

A Computational Method Based on Bernstein Polynomials for Solving FredholmIntegro-Differential Equations under Mixed Conditions

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Abstract: In this study, a computational method for solving linear FredholmIntegro-Differential Equation (FIDE) of the first order under the mixed conditions using the Bernstein polynomials. First, we present some properties of these polynomials and the method is explained. These properties are then used to convert the integro-differential equation to a system of linear algebraic equations with unknown Bernstein coefficients. Using Galerkin method, we give an approximate solution. This method seems very attractive and simple to use. Illustrative examples show the efficiency and validity of the method we discuss the results using error analysis, the results are discussed.

Keywords: Bernstein Polynomials, Linear FredholmIntegro-Differential Equations of the First Order, Mixed Conditions, Galerkin Method, Numerical Analysis, Error Estimates

Introduction

Integro-differential equations, which are composed by integral and differential equations, are a well-known mathematical tool and an important branch of modern mathematics. Among these equations, FredholmIntegro-Differential Equations (FIDE) are encountered in several areas such as biology, economics, engineering and many others, so as usual, there is no specific analytic method to solve this equations, several numerical methods are presented to approximate the solution of FIDEs. Recently, different methods based on basic functions have been proposed to approximate the solution of FIDE, such as orthogonal basis and wavelets. At the same time, various numerical methods take an important place in solving FIDE numerically, such as Legendre polynomials, which have been used for high-order linear FIDE (Yalçınbaş *et al.*, 2009), rationalized Haar functions and Walsh series, differential transform method (Golubov *et al.*, 1991) and many other known methods in the literature. Among these methods, the polynomials of Bernstein that have been widely used to solve both linear and non linear integro-differential equations. Very few paper using Bernstein polynomials to solve FIDE with mixed conditions have been considered in the literature.

The aim of this paper is to find the solution of the following integro-differential equation:

$$Q(x)y'(x) + R(x)y(x) + \lambda \int_0^1 K(x,t)y(t)dt = f(x), 0 \leq x \leq 1 \quad (1.1)$$

under the following mixed conditions:

$$\sum_{s=0}^m e_s r(r_s) = \mu \quad 0 \leq r_s \leq 1 \quad (1.2)$$

where, Q, R, f and K are continuous functions on the interval $[0,1]$. Equation 1.1 can be found in (IdreesBhattia and Bracken, 2007; Swarup, 2007).

This paper is organized as follows. In section 2, we introduce the properties of the Bernstein polynomials. In section 3, we present the numerical method to solve the problem (1.1-1.2). Illustrative examples are presented in section 4 and finally, in section 5 we give a brief conclusion.

Properties of Bernstein Polynomials

The Bernstein polynomials of degree n on $[a,b]$ are defined by:

$$b_{i,n}(x) = \binom{n}{i} \frac{(x-a)^i (b-x)^{n-i}}{(b-a)^n}, i = 0, 1, \dots, n \quad (2.1)$$

where:

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

In what follows, we give some basic properties of the Bernstein polynomials. Formore details, we refer to (Isik *et al.*, 2012; Boyer and Thiel, 2002):

$$b_{i,n}(x) = 0, \text{ if } i < 0 \text{ or } i > n \quad (2.2)$$

$$\sum_{i=0}^n b_{i,n}(x) = 1, \forall x \in [0, 1], i = 0, 1, \dots, n \quad (2.3)$$

$$b_{i,n}(a) = b_{i,n}(b) = 0, 1 \leq i \leq n-1 \quad (2.4)$$

In particular if $[a, b] = [0, 1]$, we obtain:

$$b_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}, 0 \leq x \leq 1, i = 0, 1, 2, \dots, n \quad (2.5)$$

The linear combination of the basic Bernstein polynomials is given by the following formula:

$$B(x) = \sum_{i=0}^n c_i b_{i,n}(x) \quad (2.6)$$

where, c_i are the Bernstein coefficients. We have also the following properties:

$$\begin{aligned} b_{i,n}(x) &\geq 0, x \in [0, 1] \text{ (positivity)} \\ b_{i,n}(x) &= b_{n-i,n}(1-x) \text{ (symmetry)} \end{aligned} \quad (2.7)$$

and the recurrent formula:

$$b_{i,n}(x) = \begin{cases} (1-x)b_{i,n-1}(x), & i = 0 \\ (1-x)b_{i,n-1}(x) + xb_{i-1,n-1}, & i = 1, \dots, n-1 \\ xb_{i-1,n-1}(x), & i = n \end{cases} \quad (2.8)$$

The derivatives of different degrees of Bernstein polynomials have an important role in the numerical solution of differential and integro-differential equations.

