

Original Research Paper

Numerical Solution of First Order Initial Value Problems Using a Self-Starting Implicit Two-Step Obrechhoff-Type Block Method

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Abstract: The conventional two-step implicit Obrechhoff method is a discrete scheme that requires additional starting values when implemented for the numerical solution of first order initial value problems. This paper therefore presents a two-step implicit Obrechhoff-type block method which is self-starting for solving first order initial value problems, hence bypassing the rigour of developing and implementing new starting values for the method. Numerical examples are considered to show the new method performing better when compared with previously existing methods in literature.

Keywords: Two-Step, Obrechhoff Method, Block Method, Self-Starting

Introduction

In a bid to model real-life problems in areas of engineering, biological sciences, physical sciences, electronics and many others, initial value problems are most times encountered (Shokri and Shokri, 2013). A sample first order initial value problem takes the form given below:

$$y' = f(x, y), y(a) = \alpha \quad (1)$$

However in most cases, these initial value problems cannot be solved analytically and hence the need for numerical methods. These numerical methods are adopted to obtain an approximate solution to the initial value problem under consideration (James *et al.*, 2013). Numerous numerical approaches have been proposed by scholars for the numerical approximation of initial value problems. These methods range from discrete schemes (Lambert, 1973; Butcher, 2008; Fatunla, 1988), to predictor-corrector methods (Kayode and Adeyeye, 2011; Adesanya *et al.*, 2008; Awoyemi and Idowu, 2005) and then block methods (Omar and Kuboye, 2015; Hasni *et al.*, 2013; Areo and Adeniyi, 2013).

One distinct family of methods for the numerical approximation of (1) above is the Obrechhoff methods. This family of methods is regarded to be distinct due to the presence of higher derivatives in the method. The general form of the k -step Obrechhoff method with l derivatives of y is given by Lambert (1973) as:

$$\sum_{j=0}^k \alpha_j y_{n+j} = \sum_{i=1}^l h^i \sum_{j=0}^k \beta_{ij} y_{n+j}^{(i)}; \alpha_k = +1 \quad (2)$$

with the implicit $k = 2$; $l = 2$ Adams-type method given as:

$$y_{n+2} - y_{n+1} = \frac{h}{240} (101y_{n+2}^{(1)} + 128y_{n+1}^{(1)} + 11y_n^{(1)}) + \frac{h^2}{240} (-13y_{n+2}^{(2)} + 40y_{n+1}^{(2)} + 3y_n^{(2)}) \quad (3)$$

However, adopting the method presented in Equation 3 above requires the introduction of additional one-step numerical methods such as Euler or Trapezoidal method. This approach most times does not provide optimal accuracy and hence the need to adopt block methods. Block methods have been found to give better approximation as seen in the studies by (Jator and Li, 2012; Omar, 2004; Adesanya *et al.*, 2014) amongst many others. Hence, this work presents a self-starting block method that will bypass the rigour of developing separate starting values and also with expected better accuracy when compared to previously existing methods in literature.

The second section of this article will show how the block method is derived and also discuss some basic properties of the block method while the next section will present some numerical examples and results. The fourth section of this article concludes this work.

The Two-Step Implicit Obrechhoff-Type Block Method

Consider an implicit $k = 2; l = 2$ Nyström-type form (Lambert, 1973) of Equation 2 above which can be written as:

