_{p,q}}\sum_{n=0}^{\infty}\sum_{u=0}^{n}\begin{bmatrix} n \\ u \end{bmatrix}_{p,q}$$
$$\left\{\sum_{s=0}^{u}\binom{u}{s}\mathcal{B}_s^{[m-1,\alpha]}(x,0:p,q)l^{s-u}p^{\binom{u-s}{2}}+\mathcal{B}_u^{[m-1,\alpha]}(x,0:p,q)\right\}$$
$$+\mathcal{G}_{n-u}(0,ly:p,q)l^{u-n}\frac{z^{n-1}}{[n]_{p,q}!}.$$

Equating the coefficients of $\frac{z^n}{[n]_{p,q}!}$, we get desired result for (2.15). The others in this theorem can be proved in a like manner.

Here we give the following theorem.

Theorem 5

We have:

$$\mathcal{B}_n^{[m-1,\alpha]}(x,y:p,q)=\frac{1}{[2]_{p,q}}\sum_{u=0}^{n}\begin{bmatrix} n \\ u \end{bmatrix}_{p,q}\mathcal{E}_{n-u}(lx,0:p,q)l^{u-n}$$
$$\times\left(\sum_{s=0}^{u}\begin{bmatrix} u \\ s \end{bmatrix}_{p,q}\mathcal{B}_s^{[m-1,\alpha]}(0,y:p,q)l^{s-u}p^{\binom{u-s}{2}}+\mathcal{B}_u^{[m-1,\alpha]}(0,y:p,q)\right),$$
$$\mathcal{B}_n^{[m-1,\alpha]}(x,y:p,q)$$
$$=\frac{1}{[2]_{p,q}[n+1]_{p,q}}\sum_{u=0}^{n+1}\begin{bmatrix} n+1 \\ u \end{bmatrix}_{p,q}\mathcal{G}_{n-u+1}(lx,0:p,q)l^{u-n}$$
$$\times\left(\sum_{s=0}^{u}\begin{bmatrix} u \\ s \end{bmatrix}_{p,q}\mathcal{B}_s^{[m-1,\alpha]}(0,y:p,q)l^{s-u}p^{\binom{u-s}{2}}+\mathcal{B}_u^{[m-1,\alpha]}(0,y:p,q)\right),$$
$$\mathcal{E}_n^{[m-1,\alpha]}(x,y:p,q)=\frac{1}{[n+1]_{p,q}}\sum_{u=0}^{n+1}\begin{bmatrix} n+1 \\ u \end{bmatrix}_{p,q}\mathcal{B}_{n-u+1}(lx,0:p,q)l^{u-n}$$
$$\times\left(\sum_{s=0}^{u}\begin{bmatrix} u \\ s \end{bmatrix}_{p,q}\mathcal{E}_s^{[m-1,\alpha]}(0,y:p,q)l^{s-u}p^{\binom{u-s}{2}}-\mathcal{E}_u^{[m-1,\alpha]}(0,y:p,q)\right),$$
$$\mathcal{E}_n^{[m-1,\alpha]}(x,y:p,q)$$
$$=\frac{1}{[2]_{p,q}[n+1]_{p,q}}\sum_{u=0}^{n+1}\begin{bmatrix} n+1 \\ u \end{bmatrix}_{p,q}\mathcal{G}_{n-u+1}(lx,0:p,q)l^{u-n}$$
$$\times\left(\sum_{s=0}^{u}\begin{bmatrix} u \\ s \end{bmatrix}_{p,q}\mathcal{E}_s^{[m-1,\alpha]}(0,y:p,q)l^{s-u}p^{\binom{u-s}{2}}+\mathcal{E}_u^{[m-1,\alpha]}(0,y:p,q)\right),$$
$$\mathcal{G}_n^{[m-1,\alpha]}(x,y:p,q)=\frac{1}{[n+1]_{p,q}}\sum_{u=0}^{n+1}\begin{bmatrix} n+1 \\ u \end{bmatrix}_{p,q}\mathcal{B}_{n-u+1}(lx,0:p,q)l^{u-n}$$
$$\times\left(\sum_{s=0}^{u}\begin{bmatrix} u \\ s \end{bmatrix}_{p,q}\mathcal{G}_s^{[m-1,\alpha]}(0,y:p,q)l^{s-u}p^{\binom{u-s}{2}}-\mathcal{G}_u^{[m-1,\alpha]}(0,y:p,q)\right),$$
$$\mathcal{G}_n^{[m-1,\alpha]}(x,y:p,q)=\frac{1}{[2]_{p,q}}\sum_{u=0}^{n}\begin{bmatrix} n \\ u \end{bmatrix}_{p,q}\mathcal{E}_{n-u}(lx,0:p,q)l^{u-n}$$
$$\times\left(\sum_{s=0}^{u}\begin{bmatrix} u \\ s \end{bmatrix}_{p,q}\mathcal{G}_s^{[m-1,\alpha]}(0,y:p,q)l^{s-u}p^{\binom{u-s}{2}}+\mathcal{G}_u^{[m-1,\alpha]}(0,y:p,q)\right).$$