Lemma 2.1

The derivative of order $k \geq 1$ of the Bernstein polynomials is given by:

$$b_{i,n}^{(k)}(x) = \left(\prod_{s=0}^{s=k-1} (n-s) \right) \left(\sum_{m=0}^{m=k} (-1)^{m+2} \binom{k}{m} b(x)_{i-k+m, n-k} \right), (i \geq k \geq m) \quad (2.9)$$

$$b_{i,n}^{(k)}(x) = 0 \text{ if } n < k \quad (2.10)$$

Proof

We prove the relation by induction. We check the property for $k = 1$:

$$b'_{i,n}(x) = \left(\prod_{s=0}^{s=0} (n-s) \right) \left(\sum_{m=0}^{m=1} (-1)^{m+2} \binom{1}{m} b(x)_{i-1+m, n-1} \right) \quad (2.11)$$

$$b'_{i,n}(x) = n \left(\binom{1}{0} b_{i-1, n-1}(x) - \binom{1}{1} b_{i, n-1}(x) \right)$$

$$b'_{i,n}(x) = n(b_{i-1, n-1}(x) - b_{i, n-1}(x)) \quad (2.12)$$

Thus the relation holds for $k = 1$.

It is assumed that the relation is true for $k \in \mathbb{N}^*$ and we should prove it for $(k + 1)$.

Indeed, we have:

$$b_{i,n}^{(k+1)}(x) = [b_{i,n}^{(k)}(x)]' \quad (2.13)$$

$$b_{i,n}^{(k)}(x) = \left(\prod_{s=0}^{s=k-1} (n-s) \right) \left(\begin{aligned} &b_{i-k, n-k}(x) - kb_{i-k+1, n-k}(x) \\ &+ \frac{k(k-1)}{2!} b_{i-k+2, n-k}(x) \\ &+ \dots + (-1)^{k+2} b(x)_{i, n-k}(x) \end{aligned} \right) \quad (2.14)$$

$$\begin{aligned} &[b_{i,n}^{(k)}(x)]' \\ &= \left(\prod_{s=0}^{s=k-1} (n-s) \right) \left(\begin{aligned} &b'_{i-k, n-k}(x) - kb'_{i-k+1, n-k}(x) \\ &+ \frac{k(k-1)}{2!} b'_{i-k+2, n-k}(x) \\ &+ \dots + (-1)^{k+2} b'(x)_{i, n-k}(x) \end{aligned} \right) \end{aligned} \quad (2.15)$$

According to the above relations, we have:

$$\begin{aligned} b'_{i-k, n-k}(x) &= (n-k) \left(b_{i-k-1, n-k-1}(x) - b_{i-k, n-k-1}(x) \right) \\ b'_{i-k+1, n-k}(x) &= (n-k) \left(b_{i-k, n-k-1}(x) - b_{i-k+1, n-k-1}(x) \right) \\ &\vdots \\ b'_{i, n-k}(x) &= (n-k) \left(b_{i-1, n-k-1}(x) - b_{i, n-k-1}(x) \right) \end{aligned} \quad (2.16)$$

Substituting (2.16) in the relation (2.15), we obtain:

$$\left[b_{i,n}^{(k)}(x) \right]' = \prod_{s=0}^{s=k-1} (n-s)(n-k) \left(b(x)_{i-(k+1),n-k-1} - (k+1)b(x)_{i-k,n-(k+1)} + \frac{k(k-1)}{2!} b(x)_{i-(k+1),n-(k+1)} + \dots + (-1)^{k+2} b(x)_{i,n-(k+1)} \right) \quad (2.17)$$

It is easy to check that:

$$\left(\prod_{s=0}^{s=k-1} (n-s) \right) (n-k) = \left(\prod_{s=0}^{s=k} (n-s) \right) \quad (2.18)$$

$$1 = \binom{k+1}{0}, \binom{k+1}{1} = \binom{k+1}{1}, \frac{k(k-1)}{2!} \\ = \binom{k+1}{2}, \dots, 1 = \binom{k+1}{k+1} \quad (2.19)$$

