Asymptotic Behavior for Solution of Reaction-Diffusion Systems

[1]M. Yahi and [2]K. Saoudi
[1] Scientific and Technical Research Center in Welding and Testing, 16 000. Algeria
[2]Department of Mathematics, Khenchela University, 40 000. Algeria

Abstract: The existence, uniqueness, and asymptotic behavior of the solution of a balanced two-component reaction-diffusion system have been investigated. It was shown that a global and unique solution existed and its second component can be estimated using the Lyapunov Functional. It was, also, demonstrated that each component of the solution converged, at infinity, to a constant which can be found in terms of the reacting function and the initial data. The results of the current research can be used in several areas of applied mathematics, especially when the system equations originate from mathematical models of real systems such as in Biology, Chemistry, Population Dynamics, and other disciplines.

Keywords: Reaction-Diffusion systems, Lyapunov Functional, Asymptotic Behavior, Global Existence.

INTRODUCTION

This article deals with some aspects of the diagonal reaction-diffusion system (S_1)

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1 \Delta u = g(u,v) & \text{in} \quad]0,+\infty[\times\Omega \\[2mm] \dfrac{\partial v}{\partial t} - d_2 \Delta v = f(u,v) & \text{in} \quad]0,+\infty[\times\Omega \\[2mm] \dfrac{\partial u}{\partial \eta} = \dfrac{\partial v}{\partial \eta} = 0 & \text{in} \quad]0,+\infty[\times\partial\Omega \\[2mm] u(0,x) = u_0, v(0,x) = v_0 & \text{in} \quad \Omega. \end{cases}$$

After stating local existence and uniqueness, we see some criteria for existence in the large (theorem 3.1 and 3.2) and we state the principal result (theorem 4.1) for the asymptotic behavior of the unique solution of (S_1). This generalized results of [3], [4] and [5].

Notations and Assumptions: Therein, $\Delta = \sum_{k=1}^{n} \partial_k^2$ denotes the Laplacian operator in the Euclidian coordinates, Ω is a domain (bounded connected open set) of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $\partial/\partial\eta$ is the outer normal on $\partial\Omega$, d_1 and $d_2 \in \mathbb{R}_+^*$. Two of the most fundamental assumptions associated with (S_1) are preservation of positivity and conservation of total mass. So, we suppose that the balance condition is satisfied which meant $f(u,v) + g(u,v) = 0$ and the reacting term f is, also, supposed to be a nonnegative and nonlinear continuously differentiable function in $\mathbb{R}_+^2$ which

verifies

$$\begin{cases} f(0,s) = 0 \quad \forall s \geq 0 \\[2mm] \lim_{s\to\infty} \left[\dfrac{\ln(1 + f(r,s))}{s} \right] \leq \alpha^* \forall r \geq 0 \end{cases} \tag{1.1}$$

where

$$\alpha^* = \frac{8 d_1 d_2}{n \| u_0 \|_\infty (d_1 - d_2)^2}. \tag{1.2}$$

Existence of a local solution in time: In the Banach space $C(\overline{\Omega})$ we define the norm $\| u \|_\infty = \sup_{x\in\overline{\Omega}} | u(x) |$. The product space $X = C(\overline{\Omega}) \times C(\overline{\Omega})$ endowed with the norm $\| (u,v) \|_X = \| u \|_\infty + \| v \|_\infty$ becomes a Banach space. If $w(t) = (u(t), v(t))$, then we can convert the system (S_1) to an abstract first-order system in the Banach space X of the form

$$\begin{cases} \dfrac{d}{dt}[w(t)] = \Pi w(t) + F[w(t)], t > 0 \\[2mm] w(0) = w_0 = (u_0, v_0) \in X \end{cases}$$

With $F[w(t)] = (-f(u,v), f(u,v))$ *i.e.* F is a locally lipschitz function on the space X. The operator $\Pi : D(\Delta) \times D(\Delta) \to X$ is defined by the matrix

$$\begin{pmatrix} d_1 \Delta & 0 \\ 0 & d_1 \Delta \end{pmatrix}$$

Corresponding Author: Yahi Mustefa, Scientific & Technical Research Center in Welding & Testing, 16000. Algeria

where

$$D(\Delta) = \left\{ u \in C(\overline{\Omega}) : \Delta u \in C(\overline{\Omega}), \frac{\partial u}{\partial \eta} = 0 \right\}.$$

