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MAXIMUM LIKELIHOOD ESTIMATION FOR SPATIAL DURBIN MODEL

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ABSTRACT

Spatial Durbin Model (SDM) is one method of spatial autoregressive. This model was developed because the dependencies in the spatial relationships not only occur in the dependent variable, but also on the independent variables. In the assessment of parameter estimation, the process is carried out by Maximum Likelihood Estimation (MLE). This estimation can be approximation by Spatial Autoregressive Models (SAR). By MLE, the matrix of independent variable in SAR is X and in SDM is [I X W_1X], so that the estimation in SDM was done by replace matrix X in SAR by [I X W_1X]. This estimation perform the unbiased estimator for β and σ^2. Estimate ρ was done by optimize the concentrated log-likelihood function with respect to ρ.

Keywords: Maximum Likelihood Estimation, Spatial Autoregressive Models, Spatial Durbin Model

1. INTRODUCTION

Spatial Autoregressive Model (SAR) is very popular methods in spatial analysis. Spatial method is a method to get information of observations influenced by space or location effect. Lesage and Pace (2009) stated that the autoregressive process is indicated by the dependency relationship among a set of observations or locations.

Lesage and Pace (2009) has shown that one model of spatial autoregressive is Spatial Autoregressive Models (SAR), which the function is $y = \rho W_1 y + X\beta + \varepsilon$. It shows the spatial lag effect on the dependent variable. Spatial relationship among observations is expressed by the weight matrix (W_1) and parameter ρ which is the spatial lag parameter on dependent variable. Anselin in Lesage and Pace (2009) also called the model as Mixed Regressive-Autoregressive. Special cases of SAR model is Spatial Durbin Model which add lag effect of the independent variables, so that the model is $y = \rho W_1 y + \beta_0 + X\beta_1 + W_1 X\beta_2 + \varepsilon$.

The researches about Spatial Durbin Model (SDM) were Bekti and Sutikno (2012) about diarrhea modeling, Triki and Maktouf (2012) who studies the factors associated with the emergence of banking crises during the process of financial liberalization. Joshi and Gebremedhin (2012) were study about relationship between poverty and income inequality in the Appalachian region.

Several references have written about parameter estimation of spatial autoregressive model. There was Maximum Likelihood Estimation (MLE) which noted by Ord and Anselin in Lesage and Pace (2009) also Anselin and Rey (2010). Pushparaj (2013) and Seya *et al.* (2012) were use Bayesian estimation. Liu *et al.* (2010) have been noted Generalized Method of Moments (GMM) estimation of the regression and MRSAR models with SAR disturbance. Baltagi and Bresson (2011) has been study about maximum likelihood estimation and lagrange multiplier tests for panel Seemingly Unrelated Regressions with spatial lag and spatial errors.

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Lesage and Pace (2009) noted that estimation of spatial models by least squares can be lead to inconsistent estimates of the regression parameters for models with spatially lagged dependent variables, inconsistent estimation of the spatial parameters and inconsistent estimation of standard errors. In contrast, maximum likelihood is consistent for these models. Lu and Zhang (2010) stated that GMM was close to MLE in terms of model fitting, much easier in computation and robust to non-normality and outliers. Lu and Zhang (2010) were also showed that the Bayesian method with heteroscedasticity did not effectively estimate the spatial autoregressive parameters but produced very small biases for the regression coefficients of the model when few outliers existed.

In many cases and references, the estimation parameter of SDM was performed by estimation in SAR. The matrix of independent variable in SAR is X and in SDM is Z=[I X W_1X], so that the estimation in SDM was done by replace matrix X in SAR by Z = [I X W_1X]. Therefore, this study is performing parameter estimation in SDM.

2. MATERIALS AND METHODS

General model of spatial linear regression model for cross-section data is specified as:

$$y = \rho W_1 y + X\beta + u \quad (1)$$

And:

$$u = \lambda W_2 u + \varepsilon \quad \varepsilon \sim N(0,\sigma^2 I) \quad (2)$$

Where:

y	= Vector of dependent variable (n×1),
X	= Matrix of independent variable (n× (k+1))
β	= Vector of regression coefficient parameter ((k+1) ×1)
ρ	= Spatial lag coefficient parameter on dependent variable,
λ	= Spatial lag coefficient parameter on error u ε = error (n×1)
W_1 and W_2	= Weighted matrix (n×n)
I	= Identity matrix (n×n)
n	= Number of observations or locations (i= 1,2,3,...,n)
k	= Number of independent variable (k = 1,2,3,...,l)

