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# On (2, 3, t)-Generations for the Conway Group $Co_2$

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**Abstract: Problem statement:** In this article we investigate all the (2, 3, t)-generations for the Conway's second largest sporadic simple group $Co_2$, where t is an odd divisor of order of $Co_2$. **Approach:** An (l, m, n)-generated group G is a quotient group of the triangle group T (l, m, n) = (x, y, z|$x^l = y^m = z^n = xyz = 1$). A group G is said to be (2, 3, t)-generated if it can be generated by two elements x and y such that o(x) = 2, o(y) = 3 and o (xy) = t. Computations are carried out with the aid of computer algebra system GAP-Groups, Algorithms and Programming. **Results and Conclusion:** The Conway group $Co_2$ is (2, 3, t)-generated for t an odd divisor of order of $Co_2$ except when t = 5, 7, 9.

**Key words:** Conway group, sporadic simple group, generation, subject classification, sporadic group

## INTRODUCTION

This study is intended as a sequel to author's earlier work on the determination of (2, 3, t)-generations for the sporadic simple groups. In a series of papers (Al-Kadhi, 2008a; 2008b; Al-Kadhi and Ali, 2010; Conway, 1985), the author with others established the (2, 3, t)-generations for the sporadic simple groups He, HS, $J_1$, $J_2$ and $Co_3$. Recently, the study of the Conway groups has received considerable amount of attention. Moori (1991) determined the (2, 3, p)-generations of the smallest Fischer group $Fi_{22}$. Ganief and Moori (1995) established (2, 3, t)-generations of the third Janko group $J_3$. More recently, Ali and Ibrahim (2012) computed the (2, 3, t)-generations for the Held's sporadic simple group He.

The present paper is devoted to the study of (2, 3, t)-generations of the Conway's sporadic simple group $Co_2$, where t is any odd divisor of $|Co_2|$. For more information regarding the study of (2, 3, t)-generations as well as the computational techniques, the reader is referred to (Ali and Ibrahim, 2005a; 2005b; Al-Kadhi, 2008a; 2008b; Al-Kadhi and Ali, 2010; Ganief and Moori, 1995; Moori, 1991; Liebeck and Shalev, 1996).

A group G is said to be (2, 3)-generated if it can be generated by an involution x and an element y of order 3. If o(xy) = t, we also say that G is (2, 3, t)-generated. The (2, 3)-generation problem has attracted a vide attention of group theorists. One reason is that (2, 3)-generated groups are homomorphic images of the modular group PSL (2, Z), which is the free product of two cyclic groups of order two and three. The connection with Hurwitz groups and Riemann surfaces also play a role. Recall that a (2, 3, 7)-generated group G which gives rise to compact Riemann surface of genus greater than 2 with automorphism group of maximal order, is called Hurwitz group.

## MATERIALS AND METHODS

Throughout this study our notation is standard and taken mainly from (Ali and Ibrahim, 2005a; Al-Kadhi and Ali, 2010; Moori, 1991). In particular, for a finite group G with $C_1,C_2,...,C_k$ conjugacy classes of its elements and $g_k$ a fixed representative of $C_k$, we denote $\Delta(G) = \Delta_G(C_1,C_2,...,C_k)$ the number of distinct tuples $(g_1, g_2,..., g_{k-1})$ with $g_i \in C_i$ such that $g_1g_2...g_{k-1} = g_k$. It is well known that $\Delta_G(C_1,C_2,...,C_k)$ is structure constant for the conjugacy classes $C_1,C_2,...,C_k$ and can easily be computed from the character table of G by the following formula:

$$\Delta_G(C_1,C_2,....,C_k) = \frac{|C_1||C_2|...|C_{k-1}|}{|G|}\times \sum_{i=1}^{m}\frac{X_i(g_1)X_i(g_2)...X_i(g_{k-1})\overline{X_i(g_k)}}{[X_i(1_G)]^{k-2}}$$

where, $X_1$, $X_2$,..., $X_m$ are the irreducible complex characters of G. Further let $\Delta^*(G) = \Delta^*_G(C_1, C_2,...,C_k)$ denote the number of distinct tuples $(g_1, g_2,..., g_{k-1})$ with $g_i \in C_i$ and $g_1, g_2...g_{k-1} = g_k$ such that G = < $g_1$, $g_2$,...,

