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Numerical Solution for 2D European Option Pricing Using Quarter-Sweep Modified Gauss-Seidel Method

[1]Koh Wei Sin, [1]Jumat Sulaiman and [2]Rasid Mail
[1]Mathematics with Economics Programme, School of Science and Technology,
[2]Accountancy Programme, School of Business and Economics,
University Malaysia Sabah, 88400 Kota Kinabalu, Sabah, Malaysia

Abstract: Problem statement: This study presents the numerical solution of two-dimensional European option pricing problem based on Quarter-Sweep Modified Gauss-Seidel (QSMGS) iterative method. In fact, the pricing of European option with two-underlying assets can be governed by two-dimensional Black-Scholes Partial Differential Equation (PDE). **Approach:** The PDE needs to be discretized by using full-, half- and quarter-sweep second-order Crank-Nicolson schemes to generate a system of linear equations. Then, the Modified Gauss-Seidel, a preconditioned iterative method is applied to solve the generated linear system. **Results:** In order to examine the effectiveness of QSMGS method, several numerical experiments of Full-Sweep Gauss-Seidel (FSGS), Half-Sweep Gauss-Seidel (HSGS) and Quarter-Sweep Gauss-Seidel (QSGS) methods are also included for comparison purpose. Thus, the numerical experiments show that the QSMGS iterative method is the fastest in computing as well as having the least number of iterations. In the error analysis, QSMGS method shows good and consistent results. **Conclusion:** Finally, it can be concluded that QSMGS method is superior in increasing the convergence rate.

Key words: Modified Gauss-Seidel, quarter-sweep iteration, two-dimensional Black-Scholes PDE, Crank-Nicolson scheme

INTRODUCTION

Option is a financial derivative which gives the holder the right to trade the underlying asset by a certain date for a certain price. In trading the option, the right to buy is known as call option while the vice versa is put option. Black and Scholes (1973) and Merton (1973) derived the Black-Scholes Partial Differential Equation (PDE) for option pricing which earned them the 1997 Nobel Prize in Economics. In this study, we focus on a two-dimensional Black-Scholes PDE as follows Eq. 1 (Stulz, 1982; Jeong *et al.*, 2009):

$$\begin{aligned}\frac{\partial v}{\partial t} &= -\frac{1}{2}\sigma_1^2 s_1^2 \frac{\partial^2 v}{\partial s_1^2} - \frac{1}{2}\sigma_2^2 s_2^2 \frac{\partial^2 v}{\partial s_2^2} \\ &\quad - \rho\sigma_1\sigma_2 s_1 s_2 \frac{\partial^2 v}{\partial s_1 \partial s_2} - r s_1 \frac{\partial v}{\partial s_1} - r s_2 \frac{\partial v}{\partial s_2} - rv\end{aligned} \tag{1}$$

Where:
v = Value of the option
t = Time
s_1 = First asset's prices
s_2 = Second asset's price
σ_1 = Volatility of the first asset's price
σ_2 = Volatility of the second asset's price
ρ = Correlation between the two assets' price
r = Risk free interest rate

A European call option on the maximum of two underlying assets is evaluated by solving the two-dimensional Black-Scholes PDE. The final time condition for this problem is Eq. 2 (Stulz, 1982; Haug, 2007):

$$v(s_1, s_2, T) = \max\left(\max(s_1, s_2) - K, 0\right) \tag{2}$$

Where:
K = Exercise price
T = Maturity time

Besides that, the boundary conditions are Eq. 3-6:

$$v(s_1, 0, t) = s_1 N(d_1) - Ke^{-rt} N(d_2) \tag{3}$$

Corresponding Author: Koh Wei Sin, Mathematics with Economics Programme, School of Science and Technology, University Malaysia Sabah, 88400 Kota Kinabalu, Sabah, Malaysia

$$v(0,s_2,t)=s_2N(d_1)-Ke^{-rt}N(d_2) \quad (4)$$

$$v(S_1,s_2,t)=S_1-Ke^{-rt} \quad (5)$$

$$v(s_1,S_2,t)=S_2-Ke^{-rt} \quad (6)$$

Where:

$$d_1=\frac{\ln\left(\frac{s}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2=\frac{\ln\left(\frac{s}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}=d_1-\sigma\sqrt{T-t}$$