$$\begin{aligned}
 & y(x_n) + hy'(x_n) + \frac{(2h)^2}{2!} y''(x_n) + \frac{(2h)^3}{3!} y'''(x_n) + \frac{(2h)^4}{4!} y^{(4)}(x_n) + y^{iv}(x_n) \frac{(2h)^5}{5!} + y^{v}(x_n) \frac{(2h)^6}{6!} y^{vi}(x_n) \\
 & = \alpha_0 y(x_n) + h \left[\beta_{10} y'(x_n) + \beta_{11} \left(y'(x_n) + hy''(x_n) + \frac{h^2}{2!} y''(x_n) + \frac{h^3}{3!} y^{iv}(x_n) + \frac{h^4}{4!} y^{v}(x_n) + \frac{h^5}{5!} y^{vi}(x_n) \right) \right. \\
 & \quad \left. + \beta_{12} \left(y'(x_n) + 2hy''(x_n) + \frac{(2h)^2}{2!} y''(x_n) + \frac{(2h)^3}{3!} y^{iv}(x_n) + \frac{(2h)^4}{4!} y^{v}(x_n) + \frac{(2h)^5}{5!} y^{vi}(x_n) \right) \right] \\
 & + h^2 \left[\beta_{20} \left(y''(x_n) + \beta_{21} (y'')(x_n) + hy'''(x_n) + \frac{h^2}{2!} y^{iv}(x_n) + \frac{h^3}{3!} y^{v}(x_n) + \frac{h^4}{4!} y^{vi}(x_n) \right) + \beta_{22} \left(y''(x_n) + 2hy'''(x_n) + \frac{(2h)^2}{2!} y^{iv}(x_n) \right) \right. \\
 & \quad \left. + \frac{(2h)^3}{3!} y^{v}(x_n) + \frac{(2h)^4}{4!} y^{vi}(x_n) \right]
 \end{aligned} \tag{5}$$

Where:

$$\begin{aligned}
 & y_{n+a} = y(x_n + ah) = y(x_n) \\
 & + ah y'(x_n) + \frac{(ah)^2}{2!} y''(x_n) + \frac{(ah)^3}{3!} y'''(x_n) + \dots
 \end{aligned}$$

Rewriting this expression in matrix form, where the coefficients of $h^i y^{(i)}(x_n)$ are equated gives:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & \frac{(1)^2}{2!} & \frac{(2)^2}{2!} & 0 & 1 & 2 & 2 \\ 0 & 0 & \frac{(1)^3}{3!} & \frac{(2)^3}{3!} & 0 & \frac{(1)^2}{2!} & \frac{(2)^2}{2!} & \frac{(2)^2}{2!} \\ 0 & 0 & \frac{(1)^4}{4!} & \frac{(2)^4}{4!} & 0 & \frac{(1)^2}{3!} & \frac{(2)^3}{3!} & \frac{(2)^3}{3!} \\ 0 & 0 & \frac{(1)^5}{5!} & \frac{(2)^5}{5!} & 0 & \frac{(1)^4}{4!} & \frac{(2)^4}{4!} & \frac{(2)^4}{4!} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \beta_{10} \\ \beta_{11} \\ \beta_{12} \\ \beta_{20} \\ \beta_{21} \\ \beta_{22} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ \frac{(2)^2}{2!} \\ \frac{(2)^3}{3!} \\ \frac{(2)^4}{4!} \\ \frac{(2)^5}{5!} \\ \frac{(2)^6}{6!} \end{pmatrix}$$

with values of $(\alpha_0, \beta_{10}, \beta_{11}, \beta_{12}, \beta_{20}, \beta_{21}$ and $\beta_{22})$ obtained to be:

$$\begin{aligned}
 & (\alpha_0, \beta_{10}, \beta_{11}, \beta_{12}, \beta_{20}, \beta_{21} \text{ and } \beta_{22})^T \\
 & = \left(1, \frac{7}{15}, \frac{16}{15}, \frac{7}{15}, \frac{1}{15}, 0, -\frac{1}{15} \right)^T \tag{6}
 \end{aligned}$$

Substituting the values of $\alpha_0, \beta_{10}, \beta_{11}, \beta_{12}, \beta_{20}, \beta_{21}$ and β_{22} obtained in Equation 6 above back in Equation 4 gives the two-step implicit method:

$$y_{n+2} = \sum_{j=0}^{k-2} \alpha_j y_{n+j} + \sum_{i=1}^l h^i \sum_{j=0}^k \beta_{ij} y_{n+j}^{(i)} \tag{4}$$

Expanding each term in (4) above using Taylor series gives the following expression:

$$y_{n+2} = y_n + \frac{h}{15} (7y_n^{(1)} + 16y_{n+1}^{(1)} + 7y_{n+2}^{(1)}) + \frac{h^2}{15} (y_n^{(2)} - y_{n+2}^{(2)}) \tag{7}$$

However, to implement the method derived in Equation 7, an additional method is needed.