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$g_{k-1}$ >. If $\Delta^*_G(C_1,C_2,...,C_k) > 0$, then we say that G is $(C_1,C_2,...,C_k)$-generated. If H any subgroup of G containing the fixed element $g_k \in C_k$, then $\Sigma_H(C_1,C_2,...,C_{k-1}, C_k)$ denotes the number of distinct tuples $(g_1, g_2,..., g_{k-1}) \in (C_1\times C_2\times...\times C_{k-1})$ such that $g_1g_2...g_{k-1} = g_k$ and $\langle g_1, g_2,..., g_{k-1}\rangle \le H$ where $\Sigma_H(C_1,C_2,...,C_k)$ is obtained by summing the structure constants $\Delta_H(c_1, c_2,..., c_k)$ of H over all H-conjugacy classes $c_1, c_2,..., c_{k-1}$ satisfying $c_i \subseteq H \cap C_i$ for $1\le i \le k - 1$.

The following results in certain situations are very effective at establishing non-generations.

**Theorem 1.1: (Scott's Theorem (Scott, 1977):** Let $x_1$, $x_2$,..., $x_m$ be elements generating a group G with $x_1x_2...x_n = 1_G$ and V be an irreducible module for G of dimension n ≥ 2. Let $C_V(x_i)$ denote the fixed point space of $\langle x_i \rangle$ on V and let $d_i$ is the codimension of $V/C_V(x_i)$. Then $d_1 + d_2 +... + d_m \ge 2_n$.

**Lemma 1.2: (Conder *et al*., 1992):** Let G be a finite centerless group and suppose lX, mY, nZ are G-conjugacy classes for which $\Delta^*(G) = \Delta^*_G$ (lX, mY, nZ) < $|C_G(z)|$, z ∈ nZ. Then $\Delta^*(G) = 0$ and therefore G is not (lX, mY, nZ)-generated. (2, 3, t)-Generations for $Co_2$.

## RESULTS AND DISCUSSION

The Conway group $Co_2$ is a sporadic simple group of order $2^{18}.3^6.5^3.7.11.23$ with 11 conjugacy classes of maximal subgroups. It has 60 conjugacy classes of its elements including three conjugacy classes of involutions, namely 2A, 2B and 2C. The group $Co_2$ acts primitively on a set of 2300 points. The points stabilizer of this action is isomorphic to $U_6(2)$:2 and the orbits have length 1, 891 and 1408. The permutation character of $Co_2$ on the cosets of $U_6(2)$:2 is given by $XU_6(2)$:2 = 1a + 275a + 2024a for basic properties of $Co_2$ and computational techniques, the reader is encouraged to consult (Ali and Ibrahim, 2005a; 2005b; Ganief, 1997; Ganief and Moori, 1995).

We now compute the (2, 3, t)-generations for the second Conway group $Co_2$. It is well know that if the group $Co_2$ is (2, 3, t)-generated then $\frac{1}{2}+\frac{1}{3}+\frac{1}{t}<1$. Further since we are concerned only with odd divisor of the order of $Co_2$, we only need to consider the cases when t = 7, 9, 15, 23. However, the case when t is prime has already been studied in Ganief (1997) so the remaining cases are t = 9, 15.

**Lemma 2.1:** The Conway group $Co_2$ is not (2X, 3Y, 9A)-generated where X ∈ {A, B, C}, Y ∈ {A,B}.

**Proof:** Using GAP we compute the algebra structure constants and obtain that:

$$\Delta_{Co_2}(2A,3Y,9A) = \Delta_{Co_3}(2B,3Y,9A) \\ <|C_{Co_2}(9A)|$$

Now by applying Lemma 2.2, we obtain:

$$\Delta^*_{Co_2}(2A,3Y,9A) = 0 = \Delta^*_{Co_2}(2B,3Y,9A)$$

Therefore (2A, 3Y, 9A) and (2B, 3Y, 9A) are not the generating triples for $Co_2$.

