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On (2, 3, t)-Generations for the Conway Group Co_2

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Abstract: Problem statement: In this article we investigate all the (2, 3, t)-generations for the Conway's second largest sporadic simple group Co_2, where t is an odd divisor of order of Co_2. **Approach:** An (l, m, n)-generated group G is a quotient group of the triangle group T (l, m, n) = (x, y, z|$x^l = y^m = z^n$ = xyz = 1). A group G is said to be (2, 3, t)-generated if it can be generated by two elements x and y such that o(x) = 2, o(y) = 3 and o (xy) = t. Computations are carried out with the aid of computer algebra system GAP-Groups, Algorithms and Programming. **Results and Conclusion:** The Conway group Co_2 is (2, 3, t)-generated for t an odd divisor of order of Co_2 except when t = 5, 7, 9.

Key words: Conway group, sporadic simple group, generation, subject classification, sporadic group

INTRODUCTION

This study is intended as a sequel to author's earlier work on the determination of (2, 3, t)-generations for the sporadic simple groups. In a series of papers (Al-Kadhi, 2008a; 2008b; Al-Kadhi and Ali, 2010; Conway, 1985), the author with others established the (2, 3, t)-generations for the sporadic simple groups He, HS, J_1, J_2 and Co_3. Recently, the study of the Conway groups has received considerable amount of attention. Moori (1991) determined the (2, 3, p)-generations of the smallest Fischer group Fi_{22}. Ganief and Moori (1995) established (2, 3, t)-generations of the third Janko group J_3. More recently, Ali and Ibrahim (2012) computed the (2, 3, t)-generations for the Held's sporadic simple group He.

The present paper is devoted to the study of (2, 3, t)-generations of the Conway's sporadic simple group Co_2, where t is any odd divisor of $|Co_2|$. For more information regarding the study of (2, 3, t)-generations as well as the computational techniques, the reader is referred to (Ali and Ibrahim, 2005a; 2005b; Al-Kadhi, 2008a; 2008b; Al-Kadhi and Ali, 2010; Ganief and Moori, 1995; Moori, 1991; Liebeck and Shalev, 1996).

A group G is said to be (2, 3)-generated if it can be generated by an involution x and an element y of order 3. If o(xy) = t, we also say that G is (2, 3, t)-generated. The (2, 3)-generation problem has attracted a vide attention of group theorists. One reason is that (2, 3)-generated groups are homomorphic images of the modular group PSL (2, Z), which is the free product of two cyclic groups of order two and three. The connection with Hurwitz groups and Riemann surfaces also play a role. Recall that a (2, 3, 7)-generated group G which gives rise to compact Riemann surface of genus greater than 2 with automorphism group of maximal order, is called Hurwitz group.

MATERIALS AND METHODS

Throughout this study our notation is standard and taken mainly from (Ali and Ibrahim, 2005a; Al-Kadhi and Ali, 2010; Moori, 1991). In particular, for a finite group G with $C_1,C_2,...,C_k$ conjugacy classes of its elements and g_k a fixed representative of C_k, we denote $\Delta(G) = \Delta_G(C_1,C_2,...,C_k)$ the number of distinct tuples $(g_1, g_2,..., g_{k-1})$ with $g_i \in C_i$ such that $g_1g_2...g_{k-1} = g_k$. It is well known that $\Delta_G(C_1,C_2,...,C_k)$ is structure constant for the conjugacy classes $C_1,C_2,...,C_k$ and can easily be computed from the character table of G by the following formula:

$$\Delta_G(C_1,C_2,....,C_k) = \frac{|C_1||C_2|...|C_{k-1}|}{|G|} \times \sum_{i=1}^{m} \frac{X_i(g_1)X_i(g_2)...X_i(g_{k-1})\overline{X_i(g_k)}}{[X_i(1_G)]^{k-2}}$$

where, X_1, X_2,..., X_m are the irreducible complex characters of G. Further let $\Delta^*(G) = \Delta^*_G(C_1, C_2,...,C_k)$ denote the number of distinct tuples $(g_1, g_2,..., g_{k-1})$ with $g_i \in C_i$ and $g_1, g_2...g_{k-1} = g_k$ such that G = < g_1, g_2,...,