Proof

The proof of this theorem can be easily completed by using the same proof method in the proof of Theorem 4. So we omit them.

From Corollary 1 and Theorem 5, we have the following Corollary 3.

Corollary 3

We have:

$$\mathcal{B}_n^{[m-1,\alpha]}(x,y:p,q)=\frac{1}{[2]_{p,q}}\sum_{u=0}^{n}\begin{bmatrix}n\\u\end{bmatrix}_{p,q}l^{u-n}\mathcal{E}_{n-u}(lx,0:p,q)$$
$$\times\left(\mathcal{B}_u^{[m-1,\alpha]}\left(\frac{1}{l},y:p,q\right)+\mathcal{B}_u^{[m-1,\alpha]}(0,y:p,q)\right),$$
$$\mathcal{B}_n^{[m-1,\alpha]}(x,y:p,q)$$
$$=\frac{1}{[2]_{p,q}[n+1]_{p,q}}\sum_{u=0}^{n+1}\begin{bmatrix}n+1\\u\end{bmatrix}_{p,q}l^{u-n}\mathcal{G}_{n-u+1}(lx,0:p,q)$$
$$\times\left(\mathcal{B}_u^{[m-1,\alpha]}\left(\frac{1}{l},y:p,q\right)+\mathcal{B}_u^{[m-1,\alpha]}(0,y:p,q)\right),$$
$$\mathcal{E}_n^{[m-1,\alpha]}(x,y:p,q)$$
$$=\frac{1}{[n+1]_{p,q}}\sum_{u=0}^{n+1}\begin{bmatrix}n+1\\u\end{bmatrix}_{p,q}l^{u-n}\mathcal{B}_{n-u+1}(lx,0:p,q)$$
$$\times\left(\mathcal{E}_u^{[m-1,\alpha]}\left(\frac{1}{l},y:p,q\right)-\mathcal{E}_u^{[m-1,\alpha]}(0,y:p,q)\right),$$
$$\mathcal{E}_n^{[m-1,\alpha]}(x,y:p,q)$$
$$=\frac{1}{[2]_{p,q}[n+1]_{p,q}}\sum_{u=0}^{n+1}\begin{bmatrix}n+1\\u\end{bmatrix}_{p,q}l^{u-n}\mathcal{G}_{n-u+1}(lx,0:p,q)$$
$$\times\left(\mathcal{E}_u^{[m-1,\alpha]}\left(\frac{1}{l},y:p,q\right)+\mathcal{E}_u^{[m-1,\alpha]}(0,y:p,q)\right),$$
$$\mathcal{G}_n^{[m-1,\alpha]}(x,y:p,q)$$
$$=\frac{1}{[n+1]_{p,q}}\sum_{u=0}^{n+1}\begin{bmatrix}n+1\\u\end{bmatrix}_{p,q}l^{u-n}\mathcal{B}_{n-u+1}(lx,0:p,q)$$
$$\times\left(\mathcal{G}_u^{[m-1,\alpha]}\left(\frac{1}{l},y:p,q\right)-\mathcal{G}_u^{[m-1,\alpha]}(0,y:p,q)\right),\tag{2.16}$$
$$\mathcal{G}_n^{[m-1,\alpha]}(x,y:p,q)$$
$$=\frac{1}{[2]_{p,q}}\sum_{u=0}^{n}\begin{bmatrix}n\\u\end{bmatrix}_{p,q}l^{u-n}\mathcal{E}_{n-u}(lx,0:p,q)$$
$$\times\left(\mathcal{G}_u^{[m-1,\alpha]}\left(\frac{1}{l},y:p,q\right)\mathcal{G}_u^{[m-1,\alpha]}(0,y:p,q)\right).$$

Theorem 1 and Corollary 3 yield to the following results as stated in Theorem 6.