Model in Equation (1) and Equation (2) were shown the autoregressive process in dependent variable and error. From Equation (1) and Equation (2), when X = 0 and W_2 = 0, then will be obtained spatial autoregressive model in first order, such Equation (3):

$$y = \rho W_1 y + \varepsilon \quad (3)$$

When W_1 = 0 or ρ = 0, Equation (1) will be Spatial Error Model (SEM) in Equation (4):

$$y = X\beta + \lambda W_2 u + \varepsilon \quad (4)$$

When W_2 = 0 or λ = 0, Equation (2) will be Spatial Autoregressive Model (SAR) or Mixed Regressive-Autoregressive in Equation (5):

$$y = \rho W_1 y + X\beta + \varepsilon \quad (5)$$

When ρ = 0 or λ = 0, Equation (1) and Equation (2) will be general linear regression in Equation (6). There is no spatial effect in this model:

$$y = \rho W_1 y + X\beta + \varepsilon \quad (6)$$

SDM is special cases of SAR, which adding spatial lag on independent variable (Lesage and Pace, 2009). This model was developed because the dependencies in the spatial relationships not only occur in the dependent variable, but also in the independent variable.

SDM model is specified as Equation (7). Parameter β_0 is intercept, β_1 is vector of regression coefficient parameter without weighted, β_2 is vector of regression coefficient parameter with weighted:

$$y = \rho W_1 y + \beta_0 + X\beta_1 + W_1 X\beta_2 + \varepsilon \quad (7)$$

Or:

$$y_i = \rho\sum_{j=1}^{n} w_{ij} y_j + \beta_0 + \sum_{k=1}^{l}\beta_{1k} x_{ki} + \sum_{k=1}^{l}\beta_{2k}\sum_{j=1}^{n} w_{ij} x_{kj} + \varepsilon_i$$

Vector coefficient parameter of spatial lag on independent variable is β_2. SDM can be formed into Equation (8):

$$y = (I - \rho W_1)^{-1} Z\beta + \varepsilon \quad (8)$$

Where:

$$y \sim N\left((I - \rho W_1)^{-1} Z\beta, \sigma^2 I\right) \quad Z = [I\ X\ WX] \quad \beta = [\beta_0\ \ \beta_1\ \ \beta_2]^T \quad (9)$$

Matrix Z in Equation (9) was shown that matrix X in SAR can be replaced by Z = [I X W_1 X] for SDM estimation.

The role of weighting is important because it represents relationships among locations. It also represents the neighboring among observations, so it needs the accuracy weighting method. The weight matrix is:

$$W = \begin{bmatrix} w_{11} & w_{12} & \dots & w_{1n} \\ w_{21} & w_{22} & \dots & w_{2n} \\ \dots & \dots & \dots & \dots \\ w_{n1} & w_{n2} & \dots & w_{nn} \end{bmatrix}$$

The value $w_{11}, \dots, w_{nn}$ can be take binary form, row standardization, or variance stabilization. Lesage and Pace (2012) has showed that the matrix W is typically row-stochastic, so the n×1 spatial lag vector Wy contains values constructed from an average of neighboring observations. Smith (2009) and Stakhovych and Bijmolt (2009) was noted that that for both spatial lag and spatial autoregressive models with strongly connected weight matrices, maximum likelihood estimates of the spatial dependence parameter are necessarily biased downward.

This study use the data to simulated the SDM estimation by MLE. The data was collected from Central Bureau of Statistics Indonesia in 2009 (BPS, 2010). It about relationships of rate of illiteracy (ILLITERACY) and the percentage of pouseholds owning a mobile phone (PHONE) in East Java, Indonesia. The locations were 38 regencies. In this simulation, ILLITERACY was the dependent variable and PHONE was the independent variable.