By substituting (2.18) and (2.19) in the relation (2.17), we obtain:

$$\left[b_{i,n}^{(k)}(x) \right]' = \left(\prod_{s=0}^{s=k-1} (n-s) \right) \left(b(x)_{i-(k+1),n-k-1} - (k+1)b(x)_{i-k,n-(k+1)} \right) \quad (2.20)$$

$$+ \frac{k(k-1)}{2!} b(x)_{i-(k+1),n-(k+1)} + \dots + \left((-1)^{m+2} b(x)_{i,n-(k+1)} \right) \quad (2.21)$$

$$= \left(\prod_{s=0}^{s=k} (n-s) \right) \left(\binom{k+1}{0} b(x)_{i-(k+1),n-k-1} - \binom{k+1}{1} b(x)_{i-k,n-(k+1)} \right) \quad (2.22)$$

$$+ \binom{k+1}{2} b(x)_{i-(k+1),n-(k+1)} + \dots + \left((-1)^{m+2} \binom{k+1}{k+1} b(x)_{i,n-(k+1)} \right) \quad (2.23)$$

$$= \left(\prod_{s=0}^{s=k} (n-s) \right) \left(\sum_{m=0}^{m=k+1} (-1)^{m+2} \binom{k+1}{m} b(x)_{i-(k+1)+m,n-(k-1)} \right) \quad (2.24)$$

$$= b_{i,n}^{(k+1)}(x)$$

which proves that the relation is true for all k .

Approximation Function

A function f square integrable in the interval $[0, 1]$ can be written as:

$$f(t) = \lim_{n \rightarrow \infty} \sum_{i=0}^n c_m b_m(t) \quad (2.25)$$

where, $c_{im} = \langle f, b_{im} \rangle$ and $\langle \cdot, \cdot \rangle$ is the standard inner product on $L^2 [0, 1]$.

If the series (2.25) are truncated at $n = m$, then we have (Isik *et al.*, 2012):

$$f(t) = \sum_{i=0}^m c_{i,m} b_{i,m}(t) = C^T B(t) \quad (2.26)$$

where, C and B are $(m+1) \times 1$ matrices and are given by:

$$C = [c_{0,m}, c_{1,m}, \dots, c_{m,m}]^T \quad (2.27)$$

and:

$$B(t) = [b_{0,m}(t), b_{1,m}(t), \dots, b_{m,m}(t)]^T \quad (2.28)$$

Numerical Method

Consider the linear FIDE of the first order (1.1) with the mixed conditions (1.2). The approximation of $y(x)$ by the polynomials of Bernstein is given by:

$$y(x) = \sum_{i=0}^n a_i b_{i,n}(x) = C^T B(x) \quad (3.1)$$

$$C^T = (a_0, a_1, \dots, a_n), \quad B(x) = (b_{0,n}, b_{1,n}, \dots, b_{n,n})$$

where, $b_{i,n}(x)$ are the Bernstein polynomials of degree n defined in Equation 2.1 and a_i ($i = 0, 1, \dots, n$) are unknown parameters.

Derivating Equation 3.1, we get:

$$y'(x) = \sum_{i=0}^n n a_i (b_{i-1,n-1}(x) - b_{i,n-1}(x)) \quad (3.2)$$

Substituting (3.2) into (1.1), we get:

$$Q(x) \sum_{i=0}^n n a_i (b_{i-1,n-1}(x) - b_{i,n-1}(x)) \\ + R(x) \sum_{i=0}^n a_i b_{i,n}(x) + \lambda \int_0^1 K(x,t) \sum_{i=0}^n a_i b_{i,n}(t) dt = f(x) \quad (3.3)$$

which can be written, after simplification, as:

$$\left[R(x) b_{0,n}(x) - n Q(x) b_{0,n-1}(x) \right] a_0 \\ + \lambda \int_0^1 K(x,t) b_{0,n}(t) dt$$

$$+\sum_{i=1}^n a_i \left[\begin{array}{l} nQ(x)(b_{i-1,n-1}(x) - b_{i,n-1}(x)) \\ +R(x)b_{i,n}(x) + \lambda \int_0^1 K(x,t)b_{i,n}(t)dt \end{array} \right] = f(x) \quad (3.4)$$