This method is obtained by considering the following two-step method given as:

$$y_{n+1} = \sum_{j=0}^{k-2} \alpha_j y_{n+j} + \sum_{i=1}^l h^i \sum_{j=0}^k \beta_{ij} y_{n+j}^{(i)} \tag{8}$$

Following the same steps adopted for deriving the implicit $k = 2; l = 2$ Nyström-type method given in Equation 7, the additional method is derived to give:

$$\begin{aligned}
 & y_{n+1} - y_n = \frac{h}{240} (101y_n^{(1)} + 128y_{n+1}^{(1)} + 11y_{n+2}^{(1)}) \\
 & + \frac{h^2}{240} (-13y_n^{(2)} + 40y_{n+1}^{(2)} + 3y_{n+2}^{(2)})
 \end{aligned} \tag{9}$$

Hence, Equation 7 and 9 present the desired block method for the solution of (1).

However, to check for the order and the convergence of this block method, the following theorem and definitions are adopted.

Theorem 2.1 (Fatunla, 1988)

A linear multistep method is convergent iff it is consistent and zero-stable.

Definition 2.1 (Jator and Li, 2012)

A linear multistep method is consistent if it has order $p \geq 1$.

Definition 2.2 (Butcher, 2008)

The linear operator associated with Equation 2 is defined as:

$$L[y(x);h] = \sum_{j=0}^k \alpha_j y_{n+j} - \sum_{i=1}^l h \sum_{j=0}^k \beta_{ij} y_{n+j}^{(i)} \quad (10)$$

Expanding y_{n+j} and $y_{n+j}^{(i)}$ using Taylor series to obtain:

$$\begin{aligned}
 C_0 &= \begin{pmatrix} -1+1 \\ -1+1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
 C_1 &= \begin{pmatrix} 1 - \frac{1}{240}[101(1)+128(1)+11(1)] \\ 2 - \frac{1}{15}[7(1)+16(1)+7(1)] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
 C_2 &= \begin{pmatrix} \frac{(1)^2}{2!} - \frac{1}{240}[101(0)+128(1)+11(2)] - \frac{1}{240}[13(1)-40(1)-3(1)] \\ \frac{(2)^2}{2!} - \frac{1}{15}[7(0)+16(1)+7(2)] - \frac{1}{15}[1(1)-1(1)] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
 C_3 &= \begin{pmatrix} \frac{(1)_3}{3!} - \frac{1}{240}\left[101(0)+128\left(\frac{(1)^2}{2!}\right)+11\left(\frac{(2)^2}{2!}\right)\right] - \frac{1}{240}[13(0)-40(1)-3(2)] \\ \frac{(2)^3}{3!} - \frac{1}{15}\left[7(0)+16\left(\frac{(1)^2}{2!}\right)+7\left(\frac{(2)^2}{2!}\right)\right] - \frac{1}{15}[1(0)-1(2)] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
 C_4 &= \begin{pmatrix} \frac{(1)^4}{4!} - \frac{1}{240}\left[101(0)+128\left(\frac{(1)^3}{3!}\right)+11\left(\frac{(2)^3}{3!}\right)\right] - \frac{1}{240}\left[13(0)-40\left(\frac{(1)^2}{2!}\right)-3\left(\frac{(2)^2}{2!}\right)\right] \\ \frac{(2)^4}{4!} - \frac{1}{15}\left[7(0)+16\left(\frac{(1)^3}{3!}\right)+7\left(\frac{(2)^3}{3!}\right)\right] - \frac{1}{15}\left[1(0)-1\left(\frac{(2)^2}{2!}\right)\right] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
 C_5 &= \begin{pmatrix} \frac{(1)^5}{5!} - \frac{1}{240}\left[101(0)+128\left(\frac{(1)^4}{4!}\right)+11\left(\frac{(2)^4}{4!}\right)\right] - \frac{1}{240}\left[13(0)-40\left(\frac{(1)^3}{3!}\right)-3\left(\frac{(3)^3}{3!}\right)\right] \\ \frac{(2)^5}{5!} - \frac{1}{15}\left[7(0)+16\left(\frac{(1)^4}{4!}\right)+7\left(\frac{(2)^4}{4!}\right)\right] - \frac{1}{15}\left[1(0)-1\left(\frac{(2)^3}{3!}\right)\right] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
 C_6 &= \begin{pmatrix} \frac{(1)^6}{6!} - \frac{1}{240}\left[101(0)+128\left(\frac{(1)^5}{5!}\right)+11\left(\frac{(2)^5}{5!}\right)\right] - \frac{1}{240}\left[13(0)-40\left(\frac{(1)^4}{4!}\right)-3\left(\frac{(2)^4}{4!}\right)\right] \\ \frac{(2)^6}{6!} - \frac{1}{15}\left[7(0)+16\left(\frac{(1)^5}{5!}\right)+7\left(\frac{(2)^5}{5!}\right)\right] - \frac{1}{15}\left[1(0)-1\left(\frac{(2)^4}{4!}\right)\right] \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
 C_7 &= \begin{pmatrix} \frac{(1)^7}{7!} - \frac{1}{240}\left[101(0)+128\left(\frac{(1)^6}{6!}\right)+11\left(\frac{(2)^6}{6!}\right)\right] - \frac{1}{240}\left[13(0)-40\left(\frac{(1)^5}{5!}\right)-3\left(\frac{(2)^5}{5!}\right)\right] \\ \frac{(2)^7}{7!} - \frac{1}{15}\left[7(0)+16\left(\frac{(1)^6}{6!}\right)+7\left(\frac{(2)^6}{6!}\right)\right] - \frac{1}{15}\left[1(0)-1\left(\frac{(2)^5}{5!}\right)\right] \end{pmatrix} = \begin{pmatrix} \frac{1}{4725} \\ \frac{1}{9450} \end{pmatrix}
 \end{aligned}$$

This implies that the block method has order $p = 6$

with error constant $C_7 = \left(\frac{1}{4725}, \frac{1}{9450}\right)^T$.

Hence, the block method is consistent.

$$\begin{aligned}
 L[y(x);h] &= C_{0y}(x_n) + C_1 h y'(x_n) + C_2 h^2 y''(x_n) \\
 &+ \dots + C_q h^q y^{(q)}(x_n) + C_{q+1} h^{q+1} y^{(q+1)}(x_n)
 \end{aligned}$$

The method is said to be of order q if $C_0 = C_1 = \dots = C_q = 0, C_{q+1} \neq 0$ and C_{q+1} is the error constant.

The order of the block method in Equations 7 and 9 is gotten from:

Definition 2.3 (Jator and Li, 2012)

A block method is said to be zero-stable if after normalizing and writing the block method in the following matrix difference equation form:

$$A^0 Y_{\mu+1} = A^0 Y_{\mu} + h^m [B^0 F_{\mu+1} + B^1 F_{\mu}] + h^{m-1} C^1 \Delta_{\mu} \quad (11)$$

where, $Y_{\mu+1} = (y_{n+1}, \dots, y_{n+k})^T$, $Y_{\mu} = (y_{n-(k-1)}, \dots, y_n)^T$, $F_{\mu+1} = (f_{n+1}, \dots, f_{n+k})^T$, $F_{\mu} = (f_{n-(k-1)}, \dots, f_n)^T$, $\Delta_{\mu} = (\delta_{n-(k-1)}, \dots, \delta_n)^T$, $\mu = 0, 1, \dots$, the first characteristic polynomial takes the form:

$$\rho(R) = \det(RA^0 - A^1) \quad (12)$$

and the roots of $\rho(R) = 0$ satisfy $|R_j| \leq 1, j = 1, \dots, k$.