The group $Co_2$ acts on a 275-dimensional irreducible complex module V. Let $d_{nX}$ = $\dim(V/C_V(nX))$, the co-dimension of the fix space (in V) of a representative in nX. Using the character table of $Co_2$ and with the help of Scott's Theorem (Theorem 2.1) we compute that the values of $d_{nX}$. Our investigation conclude that the triple (2C, 3Y, 9A) violates the Scott's Theorem and thus $Co_2$ is not generated by (2C, 3Y, 9A)-generated. This completes the lemma.

**Theorem 2.2:** The sporadic simple group $Co_2$ is (2X, 3Y, 15Z)-generated where X,Z ∈ {A,B,C} and Y ∈ {A,B} if and only if (X, Y,Z) ∈ {(2C, 3Y, 15B), (2C, 3Y, 15C)}.

**Proof:** Since $\Delta_{Co_2}(2A,3Y,15Z) = \Delta_{Co_2}(2B,3Y,15Z) <|C_{Co_2}(15Z)| = 30$, by Lemma 2.2, the group $Co_2$ is not (2A, 3Y, 15Z)−, (2B, 3Y, 15Z)-generated.

Further an application of Theorem 2.1 implies that the triples (2C, 3Y, 15A) are not generating triples for $Co_2$.

Next we consider the triples (2C, 3A, 15B) and (2C, 3A, 15C). We compute that the structure constants:

$$\Delta_{Co_2}(2C,3A,15B) = 90 = \Delta_{Co_2}(2C,3A,15C)$$

Up to isomorphism, the maximal subgroups of $Co_2$ having non-empty intersection with the classes 2C, 3A and 15B or 15C (respectively) are L ≅ $(2^4 \times 2^{1+6})$. $A_8$, M ≅ $3^{1+6}:2^{1+4}.S_5$ and N ≅ $5^{1+2}:4S_4$. However, we obtain algebra constants as:

$$\Sigma_L(2C, 3A, 15B) = \Sigma_M(2C, 3A, 15B)$$
$$= \Sigma_N(2C, 3A, 15B) = 0$$

$$\Sigma_L(2C, 3A, 15C) = \Sigma_M(2C, 3A, 15C)$$
$$= \Sigma_N(2C, 3A, 15C) = 0$$

Therefore:

$$\Delta^*_{Co_2}(2C,3A,15B) = \Delta_{Co_2}(2C,3A,15B) = 90 > 0$$
$$\Delta^*_{Co_2}(2C,3A,15C) = \Delta_{Co_2}(2C,3A,15C) = 90 > 0$$

proving generation of $Co_2$ by these triples.

Finally, we consider the triples (2C, 3B, 15B) and (2C, 3B, 15C). For these triples we have $\Delta_{Co_2}$ (2C, 3B, 15B) = 75 = $\Delta_{Co_2}$ (2C, 3B, 15C). The only maximal subgroups of $Co_2$ which contains (2C, 3B, 15B)-, (2C, 3B, 15C)-generated proper subgroups, up to isomorphism, are $L \cong (2^4 \times 2^{1+6}).A_8$ and $M \cong 3^{1+6}:2^{1+4}.S_5$. Further, since $\Sigma_M(2C, 3B, 15B) = 0 = \Sigma_M$ (2C, 3B, 15C) we obtain:

$$\Delta^*_{Co_2}(2C,3B,15B) = \Delta_{Co_2}(2C,3B,15B) -$$
$$\sum{}_L(2C,3B,15B) = 75 - 15 > 0$$

$$\Delta^*_{Co_2}(2C,3B,15C) = \Delta_{Co_2}(2C,3B,15C) -$$
$$\sum{}_L(2C,3B,15C) = 75 - 15 > 0$$

Thus, $Co_2$ is (2C, 3B, 15B)- and (2C, 3B, 15C)-generated and the proof is complete.

## CONCLUSION

In this article we proved the following theorem.

**Theorem 3.1:** The Conway's second sporadic simple group is (2, 3, t)-generated for t is an odd divisor of order of $Co_2$, except when t =5, 7, 9.

**Proof:** This follows from Lemma 2.1, Theorem 2.2, results from Ganief (1997) and Ganief and Moori (1998) and the fact that triangle group T(2,3,5) is isomorphic to $A_5$.

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