Multiplying both sides in (3.4) by $b_{j,n}(x)$ and integrating with respect to x from 0 to 1 we obtain:

$$\int_0^1 \left[\begin{array}{l} R(x)b_{0,n}(x) - nQ(x)b_{0,n-1}(x) \\ + \lambda \int_0^1 K(x,t)b_{0,n}(t)dt \end{array} \right] a_0 b_{j,n}(x) dx \\ + \sum_{i=1}^n a_i \int_0^1 \left[\begin{array}{l} nQ(x)(b_{i-1,n-1}(x) - b_{i,n-1}(x)) \\ + R(x)b_{i,n}(x) + \lambda \int_0^1 K(x,t)b_{i,n}(t)dt \end{array} \right] b_{j,n}(x) dx \quad (3.5) \\ + \int_0^1 f(x)b_{j,n}(x) dx$$

which can be written by the following system:

$$a_0 \alpha_J + \sum_{i=1}^n a_i c_{i,J} = G_J \quad J = 0, 1, \dots, n$$

$$\alpha_J = \int_0^1 \left[\begin{array}{l} R(x)b_{0,n}(x) - nQ(x)b_{0,n-1}(x) \\ + \lambda \int_0^1 K(x,t)b_{0,n}(t)dt \end{array} \right] b_{j,n}(x) dx, \quad J = 0, 1, 2, 3, \dots, n \quad (3.6)$$

$$C_{i,J} = \int_0^1 \left[\begin{array}{l} nQ(x)(b_{i-1,n-1}(x) - b_{i,n-1}(x)) \\ + R(x)b_{i,n}(x) + \lambda \int_0^1 K(x,t)b_{i,n}(t)dt \end{array} \right] b_{j,n}(x) dx \quad (3.7)$$

$$G_J = \int_0^1 f(x)b_{j,n}(x) dx, \quad i = 0, 1, \dots, n, \quad J = 0, 1, \dots, n-1 \quad (3.8)$$

Condition (1.2) can be written in the following form:

$$\begin{aligned} \mu &= \sum_{s=0}^m e_s y(r_s) = \sum_{s=0}^m e_s \sum_{i=0}^n a_i b_{i,n}(r_s) \\ &= \sum_{s=0}^m e_s (a_0 b_{0,n}(r_s) + a_1 b_{1,n}(r_s) + \dots + a_n b_{n,n}(r_s)) \\ &= \left(\sum_{s=0}^m e_s b_{0,n}(r_s) \right) a_0 + \left(\sum_{s=0}^m e_s b_{1,n}(r_s) \right) a_1 \\ &+ \dots + \left(\sum_{s=0}^m e_s b_{n,n}(r_s) \right) a_n \end{aligned}$$

by setting:

$$w_k = \left(\sum_{s=0}^m e_s b_{k,n}(r_s) \right), \quad k = 0, \dots, n \quad (3.9)$$

we obtain:

$$\mu = w_0 a_0 + w_1 a_1 + \dots + w_n a_n$$

The general system can be obtained as follows:

$$\begin{cases} a_0 \alpha_J + \sum_{i=1}^n a_i c_{i,J} = G_J \quad J = 0, 1, 2, \dots, n \\ \sum_{i=1}^n a_i w_i = \mu \end{cases} \quad (3.10)$$

and the linear algebraic equation is given by:

$$\tilde{A}X = \tilde{b} \quad (3.11)$$

where:

$$\tilde{A} = \begin{pmatrix} \alpha_0 & C_{1,0} & \dots & C_{n,0} \\ \alpha_1 & C_{1,1} & \dots & C_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_0 & C_{1,n-1} & \dots & C_{n,n-1} \\ w_0 & w_1 & \dots & w_n \end{pmatrix}, \quad X = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \\ a_n \end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} G_0 \\ G_1 \\ \vdots \\ G_{n-1} \\ \mu \end{pmatrix}$$

The linear system corresponding to the solution of Equation 1.1 under the condition (1.2) is given by an approximate solution (Singh *et al.*, 2009):

$$\tilde{y} = a_0 b_{0,n}(x) + a_1 b_{1,n}(x) + \dots + a_n b_{n,n}(x) \quad (3.12)$$

Remark 3.1

The square error E is defined by the formula:

$$E = (y(x) - \tilde{y}(x))^2 \quad (3.13)$$

where, $y(x)$ and $\tilde{y}(x)$ are the exact and approximate solutions respectively.