Adopting this approach to the block methods to test for zero stability, the block method is normalized to give the first characteristic polynomial as:

$$\rho(R) = \det(RA^0 - A^1) = R(R-1)$$

The roots of $\rho(R) = 0$ satisfy $|R_j| \leq 1, j = 1, 2$. Hence, the block is said to be zero-stable.

Therefore, since the block method is consistent and zero-stable, it is likewise convergent.

The region of absolute stability is determined by obtaining the stability polynomial from:

$$\det \left[\sum_{i=0}^k A^i q^{k-i} + z \sum_{i=0}^k B^i q^{k-i} + z^2 \sum_{i=0}^k C^i q^{k-i} \right], z = \lambda h \quad (13)$$

Hence, the stability polynomial for the block method is gotten as:

$$R(q) = \frac{q^3 z^4}{90} - \frac{q^3 z^3}{10} + \frac{13q^3 z^2}{30} - q^3 z + q^3 - \frac{qz^4}{90} - \frac{qz^3}{10} - \frac{qz^2}{30} - qz - q$$

Plotting the roots of the stability polynomial in boundary locus approach displays the region of absolute stability as shown Fig. 1.

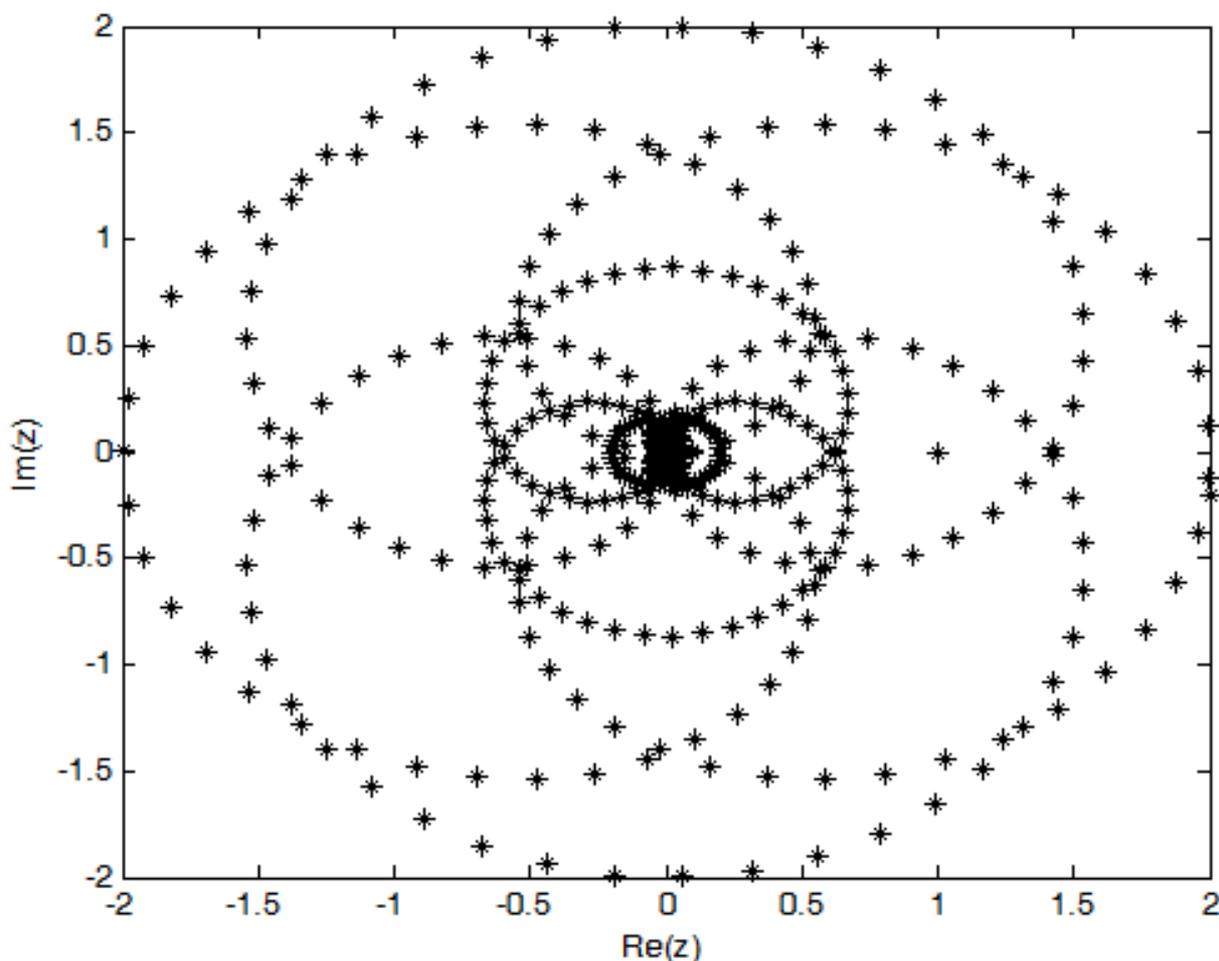


Fig. 1. Region of absolute stability for the two-step implicit Obrechhoff-Type block

Numerical Examples

This section presents some numerical examples and corresponding results of the new two-Step Obrechhoff-Type block method in comparison to the two-step method in Equation 3 and other existing methods in recent literature.

The following notations are used in the tables of results:

- 2SEM: Two-Step Implicit Obrechhoff method in Equation 3
- 2SBM: New Two-Step Obrechhoff-Type Block Method