If $d > 0$, then the operator $d\Delta$ generates an analytic semigroup of compact operators on the space $C(\overline{\Omega})$. If $S_1(t)$ and $S_2(t)$ are the analytic semigroups generated, respectively, by $d_1\Delta$ and $d_2\Delta$, on the space $C(\overline{\Omega})$, then the operator $\prod$ generates an analytic semigroup of compact operators on the space X which is given by $S(t) = \begin{pmatrix} S_1(t) & 0 \\ 0 & S_2(t) \end{pmatrix}$. The existence of a classical local solution of (2.1) in a maximal interval $[0, T_{max}[$ with initial value problem follows from standard arguments (*cf.* Henry D. [2], Lorenzi L. and all. [7] and Pazy A. [9]).

Existence of a global solution in-time: It may be surprising that question concerning the global existence of solution to the system (S_1) remain open until recently. If $d_1 = d_2$: Nonnegative solutions of (S_1) exist globally in-time. In fact, if we add the two equations of (S_1) we get the heat equation: $\frac{\partial}{\partial t}(u + v) - d_1\Delta(u + v) = 0$. Then, by the use of the balanced condition together with the maximum principle to obtain the L^∞-estimation of u and v since $\| (u + v)(t) \|_{\infty,\Omega} \leq \| u_0 + v_0 \|_{\infty,\Omega}$. So, global existence of solution holds. If $d_1 \neq d_2$: The situation is quite delicate. With the homogeneous Neumann boundary conditions, Alikakos N. D.[1] established the global existence via the embedding theorem of Sobolev and L^∞-bounds of solutions for positive initial data when $f(u,v) = -uv^\beta$ and $1 < \beta < \frac{n+2}{n}$. Masuda K.[8] Showed that the global solutions of (S_1) exist for $\beta > 1$ and converge to constant when $t \to +\infty$. If the nonlinearity f is polynomially bounded, then it is known that the solutions to (S_1) exist and remain uniformly bounded in the $L^\infty(\Omega)$-norm. Analogous result have been also established by Haraux A. and Youkana A.[3], and generalized their result to handle the nonlinearities of the function $f(u,v) = u\varphi(v)$

under condition $\alpha^* = 0$, where φ is a continuously differentiable nonnegative function not polynomially bounded but is required to grow less than exponentially.

All methods cited above (invariants regions, maximum principle, comparisons ...) have some limitations. Recall that the boundedness of the first component u of the solution is guaranteed by the maximum principle which assert, in one of its version, that: *If the component $u(t,x)$ verifies*

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1\Delta u \leq 0 & \text{in} \quad]0, T_{max}[\times \Omega \\[2mm] \dfrac{\partial u}{\partial \eta} \leq 0 & \text{in} \quad]0, T_{max}[\times \partial\Omega \end{cases}$$

Then $u(x,t) \leq \max\limits_{x \in \Omega} u_0(x) = \| u_0 \|_\infty$.

We discuss now the boundedness of the second component v of the solution. A *Lyapunov functional* L for the R-D system (S_1), is a continuously differentiable nonnegative function $L : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $L(t) = L(u(t,*),v(t,*))$ such that $\overset{\bullet}{L}(t) = \dfrac{\partial L[u(t,*),v(t,*)]}{\partial t} \leq 0$ for all solution $[u(t,*),v(t,*)]$ of the system (S_1).

Our main result, in this section, is:

Theorem 3.1: Suppose that $(u(t,*);v(t,*))$ is a solution of the system (S_1), then

$$t \to L(t) = \int_\Omega [M - u(t,x)]^{-\gamma} \exp[\beta v(t,x)] dx$$

is a decreasing functional in the interval $[0, T^*[$ iff the constants $\beta, \gamma \geq 0$ are such that

$$\beta M < \gamma < \frac{4 d_1 . d_2}{(d_1 - d_2)^2} \qquad (3.1)$$

Where M verify $\| u_0 \|_\infty < M$.