Illustrative Examples

The following examples illustrate the efficiency of the used method. Note that in these examples, the series (3.1) are truncated at levels 4 and 5, i.e., $(n = 4)$ and $(n = 5)$.

Example 4.1

Consider the following linear FIDE equation of the first order with mixed conditions:

$$\begin{cases} x^3 y'(x) + e^x y(x) + 2 \int_0^1 e^{2(x+1)t - (x+1)} y(t) dt = g(x) \\ y(0) + y(1) = 3.0860 \quad 0 \leq x \leq 1 \end{cases} \quad (4.1)$$

$$g(x) = (2x^3 + e^x) e^{2x-1} + \frac{e^{(x+2)} - e^{-(x+2)}}{x+2}$$

The exact solution is given by:

$$y(x) = e^{2x-1}$$

By a comparison of Equation 1.1 and 1.2 with the Equation 4.1, we find:

$$Q(x) = x^3, R(x) = e^x, K(x, t) = e^{2(x+t)-(x+1)}, \lambda = 2$$

$$e_0 = e_1 = 1, r_0 = 0, r_1 = 1, \mu = 3.0860$$

By using the method described in section (4), Equation 4.1 is then solved and we have:

$$\alpha_J = \int_0^1 \left[\frac{e^x b_{0,n}(x) - nx^3 b_{0,n-1}(x)}{+2 \int_0^1 e^{2(x+t)-(x+1)} b_{0,n}(t) dt} \right] b_{J,n}(x) dx \quad (4.2)$$

$$C_{i,J} = \int_0^1 \left[\frac{nx^3 (b_{i-1,n-1}(x) - b_{i,n-1}(x))}{+e^x b_{i,n}(x) + 2 \int_0^1 e^{2(x+t)-(x+1)} b_{i,n}(t) dt} \right] b_{J,n}(x) dx \quad (4.3)$$

$$G_J = \int_0^1 \left(\frac{(2x^3 + e^x) e^{2x-1}}{+ \frac{e^{(x+2)} - e^{-(x+2)}}{x+2}} \right) b_{J,n}(x) dx, \quad J = 0, 1, 2, 3, 4 \quad (4.4)$$

$$w_k = b_{k,n}(0) + b_{k,n}(1), k = 0, 1, \dots, 4$$

For $n = 4$, the matrix \tilde{A} and the vector \tilde{b} are given by:

$$\tilde{A} = \begin{pmatrix} 0.1584 & 0.1266 & 0.1235 & 0.1441 & 0.1901 \\ 0.0949 & 0.1350 & 0.1627 & 0.1911 & 0.2346 \\ 0.0560 & 0.1095 & 0.1750 & 0.2428 & 0.3208 \\ 0.0360 & 0.0758 & 0.1470 & 0.2712 & 0.4847 \\ 1.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{pmatrix}, \tilde{b} = \begin{pmatrix} 0.9458 \\ 1.1459 \\ 1.4402 \\ 1.8654 \\ 3.0860 \end{pmatrix}$$

and:

$$X = (a_0, a_1, a_2, a_3, a_4)^T$$

Since $\det \tilde{A} \neq 0$; then equation $\tilde{A}X = \tilde{b}$ has a unique solution given by:

$$X = (0.3743, 0.5165, 0.8987, 1.3507, 2.7118)^T$$

The approximate solution is:

$$\tilde{y}(x) = a_0 b_{04}(x) + a_1 b_{14}(x) + a_2 b_{24}(x) + a_3 b_{34}(x) + a_4 b_{44}(x)$$

$$\tilde{y}(x) = 1.0093x^4 - 0.6808x^3 + 1.4404x^2 + 0.5684x + 0.3743$$

The graph of the exact and approximate solutions for Equation 4.1 is represented in Fig. 1. The results, together with the square error, are reported in Table 1 for different values of x .