Where:
N = Cumulative normal distribution
S_1 = Maximum of s_1
S_2 = Maximum of s_2

Actually, the boundary conditions when one of the asset prices is zero as shown in 3,4 are obtained by using Black-Scholes formula (Black and Scholes, 1973).

The main objective of this study is to introduce Quarter-Sweep Modified Gauss-Seidel (QSMGS) method in solving the two-dimensional Black-Scholes PDE in European option pricing. Previously, QSMGS method had been used to solve the one-dimensional Black-Scholes PDE in European (Koh and Sulaiman, 2009) and American option pricing (Koh *et al*., 2010a). The results obtained have pointed out that QSMGS iterative method has a better convergence rate. For this manner, it is more motivating to study QSMGS in solving two-dimensional problem. Nevertheless, to discretize the PDE in Eq. 1, we will apply the quarter-sweep Crank-Nicolson approach. The Crank-Nicolson discretization scheme has second order accuracy and stability plus it is very famous for numerical computations in finance (Tavella and Randall, 2000). After a linear system is generated from the discretization process, the iterative method is computed to solve the linear system. Then, several numerical experiments involving FSGS, HSGS, QSGS and QSMGS methods are executed to verify the effectiveness of QSMGS method.

MATERIALS AND METHODS

Quarter-sweep Crank-Nicolson discretization scheme: We describe the full-, half- and quarter-sweep Crank-Nicolson schemes in discretizing Eq. 1. Both the full and quarter-sweep Crank Nicolson approximation equations can be derived as Eq. 7:

$$\begin{aligned}
&\frac{v_{i,j,k+1}-v_{i,j,k}}{\Delta t}= \\
&-\sigma_1^2(x_0+ip\Delta x)^2\left(\frac{v_{i-p,j,k}-2v_{i,j,k}+v_{i+p,j,k}+v_{i-p,j,k+1}-2v_{i,j,k+1}+v_{i+p,j,k+1}}{4(p\Delta x)^2}\right) \\
&-\sigma_2^2(y_0+jp\Delta y)^2\left(\frac{v_{i,j-p,k}-2v_{i,j,k}+v_{i,j+p,k}+v_{i,j-p,k+1}-2v_{i,j,k+1}+v_{i,j+p,k+1}}{4(p\Delta y)^2}\right) \\
&-\rho\sigma_1\sigma_2(x_0+ip\Delta x)(y_0+jp\Delta y) \\
&\left(\frac{v_{i+p,j+p,k}+v_{i-p,j-p,k}-v_{i-p,j+p,k}-v_{i+p,j-p,k}+v_{i+p,j+p,k+1}+v_{i-p,j-p,k+1}-v_{i-p,j+p,k+1}-v_{i+p,j-p,k+1}}{8(p\Delta x)(p\Delta y)}\right) \\
&-r(x_0+ip\Delta x)\left(\frac{v_{i+p,j,k}-v_{i-p,j,k}+v_{i+p,j,k+1}-v_{i-p,j,k+1}}{4p\Delta x}\right) \\
&-r(y_0+jp\Delta y)\left(\frac{v_{i,j+p,k}-v_{i,j-p,k}+v_{i,j+p,k+1}-v_{i,j-p,k+1}}{4p\Delta x}\right) \\
&+r\left(\frac{v_{i,j,k}+v_{i,j,k+1}}{2}\right)
\end{aligned} \quad (7)$$

Where:

$$x=s_1$$

$$y=s_2$$

As p = 1 or p = 2, it represents full- and quarter-sweep schemes respectively.