Proof: Differentiating the functional L as a function of t and using the two equations of the system (S_1) we obtain

$$\overset{\bullet}{L}(t) = \int_\omega \frac{\partial}{\partial t}\left([M - u(t,x)]^{-\gamma}\exp[\beta v(t,x)]\right) dx = I + J.$$

where

$$I = \int_\omega [\gamma d_1(M - u)^{-\gamma-1}e^{\beta v}\Delta u + \beta d_2(M - u)^{-\gamma}e^{\beta v}\Delta v] dx$$

$$J = \int_\omega [\beta(M - u)^{-\gamma} - \gamma(M - u)^{-\gamma-1}]f . e^{\beta v} dx.$$

To compute I, we use a Green formula to obtain

$$I = -\int_\Omega Q(\nabla u, \nabla v)(M - u)^{-\gamma - 2} e^{\beta v} dx$$

Where

$$Q(\nabla u, \nabla v) = \gamma d_1(\gamma + 1)|\nabla u|^2 + \beta\gamma(d_1 + d_2)(M - u)\nabla u\nabla v + \beta^2 d_2(M - u)^2 |\nabla v|^2.$$

We deduce that Q is a quadratic form in ∇u and ∇v *i.e.* $Q(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2$. If $I \le 0$ then Q will be a nonnegative quadratic form all principal determinants of its symmetric matrix

$$A_Q = \begin{pmatrix} d_1\gamma(\gamma + 1) & \dfrac{\beta\gamma(d_1 + d_2)(M - u)}{2} \\ \dfrac{\beta\gamma(d_1 + d_2)(M - u)}{2} & \beta^2 d_2(M - u)^2 \end{pmatrix}$$

associated to Q will be nonnegative. The first principal determinant of A_Q is $d_1\gamma(\gamma + 1) \ge 0$, the second is $|A_Q|$ which is nonnegative if $\gamma \le \dfrac{4d_1 d_2}{(d_1 - d_2)^2}$. If $J \le 0$ then

$$\beta(M - u)^{-\gamma} - \gamma(M - u)^{-\gamma - 1} \le 0,$$

i.e. $\beta(M - u) - \gamma \le 0$. Since we have $0 < u < M$ for all u then we obtain $\gamma > \beta M$. Thus, we have established that if $I \le 0$ and $J \le 0$ then (3.1) is satisfied. Now suppose that (3.1) is satisfied. First of all, recall that the Young inequality for a quadratic form, for $a \in \mathbb{R}_+^*$, b and $c \in \mathbb{R}_-^*$, has the form

$$ax^2 + bxy + cy^2 \le -\frac{(b^2 - 4ac)}{2}\left[\frac{x^2}{4c} + \frac{y^2}{4a}\right], \quad \text{for}$$

all $x, y \in \mathbb{R}$. To show that $I \le 0$, we observe that

$$I = \int_\Omega [a|\nabla|^2 b\nabla u.\nabla v + c|\nabla v|^2](M - u)^{-\gamma - 2} e^{\beta v} dx$$

Where
$$a = -d_1\gamma(\gamma + 1),$$
$$b = -\beta\gamma(d_1 + d_2)(M - u) \quad \text{and}$$
$$c = -\beta^2 d_2(M - u)^2.$$

By the Young inequality

$$a|\nabla|^2 + b\nabla u.\nabla v + c|\nabla v|^2 \le -\frac{(b^2 - 4ac)}{2}\left[\frac{|\nabla v|^2}{4a} + \frac{|\nabla u|^2}{4c}\right].$$

We have $d_1 d_2 - \gamma(d_1 - d_2)^2 > 0$ since $\gamma > \gamma/4$. Then we obtain

$$\frac{(b^2 - 4ac)}{8a} > \frac{d_1 d_2 - \gamma(d_1 - d_2)^2}{2d_1(\gamma + 1)} = m_2 > 0.$$