Example 4.2

Consider the following linear FIDE of the first order with mixed conditions:

$$\begin{cases} (x^2 + 1)y'(x) + xy(x) - 2 \int_0^1 e^{x+t} y(t) dt = g(x) \\ \frac{1}{\sqrt{e}} y\left(\frac{1}{2}\right) + y(1) = -0.6947 \quad 0 \leq x \leq 1 \end{cases} \quad (4.5)$$

$$g(x) = \frac{(x^2 + x + 2 - e^2)e^x}{2 - e^2}$$

The exact solution (4.2) is:

$$y(x) = \frac{e^x}{2 - e^2}$$

We do the same as in example 1 by a comparison of the Equation 1.1 and 1.2 with (4.5), we find:

$$Q(x) = x^2 + 1, R(x) = x, K(x, t) = e^{x+t}, \lambda = -2$$

$$\mu = 0.6947, e_0 = \frac{1}{\sqrt{2}}, e_1 = 1, r_0 = \frac{1}{2}, r_1 = 1$$

We solve (4.5) by using the method described in Section (4), for $(n = 5)$, we find:

$$a_0 = -0.1855765, a_1 = -0.2226918, a_2 = -0.2690726,$$

$$a_3 = -0.327809, a_4 = -0.4016988, a_5 = -0.4947668$$

The approximate solution is:

$$\tilde{y}(x) = a_0 b_{05}(x) + a_1 b_{15}(x) + a_2 b_{25}(x)$$

$$+ a_3 b_{35}(x) + a_4 b_{45}(x) + a_5 b_{55}(x)$$

$$\tilde{y}(x) = -0.00213837x^5 - 0.0065302x^4 - 0.0315287x^3$$

$$- 0.0926552x^2 - 0.1855765x - 0.1855765$$

The graph of the exact and approximate solutions for Equation (4.5) is represented in Fig. 2 and the results are reported in Table 2 for different values of x .

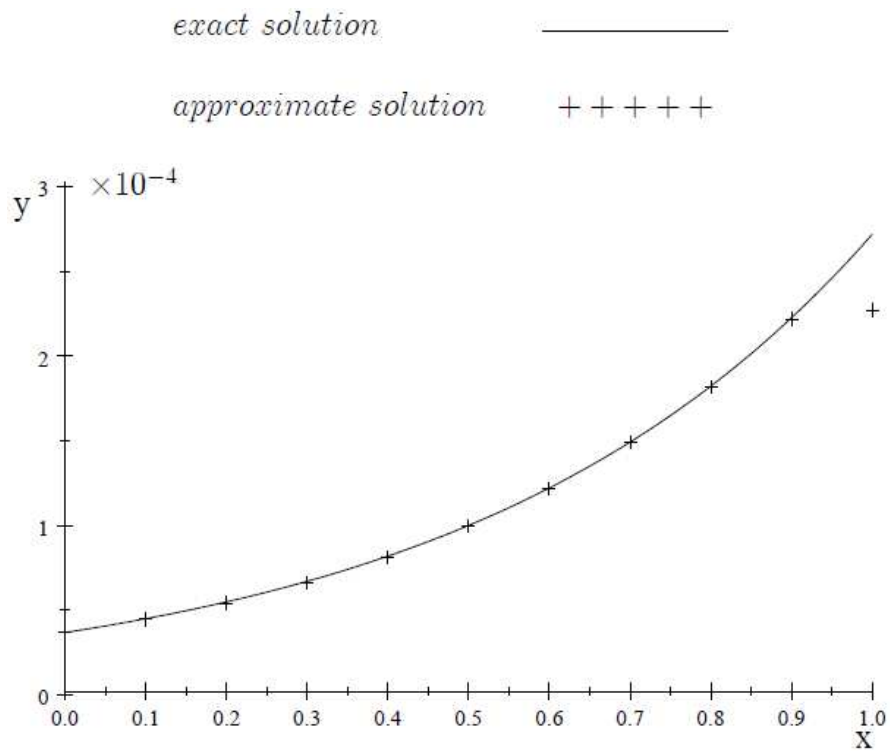


Fig. 1. Exact and approximate solution ($n = 4$) of example (4.1)

