Journal of Mathematics and Statistics 7 (2): 95-97, 2011
ISSN 1549-3644

Complete Convergence of Exchangeable Sequences

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Abstract: We prove that exchangeable sequences converge completely in the Baum-Katz sense under the same conditions as i.i.d. sequences do. **Problem statement:** The research was needed as the rate of convergence in the law of large numbers for exchangeable sequences was previously obtained under restricted hypotheses. **Approach:** We applied powerful techniques involving inequalities for independent sequences of random variables. **Results:** We obtained the maximal rate of convergence and provided an example to show that our findings are sharp. **Conclusion/Recommendations:** The technique used in the paper may be adapted in the similar study for identically distributed sequences.

Key words: Exchangeable sequences, rate of convergence, strong law of large numbers

INTRODUCTION

A sequence of random variables $\{X_n\}_{n\geq 1}$ on the probability space (Ω, F, P) is called exchangeable if for every n:

$$P[X_1 \leq x_1,...,X_n \leq x_n] = P[X_{\pi 1} \leq x_1,...,X_{\pi n} \leq x_n]$$

for any permutation π of {1, 2,…,n} and any $x_i \in R$, i = 1,…, n. In particular, exchangeable sequences are identically distributed and one can say that future samples behave like earlier samples, or any order of a finite number of samples is equally likely. Sampling without replacement, weighted averages of i.i.d. sequences, $\{Y+\varepsilon_n\}_{n\geq 1}$ and $\{Y.\varepsilon_n\}_{n\geq 1}$ are examples of exchangeable sequences, where $\{\varepsilon_n\}_{n\geq 1}$ are i.i.d. and independent of the random variable Y. By [Chow and Teicher, 2003, Theorem 7.3.3], called de Finetti's theorem, an exchangeable sequence $\{X_n\}_{n\geq 1}$ is conditionally i.i.d. given either the tail σ-field of $\{X_n\}_{n\geq 1}$ or the σ-field G of permutable events.

MATERIALS AND METHODS

Under appropriate moment conditions, strong laws of large numbers for exchangeable sequences have been obtained in (Taylor and Hu, 1987; Etemadi and Kaminski, 1996; Etemadi, 2006; Rosalsky and Stoica, 2010). The rate of convergence in the above strong laws has not been obtained in full generality; for instance, papers (Zhao, 2004) assume p = 2, r = 1 and exponential, fourth and third order moments, respectively, for $\{X_n\}_{n\geq 1}$, whereas (Taylor and Hu, 1987) requires symmetry of the X'_ns and obtains estimate (1) provided p = 2r, 2 ≤ p < 4. Using appropriate inequalities for independent sequences, the purpose of this study is to prove that exchangeable sequences converge completely in the Baum-Katz sense under the same conditions as i.i.d. sequences do.

RESULTS AND DISCUSSION

Theorem 1: Let $\{X_n\}_{n\geq 1}$ be a sequence of exchangeable random variables with $E(X_1) = 0$ and $E|X_1|^p < \infty$ for some p≥1. If 0 < r < 2, p ≥ max{r, 1} and $S_n := X_1 + \ldots + X_n$, then Eq. 1:

$$\sum_{n=1}^{\infty} n^{p/r-2} P[|S_n| \geq n^{1/r}] < \infty \quad (1)$$

The following result (cf. (Petrov, 1995)) will be used in the proof of Theorem 1.

Lemma 2: Let $\{\xi_n\}_{n\geq 1}$ a sequence of independent random variables with $E(\xi_i) = 0$ and $E|\xi_i|^p < \infty$ for all i ≥ 1 and some p ≥ 1. If 0 < r < 2 and $T_n := \xi_1 + \ldots + \xi_n$, then:

$$P[|T_n| \geq n^{1/r}] \leq C_n^{-p/r} \sum_{i=1}^{n} E|\xi_i|^p$$
$$\text{if } 1 \leq p \leq 2 \text{(von Bahr – Esseen)}$$
$$P[|T_n| \geq n^{1/r}] \leq C_n^{-p/r} \sum_{i=1}^{n} E|\xi_i|^p + C\exp(-Cn^{2/r}\sigma^{-2})$$
$$\text{if } p \geq 2 \text{ (Fuk – Nagaev)}$$

Where:

$$\sigma^2 = \sum_{i-1}^{n} E(\xi_i^2)$$

Proof of Theorem 1: By [(Chow and Teicher, 2003), Corollary 7.3.5] there exists a regular conditional distribution P^ω given the σ-field G such that for each $\omega\in\Omega$ the mixands $\{\xi_n \equiv \xi_n^\omega\} n \geq 1$, i.e., the coordinate random variables of the Borel probability space $(R^\infty, B(R^\infty), P^\omega)$ are i.i.d. Namely, for all $n \in N$, any Borel function $f : R^n \to R$ and Borel set B on R, one has Eq. 2:

$$P[f(X_1,...,X_n) \in B] = \int_\Omega P^\omega[f(\xi_1,...,\xi_n) \in B]dP \qquad (2)$$