3. RESULTS

Parameter estimate in this research was done by Maximum Likelihood Estimation. From the equation of SDM:

$$y = \rho W_1 y + Z\beta + \varepsilon$$

Develop error in this Equation (10):

$$\varepsilon = y - \rho W_1 y - Z\beta \quad (10)$$

Or:

$$\varepsilon = (I - \rho W_1)y - Z\beta$$

Then, the likelihood function is in Equation (11-12):

$$L(\sigma^2;\varepsilon) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left(-\frac{1}{2\sigma^2}\left(\varepsilon^T\varepsilon\right)\right) \quad (11)$$

$$L(\rho,\beta,\sigma^2 \mid y) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} (J)\exp\left(-\frac{1}{2\sigma^2}\left(\varepsilon^T\varepsilon\right)\right) \quad (12)$$

The Jacobian function from Equation (10) can be performed by differentiation its equation to dependent variable y Equation 13:

$$J = \left|\frac{\partial\varepsilon}{\partial y}\right| = \left|I - \rho W_1\right| \quad (13)$$

Substitute Equation (10) to Equation (12), so that likelihood function is:

$$L(\rho,\beta,\sigma^2 \mid y) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \left|I - \rho W_1\right| \exp\left(-\frac{1}{2\sigma^2}\left(\left((I-\rho W_1)y - Z\beta\right)^T\left((I-\rho W_1)y - Z\beta\right)\right)\right) \quad (14)$$

Then, the natural logarithm of Equation (14) is Equation (15-16):

$$\ln(L) = \frac{n}{2}\ln\left(\frac{1}{2\pi\sigma^2}\right) + \ln\left|I - \rho W_1\right| - \frac{1}{2\sigma^2}\left(\left((I-\rho W_1)y - Z\beta\right)^T\left((I-\rho W_1)y - Z\beta\right)\right) \quad (15)$$

$$\ln(L) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) + \ln\left|I - \rho W_1\right| - \frac{1}{2\sigma^2}\left(\left((I-\rho W_1)y - Z\beta\right)^T\left((I-\rho W_1)y - Z\beta\right)\right) \quad (16)$$

Estimate β: Parameter estimate can be performed by maximize natural logarithm in Equation (11), which differentiation this equation to β. It shows in Equation (17) and the results shows in Equation (18) and Equation (19):

$$\frac{\partial \ln(L)}{\partial\beta} = 0$$

$$\frac{\partial \ln(L)}{\partial\beta} = \frac{\partial\left(-\frac{1}{2\sigma^2}\left(\left((I-\rho W_1)y - Z\beta\right)^T\left((I-\rho W_1)y - Z\beta\right)\right)\right)}{\partial\beta}$$

$$0 = \frac{\partial\left(-\frac{1}{2\sigma^2}\left(\left((I-\rho W_1)y - Z\beta\right)^T\left((I-\rho W_1)y - Z\beta\right)\right)\right)}{\partial\beta} \quad (17)$$

$$0 = \frac{1}{\sigma^2}\left(Z^T\left(I - \rho W_1\right)y - Z^T Z\beta\right) \quad \beta = \left(Z^T Z\right)^{-1} Z^T (I - \rho W_1)y$$

So that, the estimation is:

$$\hat{\beta} = \left(Z^T Z\right)^{-1} Z^T (I - \rho W_1) y \quad (18)$$

Or:

$$\hat{\beta} = \left(Z^T Z\right)^{-1} Z^T y - \rho \left(Z^T Z\right)^{-1} Z^T W_1 y \quad (19)$$

The estimator is unbiased. It was evidenced by:

$$E\left(\hat{\beta}\right) = E\left(\left(Z^T Z\right)^{-1} Z^T (I - \hat{\rho} W_1) y\right)$$
$$= \left(Z^T Z\right)^{-1} Z^T (I - \hat{\rho} W_1)\left((I - \hat{\rho} W_1)^{-1} Z\beta\right) = \beta$$

Estimate σ^2: Such at parameter estimate β, estimate σ^2 can be performed by differentiation Equation (11) to σ^2. It shows in Equation (20) and the results shows in Equation (21):

$$\frac{\partial \ln(L)}{\partial \sigma^2} = 0 \quad \frac{\partial \ln(L)}{\partial \sigma^2} = -\frac{n}{2\sigma^2}$$
$$+ \frac{1}{2\left(\sigma^2\right)^2}\left(((I - \rho W_1) y - Z\beta)^T ((I - \rho W_1) y - Z\beta)\right)$$
$$0 = -n + \frac{1}{\sigma^2}\left(((I - \rho W_1) y - Z\beta)^T ((I - \rho W_1) y - Z\beta)\right) \quad (20)$$
$$\sigma^2 = \frac{\left(((I - \rho W_1) y - Z\beta)^T ((I - \rho W_1) y - Z\beta)\right)}{n}$$