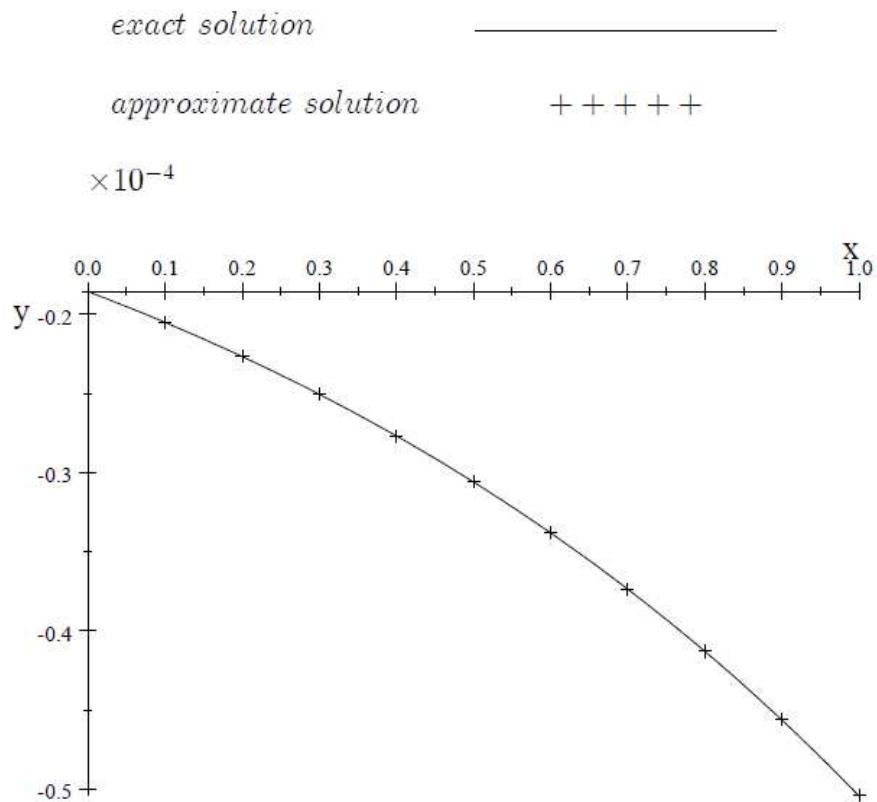


Fig. 2. Exact and approximate solutions for ($n = 5$) of example (4.2)

Table 1. Exact and approximate solutions and square error for ($n = 4$), Example (4.1)

x	Exact solution	Approximate solution	Square error $(y(x) - \tilde{y}(x))^2$
0	0.3679	0.3743	0.4096×10^{-4}
0.1	0.4493	0.4451	0.1764×10^{-4}
0.2	0.5488	0.5419	0.4761×10^{-4}
0.3	0.6703	0.6643	0.1101×10^{-4}
0.4	0.8187	0.8145	0.1764×10^{-4}
0.5	1.0000	0.9967	0.1101×10^{-4}
0.6	1.2214	1.2177	0.1369×10^{-4}
0.7	1.4917	1.4869	0.2304×10^{-4}
0.8	1.8221	1.8158	0.3469×10^{-4}
0.9	2.2255	2.2186	0.4761×10^{-4}
1	2.7183	2.7117	0.5476×10^{-4}

Table 2. Exact and approximate solutions and square error for ($n = 5$) of Example (4.2)

x	Exact solution	Approximate solution	Square error $(y(x) - \tilde{y}(x))^2$
0	-0.1856	-0.1856	0.0000×10^{-4}
0.1	-0.2051	-0.2051	0.0000×10^{-4}
0.2	-0.2266	-0.2267	0.0001×10^{-4}
0.3	-0.2505	-0.2505	0.0000×10^{-4}
0.4	-0.2768	-0.2768	0.0000×10^{-4}
0.5	-0.3059	-0.3059	0.0000×10^{-4}
0.6	-0.3381	-0.3381	0.0000×10^{-4}
0.7	-0.3737	-0.3736	0.0001×10^{-4}
0.8	-0.4130	-0.4129	0.0001×10^{-4}
0.9	-0.4564	-0.4562	0.0004×10^{-4}
1	-0.5044	-0.5040	0.0016×10^{-4}

Conclusion

We have applied the Galerkin's method by using the Bernstein polynomials to find the approximate solution of the linear FIDE of the first order. Using this procedure, the integro-differential form is reduced to solve a system of algebraic equations. Examples have been introduced to demonstrate the validity of the present technique. These examples also exhibit the accuracy and efficiency of the present method.

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Author's Contributions

Miloud Moussai: Did the research, analysed and interpreted the results. Prepared the manuscript and responsible for the paper submission.

Lakhdar Chiter: Designed the research plan, supervised and organized the structure of the manuscript and reviewed it. Responsible for the paper corrections.

Ethics

The corresponding author confirms that the co-author have read and approved the manuscript and no ethical issues involved.

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