In the same way, since $0 < u < M$ for all u (Maximum Principle), we obtain:

$$\frac{(b^2 - 4ac)}{8c} > \frac{\gamma(d_1 d_2 - \gamma(d_1 - d_2)^2)}{2d_2\beta^2 M^2} = m_1 > 0.$$

Then

$$I \le -\int_\Omega [m_1|\nabla u|^2 + m_2|\nabla v|^2](M - u)^{-\gamma - 1} e^{\beta v} dx < 0.$$

To show that $J \le 0$, use $\gamma > \beta M$ and the maximum principle to obtain $M - u \le M$. Consequently

$$J \le -(\gamma - \beta M).M^{-(\gamma + 1)}\int_\Omega f(u, v).e^{\beta v} dx.$$

Now, let $C = (\gamma - \beta M).M^{-(\gamma + 1)}$ be a nonnegative constant, then $J \le 0$. Finally, we have showed

$$\dot{L}(t) \le -\int_\Omega [m_1|\nabla u|^2 + m_2|\nabla v|^2](M - u)^{-\gamma - 1} e^{\beta v} dx - C\int_\Omega f(u, v)e^{\beta v} dx \le 0.$$

DISCUSSION

We have established that $L(t) \le k$ for some constant k and hence $(M - u)^{-\gamma} e^{\beta v} \in L^\infty([0, T^*[, L^1(\Omega))$. Since $(M - u)^{-\gamma} \ge M^{-\gamma}$ we deduce that

$$e^{\beta v} \in L^\infty([0, T^*[, L^1(\Omega)) \qquad (3.2)$$

From Smoller J. [10], it is obvious that the region E is invariant for the system (S_1). This ensures the stability for solution to be nonnegative for all time. Since the reaction f is continuous in $\mathbb{R}_+^2$ and by the maximum principle, $0 < u < M$, it follow that for all $r \in [0, M]$ we can choose $\alpha \in \mathbb{R}^+$ to have, from (1.1), $\lim_{s \to +\infty}\left[\dfrac{\log(1 + f(r, s))}{s}\right] < \alpha < \alpha^*$. Then, there is $\gamma \in \mathbb{R}^+$ so that $f(r, s) \le \gamma e^{\alpha s}$, for all $s \ge 0$ and $r \in [0, M]$. Now, from $\alpha < \alpha^*$, we have $\dfrac{n}{2}\alpha < \dfrac{4d_1 d_2}{\|u_0\|_\infty (d_1 - d_2)^2}$. Choose $p \in \mathbb{N}_*^+$ such that $p > n/2$ and sufficiently close to $n/2$ such that $p\alpha < \dfrac{4d_1 d_2}{\|u_0\|_\infty (d_1 - d_2)^2}$. Put $\beta = p\alpha$ to obtain $\beta\|u_0\|_\infty < \dfrac{4d_1 d_2}{(d_1 - d_2)^2}$. Now, from (3.2) and since $\beta = p\alpha$ and $f(u, v) \le \gamma e^{\alpha v}$, we deduce that:

The p-norm $\|f(u, v)\|_p$ is uniformly estimated on $[0, T^*[$ for $p > n/2$.

Following Henry D.[2], one has

Theorem 3.2: If $f \in C^\infty(\mathbb{R}_+^2)$ is a nonnegative continuously differentiable function such that $f(0,s) = 0$ for all $s \geq 0$ and satisfying the condition (1.1) then all solutions of (S_1) with initial data in the region E, are global in time and uniformly bounded in $[0,+\infty[\times\Omega$.

Asymptotic Behavior for solution of Reaction-diffusion system: This section deals with behavior for solutions when $t \to +\infty$ of the system (S_1). With the help of boundedness of paths in the Banach space $X = C(\overline{\Omega}) \times C(\overline{\Omega})$, the asymptotic behavior for solutions can be analyzed by application of the *"Invariance Principle"*[6] and a result on paths precompacties which asserts that: *"if u and v are solutions of (S_1) and if $f \in L^\infty([0,+\infty), C(\overline{\Omega}))$ then u and $v \in L^\infty([0,+\infty), C(\overline{\Omega}))$"*[4].