So that, the estimation is:

$$\hat{\sigma}^2 = \frac{(((I - \rho W_1) y - Z\beta)^T ((I - \rho W_1) y - Z\beta))}{n} \quad (21)$$

The estimation is biased. It was evidenced by $E\left(\hat{\sigma}^2\right) \neq \sigma^2$:

$$E\left(\hat{\sigma}^2\right) = E\left(\frac{\left(((I - \hat{\rho} W_1) y - Z\beta)^T ((I - \hat{\rho} W_1) y - Z\beta)\right)}{n}\right)$$
$$= \frac{1}{n} E\left(((I - \hat{\rho} W_1) y - Z\beta)^T ((I - \hat{\rho} W_1) y - Z\beta)\right)$$
$$= \frac{1}{n} E\left(\varepsilon^T \varepsilon\right)$$
$$= \frac{1}{n} E(SSE)$$

The unbiased estimation for σ^2 is:

$$\left(\frac{SSE}{\left(n - 2tr(S) + tr(S^T S)\right)}\right)$$

where, SSE is sum square error and:

$$S = \left(\rho W_1 + Z\left(Z^T Z\right)^{-1} Z^T (I - \rho W_1)\right)$$

Estimate ρ: Estimation of β and σ^2 are close form solutions. To produce maximum likelihood estimates for these parameters, Lesage and Pace (2009) stated that it needs to optimize the concentrated log-likelihood function with respect to ρ such as in Equation (26). Suppose that the estimation of ρ is $\hat{\rho}$, then Equation (19) become Equation (22):

$$\hat{\beta} = \left(Z^T Z\right)^{-1} Z^T y - \hat{\rho}\left(Z^T Z\right)^{-1} Z^T W_1 y \quad (22)$$

From Equation (19), can be develop two parameter estimations. There are $\hat{\delta}_0$ and $\hat{\delta}_d$ in Equation (23). Estimate δ_0 and δ_d can be develop from model $y = Z\delta_0 + e_0$ and $W_1 y - Z\delta_d + e_d$ by Ordinary Least Square:

$$\hat{\delta}_0 = \left(Z^T Z\right)^{-1} Z^T y \quad \hat{\delta}_d = \left(Z^T Z\right)^{-1} Z^T W_1 y \quad (23)$$

So:

$$\hat{\beta} = \left(Z^T Z\right)^{-1} Z^T y - \rho\left(Z^T Z\right)^{-1} Z^T W_1 y = \hat{\delta}_0 - \rho\hat{\delta}_d$$

Then, the error $e_0 = y - Z\delta_0$ and $e_d = W_1 y - Z\delta_d$ are substitute in parameter σ^2. The results shows in Equation (24):

$$\sigma^2 = \frac{\left\{[e_0 - \rho e_d]^T [e_0 - \rho e_d]\right\}}{n} \quad (24)$$

Substitute Equation (24) to Equation (11) will be performing natural logarithm to estimate ρ. The resutls shows in Equation (25):

$$\ln(L(\rho)) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2} \ln\left(\frac{[e_0 - \rho e_d]^T [e_0 - \rho e_d]}{n}\right) + \ln|I - \rho W_1| - \frac{1}{2}$$

$$\ln(L(\rho)) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\left\{[e_0 - \rho e_d]^T [e_0 - \rho e_d]\right\} - \frac{n}{2}\ln(n) + \ln|I - \rho W_1| - \frac{1}{2}$$

So:

$$f(\rho) = c - \frac{n}{2}\ln\left\{[e_0 - \rho e_d]^T [e_0 - \rho e_d]\right\} + \ln|I - \rho W_1| \quad (25)$$

Where:

$$c = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(n) - \frac{1}{2}$$

To get concentrated log-likelihood yields exactly the same as optimize maximum likelihood. There are many methods to calculate Jacobian J = $|I-\rho W_1|$ in Equation (25). Kelejian and Prucha (2007); Lesage and Pace (2009) were note these methods, such as scaling the weight matrix by its maximum eigenvalue and Monte Carlo approximation.