Based on Eq. 7, the full-sweep Crank-Nicolson approximation equation is a nine-point finite difference scheme. Hence, to derive the half-sweep Crank-Nicolson, we rotate the entire axis clockwise by 45° (Ali and Ling, 2008). As a result, a rotated nine-point approximation equation can be developed as Eq. 8:

$$
\begin{aligned}
\frac{v_{i,j,k+1}-v_{i,j,k}}{\Delta t} = \\
&-\sigma_1^{2}\left(x_0+i\Delta x\right)^{2}\left(\frac{v_{i-1,j+1,k}-2v_{i,j,k}+v_{i+1,j-1,k} +v_{i-1,j+1,k+1}-2v_{i,j,k+1}+v_{i+1,j-1,k+1}}{4\left(\Delta x\right)^{2}}\right) \\
&-\sigma_2^{2}\left(y_0+j\Delta y\right)^{2}\left(\frac{v_{i+1,j+1,k}-2v_{i,j,k}+v_{i-1,j-1,k} +v_{i+1,j+1,k+1}-2v_{i,j,k+1}+v_{i-1,j-1,k+1}}{4\left(\Delta y\right)^{2}}\right) \\
&-\rho\sigma_1\sigma_2\left(x_0+i\Delta x\right)\left(y_0+j\Delta y\right) \\
&\left(\frac{v_{i+2,j,k}+v_{i-2,j,k}-v_{i,j+2,k}-v_{i,j-2,k} +v_{i+2,j,k+1}+v_{i-2,j,k+1}-v_{i,j+2,k+1}-v_{i,j-2,k+1}}{16\left(\Delta x\right)\left(\Delta y\right)}\right) \\
&-r\left(x_0+i\Delta x\right)\left(\frac{v_{i+1,j-1,k}-v_{i-1,j+1,k} +v_{i+1,j-1,k+1}-v_{i-1,j+1,k+1}}{4\sqrt{2}\Delta x}\right) \\
&-r\left(y_0+j\Delta y\right)\left(\frac{v_{i+1,j+1,k}-v_{i-1,j-1,k} +v_{i+1,j+1,k+1}-v_{i-1,j-1,k+1}}{4\sqrt{2}\Delta x}\right) \\
&+r\left(\frac{v_{i,j,k}+v_{i,j,k+1}}{2}\right)
\end{aligned}
\tag{8}
$$

The nine-point approximation scheme for the full-, half- and quarter-sweep approaches can be illustrated in Fig. 1. According to Fig. 1, the implementations of FSGS, HSGS, QSGS and QSMGS iterative methods are performed onto the solid node points only until convergence criterion is met. Then, the remaining points are computed using direct method; see (Othman and Abdullah, 2000; Sulaiman *et al*., 2004a; 2004b; Koh and Sulaiman, 2009). Since half- and quarter-sweep approaches compute only half and quarter of the entire node points, theoretically they are faster than the standard full-sweep approach.

Modified Gauss-Seidel iterative methods: The approximations in Eq. 7 and 8 will generate large sparse linear system of form:

$$A\underset{\sim}{v}=\underset{\sim}{f} \tag{9}$$

Where:

A = Coefficient matrix

f = Known column vector, computed from the previous time level

v = Unknown column vector, values at the current time level

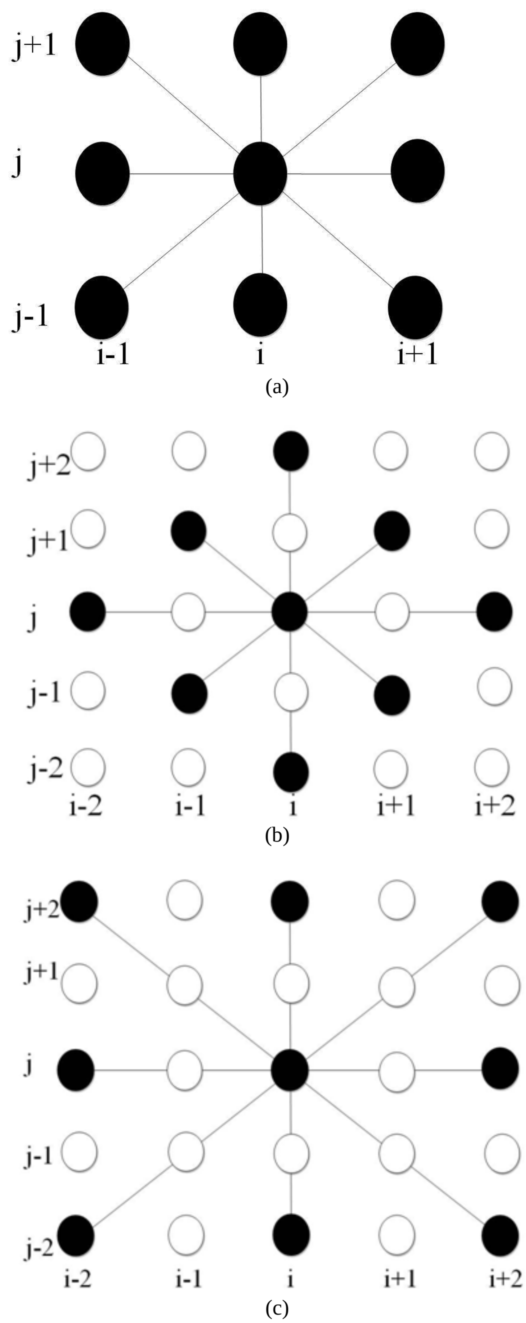

Fig. 1: Computational molecules of (a) full-sweep, (b) half-sweep and (c) quarter-sweep cases

In general, the linear system in Eq. 9 can be solved by applying Algorithm 1 (GS method).

Algorithm 1 (GS method):

* Initializing all the parameters. Set k = 0
* General iteration:

$$v_i^{(k+1)} = \frac{1}{A_{ii}}\left(f_i - \sum_{j=1}^{i-1} A_{ij} v_j^{(k+1)} - \sum_{j=i+1}^{i-n} A_{ij} v_j^{(k)}\right)$$

* Convergence test:

If the error tolerance is fulfilled, the value option at that time level is $v_i^{(k+1)}$ and the algorithm stops. Else, set k = k+1 and go to step ii.

In addition, Gunawardena *et al*. (1991) proposed the Modified Gauss-Seidel (MGS) method which uses a preconditioned matrix. The matrix when multiplied with the coefficient matrix will be able to transform the upper codiagonal to zero (Gunawardena *et al*., 1991). Following to that, we apply the preconditioned matrix to Eq. 9 and get the subsequent preconditioned linear system Eq. 10:

$$A^* \underset{\sim}{v} = \underset{\sim}{f}^* \tag{10}$$

Where:

$$A^* = PA$$

$$f^* = Pf$$

$$P = I + S$$

$$S = \begin{pmatrix} 0 & -\frac{a_{12}}{a_{11}} & 0 & \cdots & 0 \\ 0 & 0 & -\frac{a_{23}}{a_{22}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\frac{a_{n-1n}}{a_{n-1n-1}} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{(n\times n)}$$

I = Identity matrix

Then, the algorithm of MGS method can be formulated as in Algorithm 2.

Algorithm 2 (MGS method):

* Initializing all the parameters. Set k = 0.
* General iteration:
* $v_i^{(k+1)} = \frac{1}{A^*_{ii}}\left(f^*_i - \sum_{j=1}^{i-1} A^*_{ij} v_j^{(k+1)} - \sum_{j=i+1}^{i-n} A^*_{ij} v_j^{(k)}\right)$
* Convergence test:

If the error tolerance is fulfilled, the value option at that time level is $v_i^{(k+1)}$ and the algorithm stops. Else, set k = k+1 and go to step ii.