The Main result, of this section, is the following:
Theorem 4.1: Let (u,v) a global solution of the system (S_1). There is a nonnegative

constants c_1 and c_2 such that

$$\|u - c_1\|_\infty \xrightarrow{t \to \infty} 0 \quad \text{and} \quad \|v - c_2\|_\infty \xrightarrow{t \to \infty} 0.$$

Furthermore, we have $f(c_1, c_2) = 0$ and

$$c_1 + c_2 = \frac{1}{|\Omega|}\int_\Omega (u_0 + v_0)dx.$$

To establish this result, we need the followings two lemmas.

Lemma 1: Let $\varphi \in L^1(\mathbb{R}^+)$ and $\varphi(t) \geq 0$ for all $t \geq 0$. Then $\lim_{t \to +\infty} \varphi(t) = 0$.

Proof: Suppose that we have $\lim_{t \to +\infty} \varphi(t) = c$ for some constant c. Then for all $\varepsilon > 0$ there is t_ε such that for $t > t_\varepsilon$ we have $\varphi(t) - c \geq -\varepsilon$. Put

$$\varepsilon = \frac{c}{2}, \quad \text{to} \quad \text{obtain} \quad \varphi(t) \geq \frac{c}{2} \quad \text{and}$$

$$\int_0^{+\infty} \varphi(t)dt \geq \int_{t_\varepsilon}^{+\infty} \varphi(t)dt \geq \int_0^{+\infty} \frac{c}{2}dt = +\infty \quad i.e. \quad \varphi \notin L^1(\mathbb{R}^+).$$

This contradicts the hypothesis.

Lemma 2: Let $K = [0,+\infty[\times\Omega$ and (u,v) be a global solution of (S_1), then

$$\int_K \left|\frac{\partial u}{\partial t}\right|^2 dxdt < +\infty \quad \text{and} \quad \int_K \left|\frac{\partial v}{\partial t}\right|^2 dxdt < +\infty.$$

Proof: Let $F = [0,+\infty[\times\partial\Omega$. Multiply the first equation of the system (S_1) by $\frac{\partial u}{\partial t}$ and integrate on K, then apply the Green formula

$$\int_K \left|\frac{\partial u}{\partial t}\right|^2 dxdt + d_1\int_K \nabla\left(\frac{\partial u}{\partial t}\right)\nabla u dxdt - d_1\int_F \left|\frac{\partial u}{\partial t}\right|\nabla u dxdt = -\int_K \frac{\partial u}{\partial t}f(u,v)dxdt.$$

Then

$$\int_K \left|\frac{\partial u}{\partial t}\right|^2 dxdt + \frac{d_1}{2}\left[\int_\Omega |\nabla u|^2 dx - \int_\Omega |\nabla u_0|^2 dx\right] = -\int_K \frac{\partial u}{\partial t}f(u,v)dxdt$$

and hence

$$\int_K \left|\frac{\partial u}{\partial t}\right|^2 dxdt \leq \frac{d_1}{2}\int_\Omega |\nabla u_0|^2 dx + \int_K \left|\frac{\partial u}{\partial t}f(u,v)\right|dxdt.$$

Now, apply the inequality $\frac{|k|^2 + |h|^2}{2} \geq |kh|$, and

put $k = \frac{\partial u}{\partial t}$ and $h = f(u,v)$, one gets

$$\int_K \left|\frac{\partial u}{\partial t}\right|^2 dxdt \leq d_1\int_\Omega |\nabla u_0|^2 dx + \int_K |f(u,v)|^2 dxdt.$$

Since $u_0 \in L^\infty(\Omega)$ and $f \in L^\infty([0,T^*], L^\infty(\Omega))$

we deduce $\int_K \left|\frac{\partial u}{\partial t}\right|^2 dxdt < \infty$ and in the same way,

we deduce that $\int_K \left|\frac{\partial v}{\partial t}\right|^2 dxdt < \infty$. This prove the

lemma.
According to the lemma 1, put

$$\varphi : t \mapsto \varphi(t) = \int_\Omega \left|\frac{\partial u}{\partial t}\right|^2 dx \quad \text{to obtain, by lemma 2,}$$

$$\lim_{t \to +\infty} \frac{\partial u(t,x)}{\partial t} = \lim_{t \to +\infty} \frac{\partial v(t,x)}{\partial t} = 0. \quad (4.1)$$

Now, we prove the main result of theorem 4.1. With the help of the two precedent lemmas.