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g_{k-1} >. If $\Delta^*_G(C_1,C_2,...,C_k) > 0$, then we say that G is $(C_1,C_2,...,C_k)$-generated. If H any subgroup of G containing the fixed element $g_k \in C_k$, then $\Sigma_H(C_1,C_2,...,C_{k-1}, C_k)$ denotes the number of distinct tuples $(g_1, g_2,..., g_{k-1}) \in (C_1\times C_2\times...\times C_{k-1})$ such that $g_1g_2...g_{k-1} = g_k$ and $\langle g_1, g_2,..., g_{k-1}\rangle \leq H$ where $\Sigma_H(C_1,C_2,...,C_k)$ is obtained by summing the structure constants $\Delta_H(c_1, c_2,..., c_k)$ of H over all H-conjugacy classes $c_1, c_2,..., c_{k-1}$ satisfying $c_i \subseteq H \cap C_i$ for $1\leq i \leq k-1$.

The following results in certain situations are very effective at establishing non-generations.

Theorem 1.1: (Scott's Theorem (Scott, 1977): Let $x_1, x_2,..., x_m$ be elements generating a group G with $x_1x_2...x_n = 1_G$ and V be an irreducible module for G of dimension $n \geq 2$. Let $C_V(x_i)$ denote the fixed point space of $\langle x_i\rangle$ on V and let d_i is the codimension of $V/C_V(x_i)$. Then $d_1 + d_2 +... + d_m \geq 2_n$.

Lemma 1.2: (Conder *et al.*, 1992): Let G be a finite centerless group and suppose lX, mY, nZ are G-conjugacy classes for which $\Delta^*(G) = \Delta^*_G(lX, mY, nZ) < |C_G(z)|$, $z \in nZ$. Then $\Delta^*(G) = 0$ and therefore G is not (lX, mY, nZ)-generated. (2, 3, t)-Generations for Co_2.

RESULTS AND DISCUSSION

The Conway group Co_2 is a sporadic simple group of order $2^{18}.3^6.5^3.7.11.23$ with 11 conjugacy classes of maximal subgroups. It has 60 conjugacy classes of its elements including three conjugacy classes of involutions, namely 2A, 2B and 2C. The group Co_2 acts primitively on a set of 2300 points. The points stabilizer of this action is isomorphic to $U_6(2)$:2 and the orbits have length 1, 891 and 1408. The permutation character of Co_2 on the cosets of $U_6(2)$:2 is given by $XU_6(2):2$ = 1a + 275a + 2024a for basic properties of Co_2 and computational techniques, the reader is encouraged to consult (Ali and Ibrahim, 2005a; 2005b; Ganief, 1997; Ganief and Moori, 1995).

We now compute the (2, 3, t)-generations for the second Conway group Co_2. It is well know that if the group Co_2 is (2, 3, t)-generated then $\frac{1}{2}+\frac{1}{3}+\frac{1}{t}<1$. Further since we are concerned only with odd divisor of the order of Co_2, we only need to consider the cases when t = 7, 9, 15, 23. However, the case when t is prime has already been studied in Ganief (1997) so the remaining cases are t = 9, 15.

Lemma 2.1: The Conway group Co_2 is not (2X, 3Y, 9A)-generated where $X \in \{A, B, C\}$, $Y \in \{A,B\}$.

Proof: Using GAP we compute the algebra structure constants and obtain that:

$$\Delta_{Co_2}(2A,3Y,9A) = \Delta_{Co_3}(2B,3Y,9A) < |C_{Co_2}(9A)|$$

Now by applying Lemma 2.2, we obtain:

$$\Delta^*_{Co_2}(2A,3Y,9A) = 0 = \Delta^*_{Co_2}(2B,3Y,9A)$$

Therefore (2A, 3Y, 9A) and (2B, 3Y, 9A) are not the generating triples for Co_2.