Theorem 6

For $n\in\mathbb{N}_0$ and $\alpha\in\mathbb{C}$, the following relationships are valid:

$$\mathcal{B}_n^{[m-1,\alpha]}(x,y:p,q)=\frac{1}{[2]_{p,q}}\sum_{u=0}^{n}\begin{bmatrix}n\\u\end{bmatrix}_{p,q}\left([u]_{p,q}\sum_{k=0}^{u-1}\begin{bmatrix}u-1\\k\end{bmatrix}_{p,q}\mathcal{B}_k^{[m-1,\alpha]}(0,y:p,q)\mathcal{B}_{u-1-k}^{[0,-1]}(p,q)\right.$$
$$\left.+\sum_{k=0}^{\min(u,m-1)}\begin{bmatrix}u\\k\end{bmatrix}_{p,q}\mathcal{B}_{u-k}^{[m-1,\alpha]}(0,y:p,q)+\mathcal{B}_u^{[m-1,\alpha]}(0,y:p,q)\right)\mathcal{E}_{n-u}(x,0:p,q),$$
$$\mathcal{B}_n^{[m-1,\alpha]}(x,y:p,q)=\frac{1}{[2]_{p,q}[n+1]_{p,q}}\sum_{u=0}^{n+1}\begin{bmatrix}n+1\\u\end{bmatrix}_{p,q}\left([u]_{p,q}\sum_{k=0}^{u-1}\begin{bmatrix}u-1\\k\end{bmatrix}_{p,q}\mathcal{B}_k^{[m-1,\alpha]}(0,y:p,q)\mathcal{B}_{u-1-k}^{[0,-1]}(p,q)\right.$$
$$\left.+\sum_{k=0}^{\min(u,m-1)}\begin{bmatrix}u\\k\end{bmatrix}_{p,q}\mathcal{B}_{u-k}^{[m-1,\alpha]}(0,y:p,q)+\mathcal{B}_u^{[m-1,\alpha]}(0,y:p,q)\right)\mathcal{G}_{n-u+1}(x,0:p,q),$$
$$\mathcal{E}_n^{[m-1,\alpha]}(x,y:p,q)=\frac{1}{[n+1]_{p,q}}\sum_{u=0}^{n+1}\begin{bmatrix}n+1\\u\end{bmatrix}_{p,q}\left(2\sum_{k=0}^{u}\begin{bmatrix}u\\k\end{bmatrix}\mathcal{E}_k^{[m-1,\alpha-1]}(0,y:p,q)\mathcal{E}_{u-k}^{[0,-1]}(p,q)\right.$$
$$\left.-\sum_{k=0}^{\min(u,m-1)}\begin{bmatrix}u\\k\end{bmatrix}_{p,q}\mathcal{E}_{u-k}^{[m-1,\alpha]}(0,y:p,q)-\mathcal{E}_u^{[m-1,\alpha]}(0,y:p,q)\right)\mathcal{B}_{n-u+1}(x,0:p,q),$$
$$\mathcal{E}_n^{[m-1,\alpha]}(x,y:p,q)=\frac{1}{[2]_{p,q}[n+1]_{p,q}}\sum_{u=0}^{n+1}\begin{bmatrix}n+1\\u\end{bmatrix}_{p,q}\left(2\sum_{k=0}^{u}\begin{bmatrix}u\\k\end{bmatrix}\mathcal{E}_k^{[m-1,\alpha-1]}(0,y:p,q)\mathcal{E}_{u-k}^{[0,-1]}(p,q)\right.$$
$$\left.-\sum_{k=0}^{\min(u,m-1)}\begin{bmatrix}u\\k\end{bmatrix}_{p,q}\mathcal{E}_{u-k}^{[m-1,\alpha]}(0,y:p,q)+\mathcal{E}_u^{[m-1,\alpha]}(0,y:p,q)\right)\mathcal{G}_{n-u+1}(x,0:p,q),$$
$$\mathcal{G}_n^{[m-1,\alpha]}(x,y:p,q)=\frac{1}{[n+1]_{p,q}}\sum_{u=0}^{n+1}\begin{bmatrix}n+1\\u\end{bmatrix}_{p,q}\left(2[u]_{p,q}\sum_{k=0}^{u-1}\begin{bmatrix}u-1\\k\end{bmatrix}_{p,q}\mathcal{G}_k^{[m-1,\alpha-1]}(0,y:p,q)\mathcal{G}_{u-1-k}^{[0,-1]}(p,q)\right.$$
$$\left.-\sum_{k=0}^{\min(u,m-1)}\begin{bmatrix}u\\k\end{bmatrix}_{p,q}\mathcal{G}_{u-k}^{[m-1,\alpha]}(0,y:p,q)+\mathcal{G}_u^{[m-1,\alpha]}(0,y:p,q)\right)\mathcal{B}_{n-u+1}(x,0:p,q),\tag{2.17}$$
$$\mathcal{G}_n^{[m-1,\alpha]}(x,y:p,q)=\frac{1}{[2]_{p,q}}\sum_{u=0}^{n}\begin{bmatrix}n\\u\end{bmatrix}_{p,q}\left(2[u]_{p,q}\sum_{k=0}^{u-1}\begin{bmatrix}u-1\\k\end{bmatrix}_{p,q}\mathcal{G}_k^{[m-1,\alpha]}(0,y:p,q)\mathcal{G}_{u-1-k}^{[0,-1]}(p,q)\right.$$
$$\left.-\sum_{k=0}^{\min(u,m-1)}\begin{bmatrix}u\\k\end{bmatrix}_{p,q}\mathcal{G}_{u-k}^{[m-1,\alpha]}(0,y:p,q)+\mathcal{G}_u^{[m-1,\alpha]}(0,y:p,q)\right)\mathcal{E}_{n-u}(x,0:p,q).$$