To simplify optimization of the log-likelihood with respect to the scalar parameter ρ, labeled as $\rho_1, \ldots, \rho_r$ in Equation (26):

$$\begin{pmatrix} f(\rho_1) \\ f(\rho_2) \\ \ldots\ldots \\ f(\rho_r) \end{pmatrix} = \begin{pmatrix} c - \frac{n}{2}\ln\left\{[e_0 - \rho_1 e_d]^T [e_0 - \rho_1 e_d]\right\} + \ln|I - \rho_1 W_1| \\ c - \frac{n}{2}\ln\left\{[e_0 - \rho_2 e_d]^T [e_0 - \rho_2 e_d]\right\} + \ln|I - \rho_2 W_1| \\ \ldots\ldots \\ c - \frac{n}{2}\ln\left\{[e_0 - \rho_r e_d]^T [e_0 - \rho_r e_d]\right\} + \ln|I - \rho_r W_1| \end{pmatrix} \quad (26)$$

4. DISCUSSION

The SDM model for simulated data specified as:

$$y = \rho W_1 y + \beta_0 + x\beta_1 + W_1 x\beta_2 + \varepsilon$$

Table 1. Parameter estimation by SDM

Parameters					Log likelihood
β_0	β_1	β_2	ρ	σ	
Method: Eigen					
7.856*	-2.679*	1.055**	0.408*	0.3497	-14.985
(-3.096)	(-9.721)	(-1.917)	(-2,582)		
Method: Chebyshev					
7.854*	-2.679*	1.055**	0.409*	0.3497	-14.984
(-3.095)	(-9.721)	(-1.9182)	(-2,582)		
Method:Monte carlo					
8.024*	-2.680*	1.021**	0.397*	0.3502	-15.071
(-3.136)	(-9.709)	(-1.841)	(-2.486)		

Note: the first and second row in the parameter column show the parameter estimate and t-statistic, (*) significant at α = 5%, (**) significant at α = 10%, n = 38

The rate of illiteracy (ILLITERACY) was the dependent variable. The percentage of households owning a mobile phone (PHONE) was the independent variable.

The results of simulation by R Software can be seen on **Table 1**. In R, it was use spdep package which introduced by Roger Bivand (Fischer and Getis, 2009). It was use Eigen, Chebychev and Monte Carlo to calculate the Jacobian. The weighted method was row standardization. The results show that the PHONE variable in all Jacobian methods were significance on α = 5%. The lag coefficient parameter on dependent variable (ρ) is significant on α = 5%. The lag of PHONE is significant on α = 10%.

5. CONCLUSION

The parameter estimation for SDM can be approximation by SAR estimation. The matrix of independent variable in SAR is X and in SDM is Z=[I X W_1X], so that the estimation in SDM was done by replace matrix X in SAR by Z = [I X W_1X].

The likelihood function is:

$$L(\rho,\beta,\sigma^2 \mid y) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} |I - \rho W_1| \exp\left(-\frac{1}{2\sigma^2}\left(((I - \rho W_1)y - Z\beta)^T ((I - \rho W_1)y - Z\beta)\right)\right)$$

The unbiased estimator of β is:

$$\hat{\beta} = \left(Z^T Z\right)^{-1} Z^T (I - \rho W_1)y$$

The biased estimator of σ_2 is:

$$\hat{\sigma}^2 = \frac{\left((I - \rho W_1)y - Z\beta)^T ((I - \rho W_1)y - Z\beta)\right)}{n}$$

and the unbiased estimator is:

$$\left(\frac{SSE}{\left(n - 2tr(S) + tr(S^T S)\right)}\right)$$

Where:

$$S = \left(\rho W_1 + Z\left(Z^T Z\right)^{-1} Z^T (I - \rho W_1)\right)$$

To estimate ρ by optimization of the log-likelihood with respect to the scalar parameter ρ, labeled as $\rho_1, \ldots, \rho_r$ in these equation:

$$\begin{pmatrix} f(\rho_1) \\ f(\rho_2) \\ \ldots\ldots \\ f(\rho_r) \end{pmatrix} = \begin{pmatrix} c - \frac{n}{2}\ln\left\{[e_0 - \rho_1 e_d]^T [e_0 - \rho_1 e_d]\right\} + \ln|I - \rho_1 W_1| \\ c - \frac{n}{2}\ln\left\{[e_0 - \rho_2 e_d]^T [e_0 - \rho_2 e_d]\right\} + \ln|I - \rho_2 W_1| \\ \ldots\ldots \\ c - \frac{n}{2}\ln\left\{[e_0 - \rho_r e_d]^T [e_0 - \rho_r e_d]\right\} + \ln|I - \rho_r W_1| \end{pmatrix}$$

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