- Error: |Computed Solution-Exact Solution|

Tested Problem 3.1 (Mixture Model)

In an oil refinery, a storage tank contains 2000 gal of gasoline that initially has 100 lb of an additive dissolved in it. In the preparation for winter weather, gasoline containing 2 lb of additive per gallon is pumped into the tank at a rate of 40 gal min⁻¹. The well-mixed solution is pumped out at a rate of 45 gal min⁻¹. Using a numerical integrator, how much of the additive is in the tank 0.1, 0.5 and 1 min after the pumping process begins?

Let y be the amount (in pounds) of additive in the tank at time t . We know that $y = 100$ when $t = 0$. Thus, the Initial Value Problem (IVP) modeling the mixture process is:

$$y' = 80 - \frac{45y}{(2000 - 5t)}, y(0) = 100, h = 0.1$$

with theoretical solution:

$$y(t) = 2(2000 - 5t) - \frac{3900}{(2000)^9} (2000 - 5t)^9$$

Source: Sunday *et al.* (2013).

The numerical result is shown in Table 1a and 1b.

Tested Problem 3.2 (SIR Model)

The SIR model is an epidemiological model that computes the theoretical number of people infected with a contagious illness in a closed population over time. The name of this class of models derives from the fact that they involve coupled equations relating the number of susceptible people $S(t)$, number of people infected $I(t)$ and the number of people who have recovered $R(t)$. This is a good and simple model for many infectious diseases including measles, mumps and rubella. It is given by the following three coupled equations:

$$\frac{dS}{dt} = \mu(1 - S) - \beta IS \tag{14}$$

$$\frac{dI}{dt} = -\mu I - \gamma I + \beta IS \tag{15}$$

$$\frac{dR}{dt} = -\mu R + \gamma I \tag{16}$$

Table 1a. Comparison of computed results for solving tested problem 3.1

T	Exact solution	Computed solution (2SEM)	Computed solution (Sunday <i>et al.</i> , 2013)	Computed solution (2SBM)
0.1	107.7662301168309486	107.76623267141251405	107.76623011683095	107.76623011683094855
0.2	115.5149409193028512	115.51494346840455900	115.51494091930284	115.51494091930285113
0.3	123.2461630508845221	123.24616814117862409	123.24616305088452	123.24616305088452198
0.4	130.9599271090910725	130.95993218819786255	130.95992710909107	130.95992710909107252
0.5	138.6562636455413535	138.65627125250773431	138.65626364554134	138.65626364554135350
0.6	146.3352031660153396	146.33521075612409816	146.33520316601533	146.33520316601533957
0.7	153.9967761305114566	153.99678623520317743	153.99677613051145	153.99677613051145660
0.8	161.6410129533038516	161.64102303550463010	161.64101295330383	161.64101295330385157
0.9	169.2679440029996051	169.26795658656269977	169.26794400299960	169.26794400299960502
1.0	176.8775996025958863	176.87761215807155490	176.87759960259586	176.87759960259588643

Table 1b. Comparison of error for solving tested problem 3.1

T	Error (2SEM)	Error (Sunday <i>et al.</i> , 2013)	Error (2SBM)
0.1	2.554000E-06	2.700062E-13	5.000000E-17
0.2	2.549000E-06	1.278977E-13	7.000000E-17
0.3	5.090000E-06	1.421085E-13	1.200000E-16
0.4	5.079000E-06	4.263256E-13	2.000000E-17
0.5	7.607000E-06	1.136868E-13	0.000000E+00
0.6	7.590000E-06	1.705303E-13	3.000000E-17
0.7	1.010000E-05	8.526513E-14	0.000000E+00
0.8	1.008000E-05	8.526513E-14	3.000000E-17
0.9	1.258000E-05	8.526513E-14	8.000000E-17
1.0	1.256000E-05	2.273737E-13	1.300000E-16

Table 2a. Comparison of computed results for solving tested problem 3.2

T	Exact solution	Computed solution (2SEM)	Computed solution (Sunday <i>et al.</i> , 2013)	Computed solution (2SBM)
0.1	0.52438528774964299546	0.52439024390243902439	0.5243852877552174	0.52438528774960472804
0.2	0.54758129098202021342	0.54758601695347485034	0.5475812909859664	0.54758129098194536511
0.3	0.56964601178747109638	0.56965499173623217471	0.5696460117956543	0.56964601178736527269
0.4	0.59063462346100907066	0.59064317588795690893	0.5906346234953703	0.59063462346087361956
0.5	0.61059960846429756588	0.61061180145439803533	0.6105996086572718	0.61059960846413739010
0.6	0.62959088965914106696	0.62960249743238951881	0.6295908898470451	0.62959088965895722513
0.7	0.64765595514064328282	0.64767066828934612765	0.6476559553183269	0.64765595514044005788
0.8	0.66483997698218034963	0.66485398110580363236	0.6648399771546479	0.66483997698195855368
0.9	0.68118592418911335343	0.68120256739332540639	0.6811859243738679	0.68118592418887672320
1.0	0.69673467014368328820	0.69675050937978129410	0.6967346704442603	0.69673467014343242661

Table 2b. Comparison of error for solving tested problem 3.2

T	Error (2SEM)	Error (Sunday <i>et al.</i> , 2013)	Error (2SBM)
0.1	4.956150E-06	5.574430E-12	3.826740E-14
0.2	4.725970E-06	3.946177E-12	7.484830E-14
0.3	8.979940E-06	8.183232E-12	1.058240E-13
0.4	8.552430E-06	3.436118E-11	1.354510E-13
0.5	1.219300E-05	1.929473E-10	1.601760E-13
0.6	1.160780E-05	1.879040E-10	1.838420E-13
0.7	1.471310E-05	1.776835E-10	2.032250E-13
0.8	1.400410E-05	1.724676E-10	2.217960E-13
0.9	1.664320E-05	1.847545E-10	2.366300E-13
1.0	1.583920E-05	3.005770E-10	2.508620E-13