The Equation 2.17 is a (p, q)-extension of the Srivastava-Pintér addition theorem for generalized Bernoulli and Euler polynomials of order α given by:

$$
\begin{aligned}
&\mathcal{B}_n^{[m-1,\alpha]}(x+y) \\
&= \frac{1}{2}\sum_{u=0}^{n}\begin{bmatrix} n \\ u \end{bmatrix}_{p,q}\left[\mathcal{B}_u^{[m-1,\alpha]}(y)\sum_{k=0}^{\min(u,m-1)}\begin{bmatrix} u \\ k \end{bmatrix}\mathcal{B}_{u-k}^{[m-1,\alpha]}(y)\right. \\
&\left.\times u\sum_{k=0}^{u-1}\begin{bmatrix} u-1 \\ k \end{bmatrix}\mathcal{B}_k^{[m-1,\alpha]}(y)\right]\mathcal{E}_{n-u}(x)
\end{aligned}
\tag{2.18}
$$

which is obtained just by substituting $\lambda = 1$ in Theorem 3 derived by Tremblay *et al*. (2011).

Conclusion

We have introduced new generalizations of Bernoulli polynomials, Euler polynomials and Genocchi polynomials which are called generalized (p, q)-Bernoulli polynomials, (p, q)-Euler polynomials and (p, q)-Genocchi polynomials of order α. We have examined their several properties and relationships including additions theorems, difference equations, differential relations, recurrence relationships and so on. Also, we have given the (p, q)-extension of the formula of Srivastava and Pintér (2004). The results derived in this paper reduce to known properties of generalized q-polynomials of order α when $p = 1$, (Mahmudov and Keleshteri, 2014; 2013). Also, in the case $q \to p = 1$, our results in this paper reduce to ordinary results for the generalized Bernoulli, Euler and Genocchi polynomials of order α.

Acknowledgement

The authors would like to thank the reviewers for their valuable suggestions and comments, which have improved the paper's presentation substantially. Also, third author of this paper, Serkan Araci, is supported by the Research Fund of *Hasan Kalyoncu University* in 2018.

Competing Interests

The authors declare that they have no competing interests.

Authors Contributions

The authors declare that the work was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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