RESULTS

Several numerical experiments are performed to test the effectiveness of FSGS, HSGS, QSGS and QSMGS iterative methods. And the exact solution for problem 1 is given by Eq. 11 Haug (2007):

$$\begin{aligned} v(s_1,s_2,T) &= s_1 M(\varphi_1, d; \rho_1) + s_2 M\left(\varphi_2, -d + \sigma\sqrt{T}; \rho_2\right) \\ &\quad - Ke^{-rT}\left[1 - M\left(-\varphi_1 + \sigma_1\sqrt{T}, -\varphi_2 + \sigma_2\sqrt{T}; \rho\right)\right] \end{aligned} \tag{11}$$

Where:

$$d_1 = \frac{\ln\left(\frac{s_1}{s_2}\right) + \left(\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$\varphi_1 = \frac{\ln\left(\frac{s_1}{K}\right) + \left(\frac{r + \sigma_1^2}{2}\right)T}{\sigma_1\sqrt{T}}$$

$$\varphi_2 = \frac{\ln\left(\frac{s_2}{K}\right) - \left(\frac{r + \sigma_2^2}{2}\right)T}{\sigma_2\sqrt{T}}$$

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$$

$$\rho_1 = \frac{\sigma_1 - \rho\sigma_2}{\sigma}$$

$$\rho_2 = \frac{\sigma_2 - \rho\sigma_1}{\sigma}$$

Table 1: Number of iterations, execution time and RMSE for FSGS, HSGS, QSGS and QSMGS methods

	Method											
	FSGS			HSGS			QSGS			QMSGS		
Mesh size	Iter.	Time(s)	RMSE	Iter.	Time(s)	RMSE	Iter.	Time(s)	RMSE	Iter.	Time(s)	RMSE
50	18	0.43	1.05e-02	14	0.25	1.29e-01	11	0.15	8.63e-02	10	0.14	8.63e-02
100	41	3.23	2.47e-03	27	1.48	1.26e-01	18	0.53	2.37e-02	15	0.47	2.37e-02
150	76	13.43	8.74e-04	46	5.35	1.21e-01	28	1.48	7.07e-03	21	1.23	7.07e-03
200	124	48.91	6.48e-04	70	16.75	1.17e-01	41	3.67	5.74e-03	30	2.85	5.74e-03
250	185	108.45	4.05e-04	102	38.98	1.14e-01	57	8.14	3.77e-03	41	6.08	3.77e-03
300	259	220.49	2.21e-04	140	78.00	1.12e-01	76	15.91	1.78e-03	54	11.58	1.78e-03
350	345	419.78	2.11e-04	184	139.42	1.11e-01	98	27.43	1.89e-03	69	19.70	1.89e-03

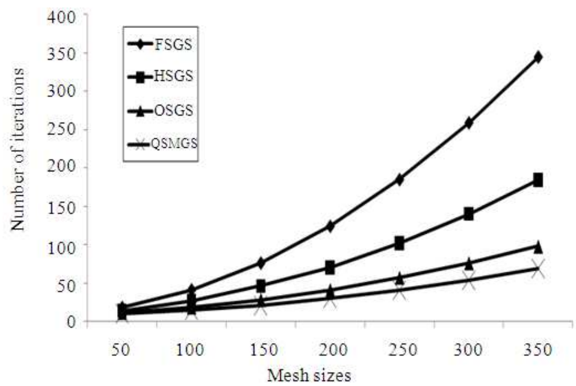

Fig. 2: Number of iterations versus mesh sizes of FSGS, HSGS, QSGS and QSMGS methods

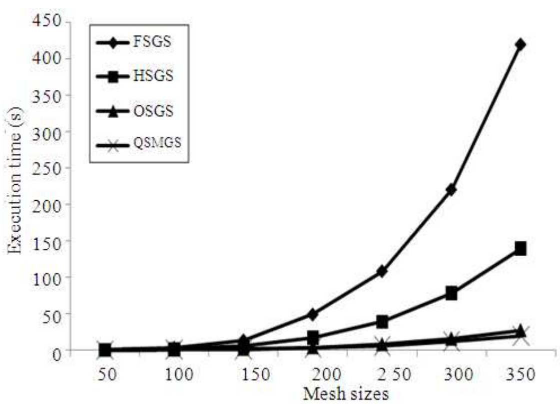

Fig. 3: Execution time (s) versus mesh sizes of FSGS, HSGS, QSGS and QSMGS methods

$M(a,b;\rho)$ Represents the cumulative bivariate normal distribution function; see (Drezner and Wesolowsky, 1990; Haug, 2007).