Proof of theorem 4.1: The paths $\{u(t), t \geq 0\}$ and $\{v(t), t \geq 0\}$ associated to the system (S_1) are precompact in the Banach space $C(\overline{\Omega}) \times C(\overline{\Omega})$[4]. Pick a sequence $(t_n)_{n \in}$ which tend to infinity when $t \to +\infty$ such that $\lim_{t \to +\infty} u(t_n, *) = u_1$ and

$\lim_{t \to +\infty} v(t_n, *) = v_1$ in $C(\overline{\Omega})$. Since the function f is continuous, we obtain $\lim_{t \to +\infty} f(u(t_n, x)), v((t_n, x)) = f(u_1, v_1)$ in $C(\overline{\Omega})$. By (4.1), the system (S_1) becomes at the infinity, a system (S_∞) of the form

$$\begin{cases} -d_1 \Delta u = -f(u,v) & \text{in} \quad \Omega \\ -d_2 \Delta v = f(u,v) & \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial \eta} = \dfrac{\partial v}{\partial \eta} = 0 & \text{in} \quad \partial\Omega. \end{cases}$$

Multiply the first equation of (S_∞) by u_1 and integrate on Ω we obtain $d_1 \int_\Omega u_1 . \Delta u_1 dx \le 0$. The Green Formula gives us $d_1 \int_\Omega |\nabla u_1|^2 \, dx = 0$, then $\nabla u_1 = 0$ and hence $u_1 = c_1$ where c_1 is a constant. On the other hand, the function v_1 verifies the homogenuous Neumann Problem

$$\begin{cases} \Delta v_1 = 0 & \text{in} \quad \Omega \\ \dfrac{\partial v_1}{\partial \eta} = 0 & \text{in} \quad \partial\Omega. \end{cases}$$

It's solution is $v_1 = c_2$ where c_2 is a constant. Since the function f is continuous on $\mathbb{R}_+^2$, at infinity, we obtain $f(c_1, c_2) = 0$. The sequence $(t_n)_{n \in \mathbb{N}}$ is arbitrary, then $\| u - c_1 \|_\infty \to 0$ and $\| v - c_2 \|_\infty \to 0$ when $t \to +\infty$. Since the inclusion $C(\overline{\Omega}) \to L^1(\overline{\Omega})$ is continuous we deduce that

$$\int_\Omega (u(t,x) + v(t,x))dx \to |\Omega|(c_1 + c_2)$$
(4.2)

The convergence in $L^1(\overline{\Omega})$ is obvious since the first and the second equation of (S_1) yields $\dfrac{\partial}{\partial t} \int_\Omega u(t,x)dx \le 0$ and $\dfrac{\partial}{\partial t} \int_\Omega v(t,x)dx \ge 0$ then $\int_\Omega u(t,x)dx$ is a decreasing function in t and $\int_\Omega v(t,x)dx$ is an increasing function in t. But u is nonnegative and v is bounded, then

$\dfrac{1}{|\Omega|} \int_\Omega u(t,x)dx$ and $\dfrac{1}{|\Omega|} \int_\Omega v(t,x)dx$ converge in time and from (4.2) we deduce

$$c_1 + c_2 = \frac{1}{|\Omega|} \int_\Omega (u(t,x) + v(t,x))dx = \frac{1}{|\Omega|} \int_\Omega (u_0 + v_0)dx.$$

Conclusion. The results of this study have given rise to an open similar problem of the existence, uniqueness and asymptotic behavior of the solution of general balanced *n*-diagonal (or *n*-tridiagonal) reaction-diffusion systems, which can be thought of as further research.

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