The group Co_2 acts on a 275-dimensional irreducible complex module V. Let $d_{nX} = \dim(V/C_V(nX))$, the co-dimension of the fix space (in V) of a representative in nX. Using the character table of Co_2 and with the help of Scott's Theorem (Theorem 2.1) we compute that the values of d_{nX}. Our investigation conclude that the triple (2C, 3Y, 9A) violates the Scott's Theorem and thus Co_2 is not generated by (2C, 3Y, 9A)-generated. This completes the lemma.

Theorem 2.2: The sporadic simple group Co_2 is (2X, 3Y, 15Z)-generated where $X,Z \in \{A,B,C\}$ and $Y \in \{A,B\}$ if and only if $(X, Y,Z) \in \{(2C, 3Y, 15B), (2C, 3Y, 15C)\}$.

Proof: Since $\Delta_{Co_2}(2A,3Y,15Z) = \Delta_{Co_2}(2B,3Y,15Z) < |C_{Co_2}(15Z)| = 30$, by Lemma 2.2, the group Co_2 is not (2A, 3Y, 15Z)−, (2B, 3Y, 15Z)-generated.

Further an application of Theorem 2.1 implies that the triples (2C, 3Y, 15A) are not generating triples for Co_2.

Next we consider the triples (2C, 3A, 15B) and (2C, 3A, 15C). We compute that the structure constants:

$$\Delta_{Co_2}(2C,3A,15B) = 90 = \Delta_{Co_2}(2C,3A,15C)$$

Up to isomorphism, the maximal subgroups of Co_2 having non-empty intersection with the classes 2C, 3A and 15B or 15C (respectively) are $L \cong (2^4 \times 2^{1+6}).A_8$, $M \cong 3^{1+6}:2^{1+4}.S_5$ and $N \cong 5^{1+2}:4S_4$. However, we obtain algebra constants as:

$$\Sigma_L(2C, 3A, 15B) = \Sigma_M(2C, 3A, 15B) = \Sigma_N(2C, 3A, 15B) = 0$$

$$\Sigma_L(2C, 3A, 15C) = \Sigma_M(2C, 3A, 15C) = \Sigma_N(2C, 3A, 15C) = 0$$

Therefore:

$$\Delta^*_{Co_2}(2C,3A,15B) = \Delta_{Co_2}(2C,3A,15B) = 90 > 0$$
$$\Delta^*_{Co_2}(2C,3A,15C) = \Delta_{Co_2}(2C,3A,15C) = 90 > 0$$

proving generation of Co_2 by these triples.

Finally, we consider the triples (2C, 3B, 15B) and (2C, 3B, 15C). For these triples we have Δ_{Co_2} (2C, 3B, 15B) = 75 = Δ_{Co_2} (2C, 3B, 15C). The only maximal subgroups of Co_2 which contains (2C, 3B, 15B)-, (2C, 3B, 15C)-generated proper subgroups, up to isomorphism, are $L \cong (2^4 \times 2^{1+6}).A_8$ and $M \cong 3^{1+6}:2^{1+4}.S_5$. Further, since Σ_M(2C, 3B, 15B) = 0 = Σ_M (2C, 3B, 15C) we obtain:

$$\Delta^*_{Co_2}(2C,3B,15B) = \Delta_{Co_2}(2C,3B,15B) - \sum\nolimits_L(2C,3B,15B) = 75 - 15 > 0$$

$$\Delta^*_{Co_2}(2C,3B,15C) = \Delta_{Co_2}(2C,3B,15C) - \sum\nolimits_L(2C,3B,15C) = 75 - 15 > 0$$

Thus, Co_2 is (2C, 3B, 15B)- and (2C, 3B, 15C)-generated and the proof is complete.

CONCLUSION

In this article we proved the following theorem.

Theorem 3.1: The Conway's second sporadic simple group is (2, 3, t)-generated for t is an odd divisor of order of Co_2, except when t =5, 7, 9.

Proof: This follows from Lemma 2.1, Theorem 2.2, results from Ganief (1997) and Ganief and Moori (1998) and the fact that triangle group T(2,3,5) is isomorphic to A_5.

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