The parameters considered are T = 0.1, K=100, r = 0.03, $\sigma_1 = 0.5$, $\sigma_2 = 0.5$, $\rho = 0.5$, $s_1 = [0,300]$ and $s_2 = [0,300]$ which are also used by Jeong *et al*. (2009). Nonetheless in this study, we have 100 time steps with the tested matrix sizes, 50, 100, 150, 200, 250, 300 and 350. Furthermore, $ET = 10^{-10}$ will be the error tolerance. In computational finance, Root Mean Squared Relative Error (RMSE) is widely used to assess the accuracy of the iterative solutions such as in Zhao *et al.* (2007), Jeong *et al*. (2009) and Koh *et al.* (2010a). The RMSE is defined by:

$$\text{RMSE} = \sqrt{\frac{1}{(N_x-1)(N_y-1)}\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\left(\frac{\tilde{v}_{i,j}-v_{i,j}}{\varepsilon+v_{i,j}}\right)^2}$$

Where:

$v_{i,j}$ = Exact value
$v_{i,j}$ = Numerical value
ε = Small positive number to avoid dividing by a too small number
N_x = Mesh size for x
N_y = Mesh size for y

These computational experiments are implemented in the Intel Core 2 Duo, 2.93GHz processor. The computational results for FSGS, HSGS, QSGS and QSMS are tabulated in Table 1 and graphically displayed in Fig. 2 and 3.

DISCUSSION

The numerical experiments analyze FSGS, HSGS, QSGS and QSMGS from the aspects of number of iterations, execution time and RMSE. Based on the numerical results presented in Table 1 as well as Fig. 2 and 3 they clearly show that QSMGS method has the least number of iterations and execution time among the tested iterative methods. The number of iterations for HSGS, QSGS and QSMGS methods reduces by 22.22-44.67%, 38.89-71.59% and 44.44-80.00% respectively compared to FSGS method. In terms of execution time, HSGS, QSGS and QSMGS methods speed up approximately 41.86-66.79%, 65.12-93.47% and 67.44-95.31% respectively relative to FSGS method. On the other hand, the accuracies of the iterative methods are in good agreement except for HSGS which is not so fine.

CONCLUSION

In this study, we examined the application of QSMGS iterative method in evaluating European option with two-underlying assets by solving the two-dimensional Black-Scholes PDE. From the results presented, they can be observed that QSMGS iterative method converges faster than the other methods by having least number of iterations. Besides that, QSMGS manages to retain the accuracy of the standard Gauss-Seidel iterative method. Thus, we can conclude that QSMGS method is a better method compared to HSGS or FSGS method in the sense of number of iterations and execution time.

For future works, other types of option pricing, for example American option with two underlying assets or even other exotic options can be explored through the applications of quarter-sweep approach. In the context of iterative method, we can improve the preconditioned iterative method by employing the Improving Modified Gauss-Seidel method, (Kohno *et al*., 1997) which had been applied in solving one-dimensional European option pricing (Koh *et al*., 2010b). Also, family of two-stage iterative methods, for example AGE (Evans and Sahimi, 1988), IADE (Sahimi *et al*., 1993), HSIADE (Sulaiman *et al*., 2004a), QSIADE (Sulaiman *et al*., 2004b), AM (Ruggiero and Galligani, 1990), HSAM (Sulaiman *et al*., 2004c) and QSAM (Muthuvalu and Sulaiman, 2010) can be considered with the preconditioned iteration concept.

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