In what follows we shall use the following notations:

$$T_n^\omega = \xi_1^\omega + ... + \xi_n^\omega;$$
$$T_{1,n}^\omega = \xi_1^\omega 1(|\xi_1^\omega| \geq n^{1/r}) + ... + \xi_n^\omega 1(|\xi_n^\omega| \geq n^{1/r});$$
$$\xi_{2,n}^\omega = \xi_1^\omega 1(|\xi_1^\omega| < n^{1/r}) + ... + \xi_n^\omega 1(|\xi_n^\omega| < n^{1/r}), \text{for } n \geq 1$$

According to (2) and the bounded convergence theorem, we have Eq. 3:

$$\sum_{n=1}^{\infty} n^{p/r-2} P[|S_n| \geq n^{1/r}] = \int_\Omega \sum_{n=1}^{\infty} n^{p/r-2} P^\omega[|T_n^\omega| \geq n_{1/r}]dP \qquad (3)$$

On one hand, using that $\sum_{n=1}^{\infty} n^{p/r-1} \leq Ck^{p/r}$, we obtain:

$$\sum_{n=1}^{\infty} n^{p/r-1} P^\omega[|\xi_1^\omega| \geq n^{1/r}] \leq \sum_{k=1}^{\infty} P^\omega[k < |\xi_1^\omega|^r \leq k+1] \sum_{n=1}^{k} n^{p/r-1}$$
$$\leq C\sum_{k=1}^{\infty} k^{p/r} P^\omega[k < |\xi_1^\omega|^r \leq k+1] \leq CE^\omega(|\xi_1^\omega|^p) \text{a.s}$$

where, E^ω denotes expectation under P^ω.
Therefore:

$$\int_\Omega \sum_{n=1}^{\infty} n^{p/r-2} P^\omega[|T_{1,n}^\omega| \geq n^{1/r}]dP \leq$$
$$\int_\Omega \sum_{n=1}^{\infty} n^{p/r-1} P^\omega[|\xi_1^\omega| \geq n^{1/r}]dP \leq CE|X_1|p < \infty \qquad (4)$$

On the other hand, by Lemma 2 we obtain:

$$\sum_{n=1}^{\infty} n^{p/r-2} P^\omega[|T_{2,n}^\omega| \geq n^{1/r}]$$
$$\leq C\sum_{n=1}^{\infty} n^{-2} \sum_{k=1}^{n} E^\omega[(|\xi_k^\omega| \leq n^{1/r})] + C\sum_{n=1}^{\infty} n^{p/r-2}$$
$$\exp\left\{-C\frac{n^{2/r}}{nE^\omega[|\xi_1^\omega|^p 1(|\xi_1^\omega| \leq n^{1/r})]}\right\}$$
$$\leq C\sum_{n=1}^{\infty} E^\omega[|\xi_1^\omega|^p 1(n-1 < |\xi_1^\omega|^r \leq n)] \sum_{j=n}^{\hbar} j^{-2}$$
$$+C\sum_{n=1}^{\infty} n^{p/r-2} \exp\left\{-C\frac{n^{2/r-1}}{E^\omega|\xi_1^\omega|^p}\right\}$$
$$\leq CE^\omega(|\xi_1^\omega|^p) + C\sum_{n=1}^{\infty} n^{p/r-2} \exp\{-Cn^{2/r-1}\} \text{a.s}$$

The latter series is convergent as r < 2.
Therefore:

$$\int_\Omega \sum_{n=1}^{\infty} n^{p/r-2} P^\omega[|T_{2,n}^\omega| \geq n^{1/r}]dP \leq C(E|X_1|^p + 1) < \infty \qquad (5)$$

Conclusion (1) now follows from (4) and (5) via (3).

CONCLUSION

It is very well known that we cannot allow p < 1 in the Baum-Katz estimate (1) for i.i.d. sequences; the following example shows that the same is true for exchangeable sequences. Consider $X_n = Y.\varepsilon_n$, where $\{\varepsilon_n\}_{n\geq1}$ are i.i.d. and independent of a Cauchy random variable Y , with $P(\varepsilon_1 = 1) = P(\varepsilon_1 = -1) = 1/2$. We have $E|X_1|^p < \infty$ for all $0 < p < 1$, but $E|X_1| = \infty$. As $X_1 \sim -X_1$, we have:

$$\frac{S_n}{n} = Y.\frac{1}{n}\sum_{i=1}^{n} \varepsilon_i \to 0 \text{ a.s}$$

i.e., the strong law of large numbers holds for the exchangeable sequence $\{X_n\}_{n\geq1}$. On the other hand:

$$\sum_{n=1}^{\infty} n^{p/r-2} P[|S_n| \geq n^{1/r}] = \sum_{n=1}^{\infty} n^{p/r-2}$$
$$P(|\varepsilon_1 + ... + \varepsilon_n| \geq n).P(|Y| \geq n^{1/r-1})$$
$$\leq C\sum_{n=1}^{\infty} n^{p/r-2} \int_{|x|\geq n^{1/r-1}} \frac{1}{1+x^2} dx$$

which diverges for all $p \geq r$ and $0 < r < 2$.

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