Table 3a. Comparison of computed results for solving tested problem 3.3

X	Exact solution	Computed solution (2SEM)	Computed solution (Badmus <i>et al.</i> , 2015)	Computed solution (2SBM)
0.1	0.90483741803595957316	0.90476190476190476190	0.904837417881202	0.90483741804503260091
0.2	0.81873075307798185867	0.81866206899176567085	0.818730752939751	0.81873075309534995788
0.3	0.74081822068171786607	0.74069425289731179744	0.740818220548903	0.74081822070486153894
0.4	0.67032004603563930074	0.67020758320587849512	0.670320045918305	0.67032004606407889464
0.5	0.60653065971263342360	0.60637828956722340034	0.606530659599218	0.60653065974444846468
0.6	0.54881163609402643263	0.54867352672129543193	0.548811635994641	0.54881163612895298782
0.7	0.49658530379140951470	0.49641890512879110507	0.496585303694640	0.49658530382799175192
0.8	0.44932896411722159143	0.44917820458666454624	0.449328964033219	0.44932896415534885121
0.9	0.40656965974059911188	0.40639932795936316088	0.406569659658082	0.40656965977917485733
1.0	0.36787944117144232160	0.36772515831292540605	0.367879441100594	0.36787944121046227174

Table 3b. Comparison of error for solving tested problem 3.3

X	Error (2SEM)	Error (Badmus <i>et al.</i> , 2015)	Error (2SBM)
0.1	7.5513E-05	1.5476E-10	9.0730E-12
0.2	6.8684E-05	1.3823E-10	1.1768E-11
0.3	1.2397E-04	1.3282E-10	2.3144E-11
0.4	1.1246E-04	1.1733E-10	2.8440E-11
0.5	1.5237E-04	1.1342E-10	3.1815E-11
0.6	1.3811E-04	9.9385E-11	3.4927E-11
0.7	1.6640E-04	9.6770E-11	3.6582E-11
0.8	1.5076E-04	8.4003E-11	3.8127E-11
0.9	1.7033E-04	8.2517E-11	3.8576E-11
1.0	1.5428E-04	7.0848E-11	3.9020E-11

where, μ , γ and β are positive parameters. Define y to be:

$$y = S + I + R$$

and adding Equations 13-15, we obtain the following evolution equation for y :

$$y' = \mu(1 - y)$$

Taking $\mu = 0.5$ and attaching an initial condition $y(0) = 0.5$ (for a particular closed population), we obtain:

$$y'(t) = 0.5(1 - y), y(0) = 0.5, h = 0.1$$

with exact solution:

$$y(t) = 1 - 0.5e^{-0.5t}$$

Source: Sunday *et al.* (2013).

The numerical result is shown in Table 2a and 2b.

Tested Problem 3.3

$$y' = -y, y(0) = 1, h = 0.1$$

$$y(x) = e^{-x}$$

Source: Badmus *et al.* (2015).

The numerical result is shown in Table 3a and 3b.

Conclusion

The work presents a two-step implicit Obrechhoff-type block method which does not require the use of starting values in its implementation. The method was used to solve some numerical examples and the results were compared with the already existing two-step Obrechhoff method which required starting values for its implementation. The results gotten in the tables above showed the new block method having a far better accuracy when compared to the exact solution. Also, the block method was compared to some recent literature and the block method also showed better accuracy although this article only considered the solution of first order initial value problems with analytical solution because the aim of the article is to show how close the numerical solution obtained (by adopting the new block method) is to the exact solution. It is worth taking note of that the methods used in comparison are also methods with equal order (order 6). Hence, the basis of comparison is fair and this new method is proposed for adoption when solving first order initial value problems.

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Author's Contributions

Zurni Omar: Worked on the drafting, reading and approval of final manuscript.

Oluwaseun Adeyeye: Proposed, derived and developed a code for implementing the method.

Ethics

The authors declare that there is no conflict of interests regarding the publication of the paper.

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